Geodesics and Circles on Real Hypersurfaces of Type A and B in a Complex Space Form

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Abstract

We denote by $M_n(c)$ a complex space form with the metric of constant holomorphic sectional curvature $4c$ and M a real hypersurface in $M_n(c)$. We will give characterizations of homogeneous real hypersurfaces of type A and B by observing the shape of geodesics and circles on M as curves in $M_n(c)$.

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1 Introduction

We denote by $M_n(c)$ a complete and simply connected complex n-dimensional Kählerian manifold of constant holomorphic sectional curvature $4c$, which is called a *complex space form*. Such an $M_n(c)$ is bi-holomorphically isometric to a complex projective space $P_n\mathbf{C}$, a complex Euclidean space $\mathbf{C}^n$ or a complex hyperbolic space $H_n\mathbf{C}$, according as $c > 0$, $c = 0$ or $c < 0$.

In this paper, we consider a real hypersurface M in $M_n(c), c \neq 0$. Typical examples of M in $P_n\mathbf{C}$ are the six model spaces of type A_1, A_2, B, C, D and E (cf. Theorem A in §2), and the ones of M in $H_n\mathbf{C}$ are the four model spaces of type A_0, A_1, A_2 and B (cf. Theorem B in §2), which are all given as orbits under certain Lie subgroups of the group consisting of all isometries of $P_n\mathbf{C}$ or $H_n\mathbf{C}$. Denote by (ϕ, ξ, η, g) the *almost contact metric structure* of M induced from the almost complex structure of $M_n(c)$ and A the shape operator of M. Eigenvalues and einvectors of A are called *principal curvatures* and *principal vectors*, respectively.

Many differential geometers have studied M from various points of view. For example, Berndt [1] and Takagi [13] investigated the homogeneity of M. Kimura [7] proved that if all principal curvatures are constant and ξ is principal vector, then M in $P_n\mathbf{C}$ is congruent to one of model spaces. Moreover, it is very interesting to characterize homogeneous real hypersurfaces of $M_n(c)$. There are many characterizations of homogeneous ones of type A since these examples have a lot of beautiful geometric properties, where *type* A means type A_1 or A_2 in $P_n\mathbf{C}$ and type A_0, A_1

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or A_2 in $H_n\mathbf{C}$. Okumura [11] and Montiel-Romero [10] proved the fact in $P_n\mathbf{C}$ and $H_n\mathbf{C}$, respectively that M satisfies $A\phi = \phi A$ if and only if M is locally congruent to type A. Also Maeda [9] gave a characterization of type A in $P_n\mathbf{C}$ (cf. Theorem E in §2). However, until now there are few results about characterizations of type B. Kim, Pyo and Ki-Nakagawa [5] characterized a real hypersurface of type B in $M_n(c)$ (cf. Theorem F in §2).

Recently, Maeda-Ogiue [8] investigated a geodesic hypersphere (i.e. type A_1 in $P_n\mathbf{C}$) by observing the shape of geodesics on M as curves in $P_n\mathbf{C}$. Motivated by this result, we are interested in characterizing M of type A in $M_n(c)$ by observing geodesics on M, and we will investigate circles on M of type B in $M_n(c)$.

The purpose of this paper is to give characterizations of homogeneous real hypersurfaces of type A and B by studing geodesics and circles on M as curves in $M_n(c)$.

2 Preliminaries

We begin with recalling the basic properties of a real hypersurface M of a complex space form $M_n(c)$. Let N be a unit normal vector field on M. The Riemannian connections $\widetilde{\nabla}$ in $M_n(c)$ and ∇ in M are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N \tag{2.1}$$

and

$$\widetilde{\nabla}_X N = -AX, \tag{2.2}$$

where g denotes the induced Riemannian metric on M. Let J the almost complex structure of $M_n(c)$. For a vector field X on M, the images of X and N under the transformation J can be represented as

$$JX = \phi X + \eta(X)N \quad , \quad JN = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on M, respectively. Moreover, it is seen that $g(\xi, X) = \eta(X)$. By the properties of the almost complex structure J, the set (ϕ, ξ, η, g) of tensors satisfies

$$\phi^2 = -I + \eta \otimes \xi \quad , \quad \phi\xi = 0 \quad , \quad \eta(\phi X) = 0 \quad , \quad \eta(\xi) = 1$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

where I denotes the identity transformation. Accordingly, this set $(\phi,\ \xi,\ \eta,\ g)$ defines the *almost contact metric structure* on M. Furthermore, the covariant derivatives of the structure tensors are given by

