Einstein Equation for an Invariant Metric on Generalized Flag Manifolds and Inner Automorphisms

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Abstract

A generalized flag manifold is a homogeneous space G/K whose isotropy subgroup K is the centralizer of a torus in G. We show that the Einstein equation for a G-invariant metric on G/K is invariant under the group of inner automorphisms of the Lie algebra of G that preserve the Lie algebra of K and a fixed Cartan subalgebra of K.

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1 Introduction

A metric g on a Riemannian manifold M is called an *Einstein metric* if $Ric(g) = cg$, where $Ric(g)$ is the Ricci tensor of the metric g, and c is a constant. In [1] we presented new G-invariant Einstein metrics for certain homogeneous spaces G/K called *generalized flag* manifolds. These metrics were non-Kähler and different from the normal metric [9]. The methodology we used there was the reduction of the Einstein equation to an algebraic system of equations through a Lie theoretic description of the Ricci curvature $Ric(g)$ and the G-invariant metric g. Let $\mathbf{g}$ and $\mathbf{k}$ be the Lie algebras of G and K respectively, and $\mathbf{h}$ a fixed Cartan subalgebra of $\mathbf{k}$. In this paper we show that the Einstein equation for a generalized flag manifold is invariant under the group of inner automorphisms of $\mathbf{g}$ that preserve $\mathbf{h}$ and $\mathbf{k}$.

2 Generalized flag manifolds: Lie theoretic description

Let G be a compact, connected and semisimple Lie group. A generalized flag manifold is a homogeneous space $M = G/K$ whose isotropy group K is the centralizer of a torus in G. Equivalently, M is the adjoint orbit $\mathrm{Ad}(G)w$ (w some element in $\mathbf{g}$) of w under

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the action of the adjoint representation Ad of G in $\mathbf{g}$ [5, 8]. Since G is semisimple and compact, the Killing form $B(X,Y) = \mathrm{trad}(X)\mathrm{ad}(Y)$ of $\mathbf{g}$ is nondegenerate and negative definite on $\mathbf{g}$, thus giving rise to an orthogonal decomposition of $\mathbf{g}$ as the direct sum $\mathbf{g} = \mathbf{k} \oplus \mathbf{m}$. This sum is $\mathrm{Ad}(K)$-invariant, i.e. $[\mathbf{m}, \mathbf{k}] \subset \mathbf{m}$. Moreover the tangent space T_pM of M can be identified with $\mathbf{m}$. This identification is given by

$$X \mapsto X^*(p) = \frac{d}{dt}(\exp tX \cdot p)\bigg|_{t=0}, \qquad X \in \mathbf{m},\ p \in M.$$

We fix a Cartan subalgebra $\mathbf{h}^{\mathbf{C}}$ of the complexified Lie algebra $\mathbf{k}^{\mathbf{C}}$; then the Cartan decompositions of $\mathbf{g}^{\mathbf{C}}$ and $\mathbf{k}^{\mathbf{C}}$ are given as follows:

$$\mathbf{g}^{\mathbf{C}} = \mathbf{h}^{\mathbf{C}} + \sum_{\alpha \in R} \mathbf{g}^{(\alpha)},\ \mathbf{k}^{\mathbf{C}} = \mathbf{h}^{\mathbf{C}} + \sum_{\alpha \in R_K} \mathbf{g}^{(\alpha)},\ \mathbf{m}^{\mathbf{C}} = \sum_{\alpha \in R_M} \mathbf{g}^{(\alpha)},$$

where R and R_K are the root systems of the pairs $(\mathbf{g}^{\mathbf{C}}, \mathbf{h}^{\mathbf{C}})$ and $(\mathbf{k}^{\mathbf{C}}, \mathbf{h}^{\mathbf{C}})$ respectively. The root system R is decomposed as $R = R_K \cup R_M$ where R_M is the set of complementary roots. The spaces $\mathbf{g}^{(\alpha)}$ are the 1-dimensional root spaces whose elements X_α are characterized by the equation $[H, X_\alpha] = \alpha(H)X_\alpha,\ H \in \mathbf{h}^{\mathbf{C}}$. We also recall that for any root α we can choose elements $E_\alpha \in \mathbf{g}^{(\alpha)}$ $(\alpha \in R)$ which have the properties $B(E_\alpha, E_{-\alpha}) = -1,\ [E_\alpha, E_{-\alpha}] = -H_\alpha$, where H_α is determined by the equation $B(H_\alpha, H) = \alpha(H)$ for all $H \in \mathbf{h}^{\mathbf{C}}$, as well as $[E_\alpha, E_\beta] = N_{\alpha\beta}E_{\alpha+\beta}$ for $\alpha,\ \beta \in R,\ \alpha + \beta \in R$, with coefficients $N_{\alpha,\beta}$ (structural constants). The set $\{E_\alpha : \alpha \in R_M\}$ constitutes a basis for $\mathbf{m}^{\mathbf{C}}$.

