On the mathematical work of Gheorghe Tzitzeica

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Abstract

The aim of this paper is to expose the main ideas in the work of G. Tzitzeica.

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The aim of this article is to give some details from the theory contained in the book "*Géométrie Différentielle Projective des Réseaux*" which was published by Tzitzeica in 1923. This masterpiece contains a great amount of beautiful geometrical results, most of which belonging to the Romanian mathematician.

1. Suppose that the equations

$$x^\alpha = f^\alpha(u,v)\ ,\ (\alpha = 0,1,...,n) \tag{0.1}$$

define a surface S in real projective space P^n. One says that the equations $du = 0$, $dv = 0$ define a *lattice of mutually conjugated lines* on the surface S if the functions f^α are solutions of a PDF equation of Laplace type:

$$E(a,b,c):\ x_{uv} + a\ x_u + b\ x_v + c\ x = 0, \tag{0.2}$$

where a, b, c are differentiable functions of u, v and x_u , x_v , x_{uv} are partial derivatives.

The solutions of the equation $du = 0$ will be named $v-$*lines* and the solutions of the equation $dv = 0$ will be named $u-$*lines*.

It is not difficult to check that, when the homogeneous coordinates x^α of the points $[x] \in S$ are solutions of the equation (2), the coordinates of the points

$$L_1(x) = x_v + a\ x\ ,\ L_{-1}(x) = x_u + b\ x \tag{0.3}$$

will be solutions of the equations $E(a_1\ , b_1\ , c_1)$, respectively $E(a_{-1}\ , b_{-1}\ , c_{-1})$, where

$$a_1 = a - \frac{h_v}{h}\ ,\ b_1 = b\ ,\ c_1 = c - a_u + b_v - b\ \frac{h_v}{h} \tag{0.4}$$

$$a_{-1} = a\ ,\ b_{-1} = b - \frac{k_u}{k}\ ,\ c_{-1} = c + a_u - b_v - a\ \frac{k_u}{k} \tag{0.5}$$

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with

$$h = a_u + ab - c \ , \ k = b_v + ab - c.$$

Let us denote by S_1 , S_{-1} the surfaces described by the points

$$x_1 = L_1(x) \ , \ x_{-1} = L_{-1}(x).$$

Under these conditions, we infer this

Proposition. *The equations $du = 0$, $dv = 0$ define on the surfaces S_1 , S_{-1} two lattices of mutually conjugated lines.*

The lattices defined in this way are named *the Laplace transforms of the lattice existing on S.*

Moreover, it is not difficult to verify the relations

$$(x_1)_u + b\ x_1 = h\ x \ , \ (x_{-1})_v + a\ x_{-1} = k\ x.$$

Since the homogeneous coordinates are defined up a non vanishing factor, it follows that:

$$x = L_{-1}\Big(L_1(x)\Big) = L_1\Big(L_{-1}(x)\Big) \ .$$

It is interesting to note that:

The point x_1 lies on the straight line which is tangent at x to the $v-$line of S that passes through x and the point x_{-1} lies on the straight line that is tangent at x to the $v-$line that passes through x.

The straight line $x\ x_1$ is therefore tangent at x to the surface S and it is tangent to the surface S_1 at the point x_1 . Similarly, the straight line $x\ x_{-1}$ is tangent to the surfaces $S,\ S_1$ at the points $x,\ x_{-1}$. The geometers use to say that these straight lines describe two *straight lines congruences*, having the surfaces $S,\ S_1$, respectively $S,\ S_{-1}$ as *focal surfaces.*

2. We will name *a Laplace sequence* any sequence $(x_n)_{n\in\mathbf{Z}}$ of lattices of mutually conjugated lines, such that, for all n,

$$x_n = L_1(x_{n-1}).$$

For Laplace sequences we shall have

$$x_n = L_{-1}(x_{n+1}).$$

The work of Tzitzeica is mainly devoted to the study of lattices of mutually conjugated lines, which are situated on hyperquadrics and to the study of *periodic Laplace sequences.*

A Laplace sequence $(x_n)_{n\in\mathbf{Z}}$ is said to be *periodic* when, for a positive integer r and for all n, one has $x_{n+r} = x_n$.