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.3}$$

$$\nabla_X \xi = \phi AX. \tag{2.4}$$

Since the ambient space is of constant holomorphic sectional curvature $4c$, the equation of Codazzi is given as follows

$$(2.5) \qquad (\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}.$$

It is well-known that there exist no totally umbilical real hypersurfaces in $M_n(c)$. So, a real hypersurface M of $M_n(c)$ is said to be *totally η-umbilical* if its shape operator A satisfies

$$AX = aX + b\eta(X)\xi$$

for some smooth functions a and b on M.

In the following, we use the same terminology and notations as the above unless otherwise stated. Now we quote the following in order to prove our results.

Theorem A ([13]). *Let M be a homogeneous real hypersurface of $P_n\mathbf{C}$. Then M is a tube of radius r over one of the following Kähler submanifolds:*

(A_1) a hyperplane $P_{n-1}\mathbf{C}$, where $0 < r < \frac{\pi}{2}$,

(A_2) a totally geodesic $P_k\mathbf{C}$ $(1 \le k \le n-2)$, where $0 < r < \frac{\pi}{2}$,

(B) a complex quadratic Q_{n-1}, where $0 < r < \frac{\pi}{4}$,

(C) $P_1\mathbf{C} \times P_{(n-1)/2}\mathbf{C}$, where $0 < r < \frac{\pi}{4}$ and $n(\ge 5)$ is odd,

(D) a complex Grassmann $G_{2,5}\mathbf{C}$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,

(E) a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.

Theorem B ([1]). *Let M be a real hypersurface of $H_n\mathbf{C}$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following:*

(A_0) a horosphere in $H_n\mathbf{C}$,

(A_1) a geodesic hypersphere $H_0\mathbf{C}$ or a tube over a hyperplane $H_{n-1}\mathbf{C}$,

(A_2) a tube over a totally geodesic $H_k\mathbf{C}$ $(1 \le k \le n-2)$,

(B) a tube over a totally real hyperbolic space $H_n\mathbf{R}$.

Theorem C ([10], [11]). *Let M be a real hypersurface of $M_n(c)$. Then M satisfies $A\phi = \phi A$ if and only if M is locally congruent to one of type A_1 and A_2 when $c > 0$, and of type A_0, A_1 and A_2 when $c < 0$.*

Theorem D ([10], [14]). *Let M be a real hypersurface of $M_n(c)$. Then M is totally η-umbilical if and only if M is locally congruent to one of type A_1 when $c > 0$, and of type A_0 and A_1 when $c < 0$.*

Theorem E ([3], [9]. *Let M be a real hypersurface of $M_n(c)$. Then the following are equivalent:*

(1) M is locally congruent to one of type A,

(2) $(\nabla_X A)Y = -c\{g(\phi X, Y)\xi + \eta(Y)\phi X\}$ for any vector fields X and Y on M.

Theorem F ([5]). *Let M be a real hypersurface of $M_n(c)$. Then the following are equivalent:*

(1) M is locally congruent to type B,

(2) $(\nabla_X A)Y = k\{2\eta(X)(A\phi - \phi A)Y + \eta(Y)(A\phi - 3\phi A)X + g((A\phi - 3\phi A)X, Y)\xi\}$ for any vector fields X and Y on M and $k \in \mathbf{R}$.

Proposition A ([4], [9]). Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If ξ is principal, then the corresponding principal curvature α is locally constant.

Here we consider the case where the structure vector ξ is *principal*, namely, $A\xi = \alpha\xi$. It follows from (2.5) that

$$2A\phi A = 2c\phi + \alpha(A\phi + \phi A) \tag{2.6}$$

and hence, if $AX = \lambda X$ for any vector field X orthogonal to ξ, then we get

$$(2\lambda - \alpha)A\phi X = (\alpha\lambda + 2c)\phi X.$$

Accordingly, it turns out that in the case where $\alpha^2 + c \neq 0$, ϕX is also principal vector with principal curvature $\mu = (\alpha\lambda + 2c)/(2\lambda - \alpha)$, that is, we obtain

$$\begin{aligned} &A\phi X = \mu\phi X, \\ &2\lambda - \alpha \neq 0, \quad \mu = (\alpha\lambda + 2c)/(2\lambda - \alpha). \end{aligned} \tag{2.7}$$

Finally, we recall the definition of helices in Riemannian geometry.