3 Invariant metrics

Having the decomposition $\mathbf{g}^{\mathbf{C}} = \mathbf{k}^{\mathbf{C}} \oplus \mathbf{m}^{\mathbf{C}}$ associated with the generalized flag manifold $M = G/K$ and the decomposition $R = R_K \cup R_M$ of the root system R, we set $\mathbf{h} = \mathbf{g} \cap \mathbf{h}^{\mathbf{C}}$ and define

$$\mathbf{t} = Z(\mathbf{k}^{\mathbf{C}}) \cap \mathbf{h} = \{X \in \mathbf{h} : \phi(X) = 0 \quad \forall \phi \in R_K\}.$$

If $\mathbf{h}^*$ and $\mathbf{t}^*$ are the dual spaces of $\mathbf{h}$ and $\mathbf{t}$, consider the restriction map

$$\kappa : \mathbf{h}^* \to \mathbf{t}^*$$
$$\alpha \mapsto \kappa(\alpha) = \alpha|_{\mathbf{t}}$$

and set $R_{\mathbf{t}} \equiv \kappa(R) = \kappa(R_M)$ (note that $\kappa(R_K) = 0$). The elements of $R_{\mathbf{t}}$ are called $\mathbf{t}$-*roots.* The benefit from these is that there is a one-to-one correspondence between $\mathbf{t}$-roots τ and irreducible $\mathrm{ad}_{\mathbf{g}}(\mathbf{k}^{\mathbf{C}})$-invariant submodules $M_\tau^{\mathbf{C}} = \sum_{\kappa(\alpha)=\tau} \mathbf{C}E_\alpha$ of $\mathbf{m}^{\mathbf{C}}$ [6, 2].

Now, a G-invariant metric on $M = G/K$ can be described by an $\mathrm{ad}_{\mathbf{g}}(\mathbf{k})$-invariant scalar product g on $\mathbf{m}$, and we extend without any change in notation to $\mathbf{m}^{\mathbf{C}}$. Let

$\{\omega^\alpha : \alpha \in R\}$ be the vector space basis in $(\mathbf{m}^{\mathbf{C}})^*$ which is dual to the basis $\{E_\alpha : \alpha \in R_M\}$. We fix a system of positive roots $R^+ = R_K^+ \cup R_M^+$, where $R_K^+ = R_K \cap R^+$, $R_M^+ = R_M \cap R^+$, and set $R_{\mathbf{t}}^+ = \kappa(R^+)$. The following proposition gives a description of the invariant metrics on M.

Proposition 1 [**3**].*Any real $ad_{\mathbf{g}}(\mathbf{k}^{\mathbf{C}})$-invariant scalar product g on $\mathbf{m}^{\mathbf{C}}$ has the form*

$$g = \sum_{\alpha \in R_M^+} g_\alpha \omega^\alpha \vee \omega^{-\alpha} = \sum_{\tau \in R_{\mathbf{t}}^+} g_\tau \sum_{\alpha \in \kappa^{-1}(\tau)} \omega^\alpha \vee \omega^{-\alpha},$$

where $\omega \vee \rho = \frac{1}{2}(\omega \otimes \rho + \rho \otimes \omega)$, $g_\alpha \in \mathbf{R}^+$, and $g_\alpha = g_\beta$ if $\alpha|_{\mathbf{t}} = \beta|_{\mathbf{t}}$ so the invariant Riemannian metrics on a generalized flag manifold $M = G/K$ depend (modulo the scale factor) on $|R_{\mathbf{t}}^+|$ parameters.

