Mathematics and crystallography

Ana-Maria Teleman

Abstract

We present a few geometrical results related to the symmetry of the crystals together with some philosophical comments.

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Refering to the hexagonal symmetry of the flowers of ice crystals which one can see in winter time on the windows and to the pentagonal symmetry of many flowers, Sir Thomas Browne said that they "declare in a delicate way how nature keeps geometrical order in all things." It is also amazing to see that each type of symmetry characterises a great variety of flowers or crystals. These tells us that the world that God has created is not only geometrically rigorous but has also the beauty given by an extraordinary artist.

The mathematical description of crystals is very important for many scientifical fields. Their symmetry is described by the crystallographic groups. These groups allow to classify the crystals from both mathematical and physical point of view.

Crystals are considered as subsets of the Euclidian space E^3. We recall

The fundamental theorem of Euclidian geometry : *The isometries of the Euclidian space E^n are functions $f : E^n \to E^n$, $f(x) = Ax + b$, for any $x \in E^n$, where $A \in O(n)$ and $b \in E^n$.*
In our case $n = 3$.
Definition 1: *A crystallographic group is a discrete group of isometries of E^3: $C = \{f = [x \to Ax + b]; A \in G, b \in \Gamma\}$, where $G \subset O(3)$ is a finite group and Γ contains an abelian free group generated by three linearly independent translations of E^3.*
Definition 2: *A group of transformations of a topological space is called a discrete group of transformations when all its orbits are discrete.*

We consider on E^3 the topology associated with the Euclidian metric.

The finite group G, associated to a crystallographic group is related to the exterior aspect of the corresponding crystal. The crystallographic group itself describes

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the internal structure of the crystal, namely the set of equilibrium positions of the atoms in the crystallographic lattice.
Definition 3: *Two crystallographic groups C and C' are called of the same type if there exists $T \in SO(3)$ such that $C' = TCT^{-1}$.*
It has been proven that:

Theorem 1: *There are exactly 230 types of crystallographic groups.*

Theorem 2: *There are exactly 32 types of finite subgroups $G \subset O(3)$ corresponding to all crystallographic groups.*

These 32 types of finite subgroups of the orthogonal group $O(3)$ are classified in 7 **crystalllographic systems**, each system containing a maximal group and a set of some of its subgroups.

The 7 crystallographic systems are: cubic (5 types), quadratic or tetragonal (7 types), orthorombic (5 types), monoclinic (3 types), triclinic (2 types), romboedric or trigonal (3 types) and hexagonal (7 types).
We give some examples of mineral crystals:
- cubic system: diamond, garnet, ...,
-quadratic system: zircon, scapolyte, ... ,
-orthorombic system: topaz, olivine, ... ,
-monoclinic system: gypsum, feldspar, ... ,
-triclinic system: axinite, disthene, ... ,
-romboedric system: tourmaline, quartz, calcite, ... ,
-hexagonal system: beryl, apatite, ...

In Romania can be found very beautiful crystals of quartz. Among them we mention the famous **diamonds of Maramureş**.

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Ana-Maria Teleman
University of Bucharest, Faculty of Mathematics and Informatics
14 Academiei St., RO-010014, Bucharest, Romania