A smooth curve $\gamma = \gamma(s)$ in a Riemannian manifold parametrized by its arc length s is called a *helix of proper order* d if there exists an orthonormal frame $\{V_1 = \dot{\gamma}, \ldots, V_d\}$ along γ and positive constants $k_1, \ldots, k_{d-1}$ which satisfy

$$\nabla_{\dot{\gamma}_i} V_j(s) = -k_{j-1}V_{j-1}(s) + k_j V_{j+1}(s), \quad j = 1, \ldots, d,$$

where $V_0 = V_{d+1} = 0$. The constants k_j $(1 \leq j \leq d-1)$ and the orthonormal frame $\{V_1, \ldots, V_d\}$ are called the *curvatures* and the *Frenet frame* of γ, respectively. And a smooth curve is called a *helix of order* d if it is a helix of proper order r $(\leq d)$.

Note that a helix of order 1 is nothing but a geodesic, and a helix of order 2 is called a *circle*. That is, a smooth curve $\gamma = \gamma(s)$ in a Riemannian manifold parametrized by its arc length s is called a *circle* if there exists a field $Y = Y(s)$ of unit vectors along γ which satisfies $\nabla_{\dot{\gamma}}\dot{\gamma} = kY$ and $\nabla_{\dot{\gamma}}Y = -k\dot{\gamma}$ for some positive constant k which is called the *curvature* of γ. Moreover, for an arbitrary point x, an arbitrary orthonormal pair (u, v) of vectors at x and an arbitrary positive number k, there exists a unique circle $\gamma = \gamma(s)$ with $\gamma(0) = x$, $\dot{\gamma}(0) = u$ and $Y(0) = v$.

3 Real hypersurfaces of type A

We denote by $M_n(c)$ a complex space form with the metric of constant holomorphic sectional curvature $4c$ and M a real hypersurface in $M_n(c)$, $c \neq 0$. In this section, we are concerned with homogeneous real hypersurfaces of type A. Then, according to Takagi's classification theorem [13] and Berndt's one [1], the principal curvatures and their multiplicities of type A in $M_n(c)$ are given as follows:

In the case $c > 0$,

(i) type A_1 has two distinct constant principal curvatures $\alpha = 2\cot 2r$ with multiplicity 1 and $\lambda = \cot r$ with multiplicity $2n - 2$,

(ii) type A_2 has three distinct constant principal curvatures $\alpha = 2\cot 2r$ with multiplicity 1, $\lambda = -\tan r$ with multiplicity $2k$ and $\mu = \cot r$ with multiplicity $2(n - k - 1)$, where $1 \leq k \leq n - 1$.

In the case $c<0$,

(i) type A_0 has two distinct constant principal curvatures $\alpha=2$ with multiplicity 1 and $\lambda=1$ with multiplicity $2n-2$,

(ii) type A_1 has two distinct constant principal curvatures $\alpha=2\coth(2r)$ with multiplicity 1 and $\lambda=\tanh(r)$ if $0<\lambda<1$ or $\lambda=\coth(r)$ if $\lambda>1$ with multiplicity $2n-2$,

(iii) type A_2 has three distinct constant principal curvatures $\alpha=2\coth(2r)$ with multiplicity 1, $\lambda=\tanh(r)$ with multiplicity $2k$ and $\mu=\coth(r)$ with multiplicity $2(n-k-1)$, where $1\le k\le n-1$.

The following discussion in the case $c>0$ is partially indebted to Maeda and Ogiue [8]:

First of all, we prove the following

Lemma 3.1 *Let M be a real hypersurface of type A in $M_n(c)$, $c\neq 0$. Take orthonormal vectors $(v_1,v_2,\ldots,v_{2n-2})$ orthogonal to ξ at an arbitrary point p of M in such a way that $(v_1,v_2,\ldots,v_{2k})$ (resp. $(v_{2k+1},v_{2k+2},\ldots,v_{2n-2})$) are principal vectors with principal curvature λ (resp. μ). Then $(v_1,v_2,\ldots,v_{2n-2})$ satisfy the following:*

(1) All geodesics γ_i on M with $\gamma_i(0)=p$ and $\dot\gamma_i(0)=v_i$ $(1\le i\le 2k)$ are circles of the curvature λ in $M_n(c)$.