4 The Ricci tensor and the Einstein equation

The Ricci tensor can now be determined by its value on the basis $\{E_\alpha : \alpha \in R_M\}$. We have the following

Proposition 2 [**2**]. *The Ricci tensor for an invariant metric g described in proposition 1 is given by*

$$Ric(E_\alpha, E_\beta) = 0 \ , \qquad \alpha, \beta \in R_M \ , \alpha + \beta \notin R_M$$

$$Ric(E_\alpha, E_{-\alpha}) = (\alpha, \alpha) + \sum_{\substack{\phi \in R_K \\ \alpha + \phi \in R}} N_{\alpha,\phi}^2 + \frac{1}{4} \sum_{\beta \in R_M^*} \frac{N_{\alpha,\beta}^2}{g_{\alpha+\beta} g_\beta} (g_\alpha^2 - (g_{\alpha+\beta} - g_\beta{}^2),$$

where $R_M^* = R_M - \kappa^{-1}(\kappa(\alpha))$. Thus the Einstein equation $Ric(g) = cg$ reduces (after normalizing either one of the g_α or c to 1) to an algebraic system of $|R_{\mathbf{t}}^+|$ equations with $|R_{\mathbf{t}}^+|$ unknowns.

5 Inner automorphisms

We recall that $\mathrm{Ad}z$, $z \in G$ is the derivative of the conjugation $C_z : g \mapsto zgz^{-1}$ in G. The group of inner automorphisms of a complex Lie algebra $\mathbf{g}$ consists of finite products of the form $\mathrm{Ad}z$, $z \in G$, and it is a subgroup of the group of all automorphisms of $\mathbf{g}$. Further, the Weyl group of the root system R is the set of all linear transformations on $\mathbf{h}_{\mathbf{R}} = \sum_{\alpha \in R} \mathbf{R} H_\alpha$ induced by inner automorphisms of $\mathbf{g}$ that preserves $\mathbf{h}$. We can now state the main theorem.

Theorem.*The set of equations that determine the Einstein condition for the generalized flag manifold $M = G/K$ as given in proposition 2 is invariant under the group of inner automorphisms of $\mathbf{g}$ that preserve $\mathbf{h}$ and $\mathbf{k}$. Equivalently, it is invariant under those elements in the Weyl group of R that preserve R_K.*

Proof. Without loss of generality we need to examine the effect of $w = \mathrm{Ad}z$ on the root elements E_α, the structural constants $N_{\alpha,\beta}$, the components g_α of the G-invariant metric g, and finally on the set of equations that determine the Einstein equation.

STEP 1 Action of $\mathrm{Ad}z$ on E_α.

Since $\mathrm{Ad}z$ preserves $\mathbf{h}$ the equation $\alpha^*(H) = B(H, \mathrm{Ad}z(H_\alpha))$ defines a root α^* so that $\mathrm{Ad}z(H_\alpha) = H_{\alpha^*}$.

Claim: $\mathrm{Ad}z(E_\alpha) = E_{\alpha^*}$.

We apply $\mathrm{Ad}z$ to the equation $[H, E_\alpha] = \alpha(H)E_\alpha$ and obtain

$$[\mathrm{Ad}z(H), \mathrm{Ad}z(E_\alpha)] = \alpha(H)\mathrm{Ad}z(E_\alpha). \tag{1}$$

By the invariance of the Killing form under $\mathrm{Ad}z$ we have that

$$\begin{aligned} \alpha(H) &= B(H, H_\alpha) = B(\mathrm{Ad}z(H), \mathrm{Ad}z(H_\alpha)) \\ &= B(\mathrm{Ad}z(H), H_{\alpha^*}) = \alpha^*(\mathrm{Ad}z(H)). \end{aligned} \tag{2}$$

From (1) and (2) we obtain

$$[\mathrm{Ad}z(H), \mathrm{Ad}z(E_\alpha)] = \alpha^*(\mathrm{Ad}z(H))\mathrm{Ad}z(E_\alpha),$$

which implies that $\mathrm{Ad}z(E_\alpha)$ is the root vector E_{α^*} corresponding to the root α^* up to a constant. However the E_α's have been chosen so that this constant is normalized to 1. Notice that α^* also satisfies the equation

$$\alpha^*(H) = B(\mathrm{Ad}z^{-1}(H), H_\alpha) = \alpha(\mathrm{Ad}z^{-1}(H)) = w \cdot \alpha(H)$$

which is the usual definition of the action of an element w in the Weyl group on the roots.

STEP 2 Transformation of $N_{\alpha,\beta}$.