3. Tzitzeica was lead to these studies starting from a deep research concerning *the deformation theory* of surfaces in three dimensional Euclidean space E^3.

Tzitzeica was interested, at the beginning, on the isometric deformations of *tetraedral surfaces*, which are algebraic surfaces defined by equations of the form

$$\alpha\ x^{2/3} + \beta\ y^{2/3} + \gamma\ z^{2/3} = 1,$$

where α, β, γ are non vanishing real numbers.

The Roumanian geometer showed that the determination of the isometric deformations of tetraedral surfaces can be reduced to the determination of the surfaces Σ in E^3 that enjoy the property

$$\frac{K}{d^4} = constant, \tag{0.6}$$

where K is the Gauss curvature of Σ at a generic point $x \in \Sigma$ and d is the distance from the origin O to the tangent plane of Σ at x.

Supposing the surface Σ is defined by the equation

$$z = F(x, y),$$

we will have

$$\frac{K}{d^4} = \frac{F_{xx}\ F_{yy} - (F_{xy})^2}{(F - xF_x - yF_y)^4}\ .$$

One could try to determine the surfaces Σ enjoying the property (6) by using this expression of K/d^4.

Tzitzeica proved that all quadrics enjoy the property (6) and he was able to determine all the ruled surfaces that enjoy this property. As far the surfaces that are not ruled surfaces, the problem arised to be very difficult. He was able to find two examples, which are defined by the equations

$$x\ y\ z = 1\ ,\ z(x^2 + y^2) = 1.$$

Trying to find other solutions, he was lead to develop an extensive theory, based on the Laplace configurations. The results obtained by Tzitzeica in this direction are contained in beautiful book entitled "The Projective Differential Geometry of lattices", first published in French, in 1927.

4. The elementary theory of surfaces enjoying the property (6) is based on the following equations, that were obtained by Tzitzeica, as an anologue of Gauss-Weingarten equations

$$x_{uu} = a\ x_u + b\ x_v + c\ x$$
$$x_{uv} = a'\ x_u + b'\ x_v + c'\ x$$
$$x_{vv} = a''\ x_u + b''\ x_v + c''\ x.$$

Generally speaking, any triple (x^1, x^2, x^3) of solutions of such a system defines a surface Σ in E^3, provided the determinant

$$D = \det \begin{pmatrix} x^1 & x^1_u & x^1_v \\ x^2 & x^2_u & x^3_v \\ x^3 & x^3_u & x^3_v \end{pmatrix}$$

is not equal to zero.

We can compare the relations above to the Gauss-Weingarten equations using the formulas

$$x = \frac{\rho}{\Delta}\Big((G\rho_u - F\rho_v)\ x_u + (E\rho_v - F\rho_u)\ x_v\Big) + d\ N,$$

where

$$\rho = \| x \| \ , \ E = \| x_u \|^2 \ , \ F = < x_u \ , \ x_v > \ , \ G = \| x_v \|^2$$

$$\Delta = EG - F^2 \ , \ d = \frac{D}{\sqrt{\Delta}} \ , \ N = \frac{x_u \times x_v}{\sqrt{\Delta}} \ , \ d = < x, N > .$$

These formulas show that

$$K = \frac{d^2 \Big(cc'' - (c')^2 \Big)}{EG - F^2} \ , \ \frac{K}{d^4} = \frac{cc'' - (c')^2}{D^2} \ .$$

When $c = c'' = 0$ and $c' \neq 0$, the curvature K of Σ is negative and the equations $du = 0$, $dv = 0$ define the asymptotic lines on the surface Σ.