(2) All geodesics γ_i on M with $\gamma_i(0)=p$ and $\dot\gamma_i(0)=v_i$ $(2k+1\le i\le 2n-2)$ are circles of the curvature μ in $M_n(c)$.

Proof. Let $\gamma_i=\gamma_i(s)$ $(1\le i\le 2n-2)$ be geodesics on M with $\gamma_i(0)=p$ and $\dot\gamma_i(0)=v_i$. Then, taking account of (2.4) and Theorem C, we have

$$\nabla_{\dot\gamma_i}(g(\dot\gamma_i,\xi))=g(\dot\gamma_i,\nabla_{\dot\gamma_i}\xi)=g(\dot\gamma_i,\phi A\dot\gamma_i)=g(\dot\gamma_i,A\phi\dot\gamma_i)=-g(\phi A\dot\gamma_i,\dot\gamma_i)=0.$$

This implies that each $\dot\gamma_i$ $(1\le i\le 2n-2)$ is perpendicular to ξ since $g(\dot\gamma_i(0),\xi)=g(v_i,\xi)=0$.

Thus, owing to Theorem E, we get

$$\nabla_{\dot\gamma_i}\parallel A\dot\gamma_i-\lambda\dot\gamma_i\parallel^2=2g((\nabla_{\dot\gamma_i}A)\dot\gamma_i,A\dot\gamma_i-\lambda\dot\gamma_i)=0,$$

where $1\le i\le 2k$. Since $A\dot\gamma_i(0)-\lambda\dot\gamma_i(0)=Av_i-\lambda v_i=0$, we obtain $A\dot\gamma_i-\lambda\dot\gamma_i=0$ $(1\le i\le 2k)$. Here we note that $k=n-1$ in type A_1 when $c>0$, and in type A_0 and A_1 when $c<0$. Therefore, we see from (2.1) and (2.2) that

$$\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i=g(A\dot\gamma_i,\dot\gamma_i)N=g(\lambda\dot\gamma_i,\dot\gamma_i)N=\lambda N$$

and

$$\widetilde\nabla_{\dot\gamma_i}N=-A\dot\gamma_i=-\lambda\dot\gamma_i.$$

This implies that γ_i $(1\le i\le 2k)$ are circles of the curvature λ and the Frenet frame $\{\dot\gamma_i,N\}$ in $M_n(c)$.

Similarly in the case where M is of type A_2, we can show that γ_i $(2k+1\le i\le 2n-2)$ are circles of the curvature μ and the Frenet frame $\{\dot\gamma_i,N\}$ in $M_n(c)$.

Theorem 3.2. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$. Then M is locally congruent to one of type A_1 when $c>0$, and of type A_0 and A_1 when $c<0$ if and only if there exist orthonormal vectors $(v_1,v_2,\ldots,v_{2n-2})$ orthogonal to ξ at an arbitrary point p of M such that all geodesics $\gamma_i=\gamma_i(s)$ on M through p in the direction v_i+v_j $(1\le i\le j\le 2n-2)$ are circles in $M_n(c)$.*

Proof. Let M be locally congruent to one of type A_1 when $c > 0$, and of type A_0 and A_1 when $c < 0$. Then Lemma 3.1 shows that, for an arbitrary unit vector $X \perp \xi$ at $p \in M$, a geodesic $\gamma = \gamma(s)$ with $\gamma(0) = p$ and $\dot\gamma(0) = X$ is a circle in $M_n(c)$. Thus there exist orthonormal vectors $(v_1, v_2, \ldots, v_{2n-2})$ orthogonal to ξ at an arbitrary point p of M such that all geodesics $\gamma_i = \gamma_i(s)$ on M through p in the direction v_i $(1 \leq i \leq 2n-2)$ are circles in $M_n(c)$.

Conversely, let $\gamma_i = \gamma_i(s)$ $(1 \leq i \leq 2n-2)$ be geodesics on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$. Then by such assumption that all geodesics $\gamma_i = \gamma_i(s)$ on M through p in the direction v_i $(1 \leq i \leq 2n-2)$ are circles in $M_n(c)$, they satisfy

$$\widetilde\nabla^2_{\dot\gamma_i}\dot\gamma_i = -k_i^2\dot\gamma_i \tag{3.1}$$

for some positive constants k_i.