The numbers $N_{\alpha,\beta}$ are determined by the equation $[E_\alpha, E_\alpha] = N_{\alpha,\beta}E_{\alpha+\beta}$ $(\alpha + \beta \in R)$. Applying $\mathrm{Ad}z$ to this equation and using step 1 we get $[E_{\alpha^*}, E_{\beta^*}] = N_{\alpha,\beta}E_{(\alpha+\beta)^*}$ $(\alpha^* + \beta^* \in R)$, or $N_{\alpha^*,\beta^*}E_{\alpha^*+\beta^*} = N_{\alpha,\beta}E_{(\alpha+\beta)^*}$. The last equation determines the action of $\mathrm{Ad}z$ on $N_{\alpha,\beta}$ implicitly.

STEP 3 Transformation of g_α.

Let w be an element in the Weyl group of R that preserves R_K. Then the diffeomorphism C_z preserves K, thus it induces a map $\overline{C}_z$ on G/K. Then $\mathrm{Ad}z$ restricts to a map $d\overline{C}_z$ on $\mathbf{m} = T_o(G/K)$, and consequently it takes an invariant metric g to a new invariant metric $\mathrm{Ad}z \cdot g$ defined by

$$\mathrm{Ad}z \cdot g(X, Y) = g(\mathrm{Ad}z(X), \mathrm{Ad}z(Y)).$$

For $X = E_\alpha$ and $Y = E_{-\alpha}$ this gives

$$\mathrm{Ad}z \cdot g_\alpha = g(E_{\alpha^*}, E_{(-\alpha)^*}) = g_{\alpha^*}.$$

STEP 4 Transformation of the system of equations.

We apply $w = \mathrm{Ad}Z$ to the system of equations in proposition 2 and obtain

$$(3)\quad (\alpha^*,\alpha^*) = \sum_{\substack{\phi^*\in R_K\\ \alpha^*+\phi^*\in R}} N^2_{\alpha^*,\phi^*} + \frac{1}{4}\sum_{\beta^*\in R^*_M} \frac{N^2_{\alpha^*,\beta^*}}{g_{\alpha^*+\beta^*}g_{\beta^*}}(g^2_{\alpha^*} - (g_{\alpha^*+\beta^*} - g_{\beta^*})^2) = g_{\alpha^*}.$$

We need to show that (3) is equivalent to

$$(4)\qquad (\alpha^*,\alpha^*) = \sum_{\substack{\phi\in R_K\\ al^*+\phi\in R}} N^2_{\alpha^*,\phi} + \frac{1}{4}\sum_{\beta\in R^*_M} \frac{N^2_{\alpha^*,\beta}}{g_{\alpha^*+\beta}g_{\beta}}(g^2_{\alpha^*} - (g_{\alpha^*+\beta} - g_{\beta})^2) = g_{\alpha^*}.$$

Since $\alpha^* = w\cdot\alpha$ we can replace ϕ and β in (4) by ψ^* and γ^* respectively (for some ψ and $\gamma\in R$). Then we can use the invariance of R_K under w to obtain equation (3).

Example. Let $G/K = SU(n)/S(U(n_1)\times\cdots\times U(n_s))$, $\quad n = \sum_{i=1}^{s} n_i$.

According to [3] and [1] the Einstein equation reduces to the following system

$$n_i + n_j + \frac{1}{2}\sum_{l\neq i,j}\frac{n_l}{g_{il}g_{jl}}(g^2_{ij} - (g_{il} - g_{jl})^2) = g_{ij}$$

of $\frac{1}{2}s(s-1)$ equations with $\frac{1}{2}s(s-1)$ unknowns g_{ij}, the components of the $SU(n)$-invariant metric g. The Weyl group of $SU(n)$ is the group of permutations w of the set $\{1,\ldots,n\}$ which acts on the set $R = \{\epsilon_i - \epsilon_j : i\neq j\}$ of roots of $SU(n)$ according to $w(\epsilon_i - \epsilon_j) = \epsilon_{w(i)} - \epsilon_{w(j)}$, and on g_{ij} by $wg_{ij} = g_{w(i),w(j)}$. The integers n_i are transformed by $wn_i = n_{w(i)}$. Since w preserves R_K the set of equivalence classes $\{[g_{ij}] : g_{ij}\sim g_{kl} \Leftrightarrow \kappa(\epsilon_i - \epsilon_j) = \kappa(\epsilon_k - \epsilon_l)\}$ is closed under w. Thus the equations are preserved.

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