Suppose that the surface Σ has the property (6) and that Σ has negative curvature at each point. Then Tzitzeica proved that, using the asymptotic lines on Σ as coordinate lines, the Gauss-Weingarten-Tzitzeica equations can be reduced to have

$$a' = b' = c = c'' = 0$$

$$c' = h \ , \ a = \frac{h_u}{h} \ , \ b = \frac{1}{h} \ , \ a'' = \frac{1}{h} \ , \ b'' = \frac{h_v}{h} \ ; \tag{0.7}$$

the integrability conditions for the corresponding system of PDE reduce to just one PDE, which is

$$(\ln h)_{uv} = h - \frac{1}{h^2} \ . \tag{0.8}$$

This equation has the obvious solution $h = 1$ and this solution corresponds to the two examples mentioned above.

In fact, from the definition of D, using the relations (7), we infer the relations

$$\frac{D_u}{D} = \frac{h_u}{h} \ , \ \frac{D_v}{D} = \frac{h_v}{h}$$

and it follows that

$$\frac{K}{d^4} = -\frac{(c')^2}{D^2} = -\frac{h^2}{D^2}$$

is a constant.

Tzitzeica was very unhappy about the impossibility to find other solutions to his equation (8), but it is this equation that lead him to the discovery of many beautiful results.

For instance, he proved this

Theorem. *A lattice of mutually conjugated lines verifying an equation of Laplace type*

$$x_{uv} = h \ x,$$

in which h is a solution to the equation (8), is a term in a periodic Laplace sequence with period three.

He also proved that *the surfaces Σ that enjoy the property (6) are linked to certain periodic Laplace sequences of period six.*

Seemingly, Tzitzeica's theory applies only to surfaces with everywhere negative Gauss curvature. In reality, Tzitzeica used to consider only surfaces defined by analytic functions, which are extendable over the complex field; under this hypothesis, it was possible for him to take into consideration, for instance the asymptotic lines of any surface, ignoring the sign of the curvature; in the case when the curvature is positive, the asymptotic lines are complex, but imaginary conjugated with one another.

This point of view was commonly used by proeminent geometers in the 19-th century, such as Sophus Lie and Gaston Darboux, under the guidance of which Tzitzeica elaborated his doctoral thesis in Paris.

5. We add one of the most beautiful theorems discovered by Tzitzeica.

We first remind that, using the Plücker coordinates of a straight line δ in P^3 and following F. Klein, we can associate with δ a point $K(\delta)$ on *the Klein hyperquadric* $Q \subset P^5$.

A Theorem of Tzitzeica. *Let x be a point on a surface $\Sigma \subset P^3$. Suppose that through the point x pass two distinct asymptotic lines of Σ and let $\delta_1(x)$, $\delta_2(x)$ be the tangent straight lines at x to these asymptotic lines. Suppose that the points $p_1 = K(\delta_1)$, $p_2 = K(\delta_2)$ describe two surfaces S_1 , S_2.*

Under these conditions, the straight line p_1p_2 is tangent to the surfaces S_1 , S_2 at the points p_1 , p_2 .

6. The mathematical work of Gh. Tzitzeica is a part of the Differential Projective Geometry as developed by the French mathematicians G. Darboux, Desmoulin, Guichard, etc.

In recent years, Tzitzeica's equation (6) has been considered from the point of view of *Soliton Theory*, since this equation is a nonlinear PDE. One obtains *soliton solutions* to the equation (8) when we consider solutions of the form

$$h(u,v) = f(w),$$

where

$$w = u + c\ v\ ,\ (c = const.)$$

For such solutions, the function f will satisfy the ordinary differential equation

$$c\ (\ln f)'' = f - \frac{1}{f^2}\ .$$

Multiplying both sides in this equation with $2(\ln f)'$ and integrating, we get the equation

$$c(f')^2 = 2f^3 + kf^2 + 1\ ,\ (k = const.)$$

As a consequence, we get

$$w = \int \frac{\pm c\ df}{\sqrt{c(2f^3 + kf^2 + 1)}}\ ,$$

and we infer that:

The soliton solutions of Tzitzeica's equation (8) are given by the inverses of certain elliptic functions.

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