On the other hand, from (2.1) and (2.2) it follows that

$$\widetilde\nabla^2_{\dot\gamma_i}\dot\gamma_i = g((\nabla_{\dot\gamma_i}A)\dot\gamma_i, \dot\gamma_i)N - g(A\dot\gamma_i, \dot\gamma_i)A\dot\gamma_i. \tag{3.2}$$

Comparing the tangential components of (3.1) with (3.2), we have

$$g(A\dot\gamma_i, \dot\gamma_i)A\dot\gamma_i = k_i^2\dot\gamma_i$$

so that we get

$$g(Av_i, v_i)Av_i = k_i^2 v_i,$$

which implies

$$Av_i = \pm k_i v_i \quad (1 \leq i \leq 2n-2). \tag{3.3}$$

Thus we obtain

$$g(Av_i, v_j) = 0 \quad (1 \leq i < j \leq 2n-2) \tag{3.4}$$

because vectors $(v_1, v_2, \ldots, v_{2n-2})$ are orthonormal.

Let $\gamma_{ij} = \gamma_{ij}(s)$ $(1 \leq i < j \leq 2n-2)$ be geodesics on M with $\gamma_{ij}(0) = p$ and $\dot\gamma_{ij}(0) = (v_i + v_j)/\sqrt{2}$. Then by the same argument as the above we have

$$g(A(v_i + v_j), (v_i + v_j))A(v_i + v_j) = 2k_{ij}^2(v_i + v_j)$$

for some positive constants k_{ij}. Hence we get

$$g(A(v_i + v_j), (v_i - v_j)) = 0 \quad (1 \leq i < j \leq 2n-2).$$

Therefore, combining this with (3.4) we have

$$g(Av_i, v_i) = g(Av_j, v_j) \quad (1 \leq i, j \leq 2n-2).$$

This, together with (3.3), implies that $AX = kX$ for all X ortogonal to ξ and for some constant k.

Moreover, ξ is also principal because $g(A\xi, X) = g(\xi, AX) = g(\xi, kX) = 0$ for all $X \perp \xi$.

Thus we see that M is η-umbilic at p and hence M is totally η-umbilic in $M_n(c)$ since p is arbitrary. Therefore, owing to Theorem D, it follows that M is locally congruent to one of type A_1 when $c > 0$, and of type A_0 and A_1 when $c < 0$.

Theorem 3.3. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. Then M is locally congruent to one of type A_1 and type A_2 with $r = \pi/4$ when $c > 0$, and of type A_0 and A_1 when $c < 0$ if and only if there exist orthonormal vectors $(v_1, v_2, \ldots, v_{2n-2})$ orthogonal to ξ at an arbitrary point p of M such that all geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$ $(1 \leq i \leq 2n-2)$ are circles in $M_n(c)$ with the same curvature.*

Proof Let M be locally congruent to one of type A_1 and type A_2 with $r = \pi/4$ when $c > 0$, and of type A_0 and A_1 when $c < 0$. By Lemma 3.1, there exist orthonormal vectors $(v_1, v_2, \ldots, v_{2n-2})$ orthogonal to ξ at an arbitrary point p of M such that all geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$ $(1 \leq i \leq 2n-2)$ are circles in $M_n(c)$ with the same curvature λ.

Conversely, let $\gamma_i = \gamma_i(s)$ $(1 \leq i \leq 2n-2)$ be geodesics on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$. Then by assumption that all geodesics $\gamma_i = \gamma_i(s)$ on M through p in the direction v_i $(1 \leq i \leq 2n-2)$ are circles in $M_n(c)$ with the same curvature k, the same argument as one in the proof of Theorem 3.2 gives

$$g(Av_i, v_i)Av_i = k^2 v_i \quad (1 \leq i \leq 2n-2),$$

where k is a positive constant. Then we get

$$Av_i = kv_i \ \text{ or } \ Av_i = -kv_i \quad (1 \leq i \leq 2n-2). \tag{3.5}$$

Thus we obtain the fact that ξ is principal because $g(A\xi, v_i) = g(\xi, Av_i) = g(\xi, \pm kv_i) = 0$ for $1 \leq i \leq 2n-2$. Therefore M is a real hypersurface in $M_n(c)$ with at most three distinct constant principal curvatures $k, -k$ and α, where we have used Proposition A. Consequently, M is locally congruent to one of homogeneous real hypersurfaces of type A_1, A_2 and B when $c > 0$, and of type A_0, A_1, A_2 and B when $c < 0$. But the shape operators of homogeneous real hypersurfaces of type A_2 of radius $r(\neq \pi/4)$ and B when $c > 0$, and of type A_2 and B when $c < 0$ do not satisfy (3.5). That is, M is locally congruent to one of type A_1 and type A_2 with $r = \pi/4$ when $c > 0$, and of type A_0 and A_1 when $c < 0$.

Replacing geodesics in Lemma 3.1 by circles, we have the following

Lemma 3.4. *Let M be a real hypersurface of type A in $M_n(c)$, $c \neq 0$. Take orthonormal vectors $(v_1, v_2, \ldots, v_{2n-2})$ orthogonal to ξ at an arbitrary point p of M in such a way that $(v_1, v_2, \ldots, v_{2k})$ (resp. $(v_{2k+1}, v_{2k+2}, \ldots, v_{2n-2})$) are principal vectors with principal curvature λ (resp. μ). Then $(v_1, v_2, \ldots, v_{2n-2})$ satisfy the following:*

- *(1) All circles γ_i of an arbitrary curvature in M with $\gamma_i(0) = p$, $\dot\gamma_i(0) = v_i$ and the Frenet frame $\{\dot\gamma_i, \xi\}$ $(1 \leq i \leq 2k)$ are circles of the curvature λ and the Frenet frame $\{\dot\gamma_i, N\}$ in $M_n(c)$.*
- *(2) All circles γ_i of an arbitrary curvature in M with $\gamma_i(0) = p$, $\dot\gamma_i(0) = v_i$ and the Frenet frame $\{\dot\gamma_i, \xi\}$ $(2k+1 \leq i \leq 2n-2)$ are circles of the curvature μ and the Frenet frame $\{\dot\gamma_i, N\}$ in $M_n(c)$.*

Proof. Let $\gamma_i = \gamma_i(s)$ $(1 \leq i \leq 2n-2)$ be circles on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$. Then, from the assumption that all γ_i have the Frenet frame $\{\dot\gamma_i, \xi\}$, it follows that each $\dot\gamma_i$ $(1 \leq i \leq 2n-2)$ is perpendicular to ξ.

Thus, owing to Theorem E, we get

$$\begin{aligned}\nabla_{\dot\gamma_i} \| A\dot\gamma_i - \lambda\dot\gamma_i \|^2 &= 2g((\nabla_{\dot\gamma_i}A)\dot\gamma_i + A(\nabla_{\dot\gamma_i}\dot\gamma_i) - \lambda\nabla_{\dot\gamma_i}\dot\gamma_i, A\dot\gamma_i - \lambda\dot\gamma_i) \\ &= 2g((\nabla_{\dot\gamma_i}A)\dot\gamma_i + A(m_i\xi) - \lambda m_i\xi, A\dot\gamma_i - \lambda\dot\gamma_i) \\ &= 2g((\nabla_{\dot\gamma_i}A)\dot\gamma_i + (\alpha-\lambda)m_i\xi, A\dot\gamma_i - \lambda\dot\gamma_i) = 0,\end{aligned}$$

where each m_i is the curvature of $\dot\gamma_i$ $(1 \le i \le 2k)$. Since $A\dot\gamma_i(0) - \lambda\dot\gamma_i(0) = Av_i - \lambda v_i = 0$, we obtain $A\dot\gamma_i - \lambda\dot\gamma_i = 0$ $(1 \le i \le 2k)$. Therefore, we see from (2.1) and (2.2) that

$$\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i = g(A\dot\gamma_i, \dot\gamma_i)N = g(\lambda\dot\gamma_i, \dot\gamma_i)N = \lambda N$$

and

$$\widetilde\nabla_{\dot\gamma_i}N = -A\dot\gamma_i = -\lambda\dot\gamma_i.$$

This implies that γ_i $(1 \le i \le 2k)$ are circles of the curvature λ and the Frenet frame $\{\dot\gamma_i, N\}$ in $M_n(c)$. Here we note that $k = n-1$ in type A_1 when $c > 0$, and in type A_0 and A_1 when $c < 0$.

Similarly in the case where M is of type A_2, we can show that γ_i $(2k+1 \le i \le 2n-2)$ are circles of the curvature μ and the Frenet frame $\{\dot\gamma_i, N\}$ in $M_n(c)$.

4 Real hypersurfaces of type B

We denote by $M_n(c)$ a complex space form with the metric of constant holomorphic sectional curvature $4c$ and M a real hypersurface in $M_n(c)$, $c \ne 0$. In this section, we are concerned with homogeneous real hypersurfaces of type B. Then, according to Takagi's classification theorem [13] and Berndt's one [1], the principal curvatures and their multiplicities of type B in $M_n(c)$ are given as follows:

(i) In the case $c > 0$, type B has three distinct constant principal curvatures $\alpha = 2\cot 2r$ with multiplicity 1, $\lambda = -\tan(r - \pi/4)$ with multiplicity $n-1$ and $\mu = \cot(r - \pi/4)$ with multiplicity $n-1$.

(ii) In the case $c < 0$, type B has three distinct constant principal curvatures $\alpha = 2\tanh(2r)$ with multiplicity 1, $\lambda = \tanh(r)$ with multiplicity $n-1$ and $\mu = \coth(r)$ with multiplicity $n-1$.

Then we first have the following

Lemma 4.1. *Let M be a real hypersurface of type B in $M_n(c)$, $c \ne 0$. Take orthonormal vectors $(v_1, v_2, \ldots, v_{2n-2})$ orthogonal to ξ at an arbitrary point p of M in such a way that $(v_1, v_2, \ldots, v_{n-1})$ (resp. $(v_n, v_{n+1}, \ldots, v_{2n-2})$) are principal vectors with principal curvature λ (resp. μ). Then $(v_1, v_2, \ldots, v_{2n-2})$ satisfy the following:*

(1) All circles γ_i of an arbitrary curvature in M with $\gamma_i(0) = p$, $\dot\gamma_i(0) = v_i$ and the Frenet frame $\{\dot\gamma_i, \xi\}$ $(1 \le i \le n-1)$ are circles of the curvature λ and the Frenet frame $\{\dot\gamma_i, N\}$ in $M_n(c)$.

(2) All circles γ_i of an arbitrary curvature in M with $\gamma_i(0) = p$, $\dot\gamma_i(0) = v_i$ and the Frenet frame $\{\dot\gamma_i, \xi\}$ $(n \le i \le 2n-2)$ are circles of the curvature μ and the Frenet frame $\{\dot\gamma_i, N\}$ in $M_n(c)$.

Proof. Let $\gamma_i = \gamma_i(s)$ $(1 \le i \le 2n-2)$ be circles on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$. Then, from the assumption that all γ_i have the Frenet frame $\{\dot\gamma_i, \xi\}$, it follows that each $\dot\gamma_i$ $(1 \le i \le 2n-2)$ is perpendicular to ξ.

Thus, owing to Theorem F, we get

$$\begin{aligned}\nabla_{\dot\gamma_i} \| A\dot\gamma_i - \lambda\dot\gamma_i \|^2 &= 2g((\nabla_{\dot\gamma_i}A)\dot\gamma_i + A(\nabla_{\dot\gamma_i}\dot\gamma_i) - \lambda\nabla_{\dot\gamma_i}\dot\gamma_i, A\dot\gamma_i - \lambda\dot\gamma_i) = \\ &= 2g((\nabla_{\dot\gamma_i}A)\dot\gamma_i + A(k_i\xi) - \lambda k_i\xi, A\dot\gamma_i - \lambda\dot\gamma_i) = \\ &= 2g((\nabla_{\dot\gamma_i}A)\dot\gamma_i + (\alpha-\lambda)k_i\xi, A\dot\gamma_i - \lambda\dot\gamma_i) = 0,\end{aligned}$$

where each k_i is the curvature of γ_i $(1 \le i \le n-1)$. Since $A\dot\gamma_i(0) - \lambda\dot\gamma_i(0) = Av_i - \lambda v_i = 0$, we obtain $A\dot\gamma_i - \lambda\dot\gamma_i = 0 \ \ (1 \le i \le n-1)$. Therefore, we see from (2.1) and (2.2) that

$$\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i = g(A\dot\gamma_i, \dot\gamma_i)N = g(\lambda\dot\gamma_i, \dot\gamma_i)N = \lambda N$$

and

$$\widetilde\nabla_{\dot\gamma_i}N = -A\dot\gamma_i = -\lambda\dot\gamma_i.$$

This implies that γ_i $(1 \le i \le n-1)$ are circles of the curvature λ and the Frenet frame $\{\dot\gamma_i, N\}$ in $M_n(c)$.

Similarly we can show that γ_i $(n \le i \le 2n-2)$ are circles of the curvature μ and the Frenet frame $\{\dot\gamma_i, N\}$ in $M_n(c)$.

Theorem 4.2. *Let M be a real hypersurface of $M_n(c)$, $c \ne 0$. Then M is locally congruent to one of type A and B if and only if there exist orthonormal vectors $(v_1, v_2, \ldots, v_{2n-2})$ orthogonal to ξ at an arbitrary point p of M such that all circles $\gamma_i = \gamma_i(s)$ in M with $\gamma_i(0) = p$, $\dot\gamma_i(0) = v_i$ and the Frenet frame $\{\dot\gamma_i, \xi\}$ $(1 \le i \le 2n-2)$ are circles in $M_n(c)$ of the same curvature c_i $(1 \le i \le 2k)$ or the same one c_j $(2k+1 \le j \le 2n-2)$.*

Proof. Let M be locally congruent to one of type A and B in $M_n(c)$.

First of all, let M be of type A in $M_n(c)$ and let $\gamma_i = \gamma_i(s)$ $(1 \le i \le 2n-2)$ be circles in M with $\gamma_i(0) = p$, $\dot\gamma_i(0) = v_i$ and the Frenet frame $\{\dot\gamma_i, \xi\}$. Then, owing to Lemma 3.4, there exist orthonormal vectors $(v_1, v_2, \ldots, v_{2n-2})$ orthogonal to ξ at an arbitrary point p of M such that these circles $\gamma_i = \gamma_i(s)$ $(1 \le i \le 2n-2)$ are circles in $M_n(c)$ of the same curvature $c_i = \lambda$ $(1 \le i \le 2k)$ or the same one $c_j = \mu$ $(2k+1 \le j \le 2n-2)$.

Next, let M be of type B in $M_n(c)$. Then by Lemma 4.1, there exist orthonormal vectors $(v_1, v_2, \ldots, v_{2n-2})$ orthogonal to ξ at an arbitrary point p of M such that all circles $\gamma_i = \gamma_i(s)$ in M with $\gamma_i(0) = p$, $\dot\gamma_i(0) = v_i$ and the Frenet frame $\{\dot\gamma_i, \xi\}$ $(1 \le i \le 2n-2)$ are circles in $M_n(c)$ of the same curvature $c_i = \lambda$ $(1 \le i \le n-1)$ or the same one $c_j = \mu$ $(n \le j \le 2n-2)$.

Conversely, assume that there exist orthonormal vectors $(v_1, v_2, \ldots, v_{2n-2})$ orthogonal to ξ at an arbitrary point p of M such that all circles $\gamma_i = \gamma_i(s)$ in M with $\gamma_i(0) = p$, $\dot\gamma_i(0) = v_i$ and the Frenet frame $\{\dot\gamma_i, \xi\}$ $(1 \le i \le 2n-2)$ are circles in $M_n(c)$ of the same curvature c_i $(1 \le i \le 2k)$ or the same one c_j $(2k+1 \le j \le 2n-2)$. Then the same argument as one in the proof of Theorem 3.2 gives

$$g(Av_i, v_i)Av_i = {c_i}^2 v_i \quad (1 \le i \le 2k)$$

and

$$g(Av_j, v_j)Av_j = {c_j}^2 v_j \quad (2k+1 \le j \le 2n-2),$$

where c_i and c_j are positive constants. Then we get

$$\text{(4.1)} \qquad Av_i = \pm c_i v_i \text{ and } Av_j = \pm c_j v_j \ (1 \le i \le 2k, 2k+1 \le j \le 2n-2).$$

Thus, by means of (4.1), we obtain the fact that ξ is principal because $g(A\xi, v_i) = g(\xi, Av_i) = 0$ for $(1 \le i \le 2n-2)$. Therefore M is a real hypersurface in $M_n(c)$ with at most five distinct constant principal curvatures $c_i, -c_i, c_j, -c_j$ and α, where we have used Proposition A. Consequently, M is locally congruent to one of homogeneous real hypersurfaces of type A_1, A_2, B, C, D and E when $c > 0$, and of type A_0, A_1, A_2 and B when $c < 0$. But the shape operators of homogeneous real hypersurfaces of type C, D and E when $c > 0$ do not satisfy (4.1). That is, M is locally congruent to one of type A and B in $M_n(c)$.

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