Semi-Symmetric Conformal Metrical N-Linear Connections in the Bundle of Accelerations

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Abstract

In the present paper we determine all semi-symmetric conformal metrical N-linear connections, which preserve the nonlinear connection N, in the bundle of accelerations. We study the group of transformations of these connections and its invariants.

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Key words: osculator bundle, curvature, torsion, semi-symmetric conformal metrical N-linear connection.

1 Introduction

The differential geometry of higher order Lagrange spaces was introduced and studied by R.Miron and Gh.Atanasiu in [8] – [13].

The applications of the Lagrange geometry of order k in Physics and Mechanics are quite numerous and important, [8].

The study of higher order Lagrange spaces is grounded on the k-osculator bundle notion. The bundle of accelerations corresponds in this study to $k = 2, [1], [10]$.

In the present paper we define the notion of semi-symmetric conformal metrical N-linear connection on $E = Osc^2M$ and we determine the set of all semi-symmetric conformal metrical N-linear connections, which preserve the nonlinear connection N, on E. (§2) The group of their transformations preserving a nonlinear connection N, gives us important invariants (§3).

As to the terminology and notations we use those from [14], which are essentially based on M.Matsumoto's book [6].

2 Notion of semi-symmetric conformal metrical N-linear connection in the bundle of accelerations

Let M be a real n-dimensional C^∞-differentiable manifold and (Osc^2M, π, M) its 2-osculator bundle, or the bundle of accelerations.

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The local coordinates on $E = Osc^2M$ are denoted by $(x^i, y^{(1)i}, y^{(2)i})$.

If N is a nonlinear connection on E, with the coefficients $\underset{(1)}{N}{}^i{}_j$, $\underset{(2)}{N}{}^i{}_j$, then let $D\Gamma(N) = (L^i_{jk}, \underset{(1)}{C}{}^i_{jk}, \underset{(2)}{C}{}^i_{jk})$ be an N- linear connection on E.

We consider a metrical d-structure on E, defined by a d-tensor field of the type $(0,2)$, marked as $g_{ij}(x^i, y^{(1)i}, y^{(2)i})$. This d-tensor field is symmetric and nondegenerate.

To the metrical d-structure g_{ij} on E, we associate Obata's operators

$$\Omega^{ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j - g_{sj}g^{ir}),\ \Omega^{*ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j + g_{sj}g^{ir}), \tag{2.1}$$

where (g^{ij}) is the inverse matrix of (g_{ij}). They have the same properties as the ones associated with a Finsler space [14].

Let $\mathcal{S}_2(E)$ be the set of all symmetric d-tensor fields of the type $(0,2)$ on E. It is easy to show that, the relation

$$a_{ij} \sim b_{ij} \Leftrightarrow \exists \rho(x, y^{(1)}, y^{(2)}) \in \mathcal{F}(E) \mid a_{ij} = e^{2\rho}b_{ij}; \quad a_{ij}, b_{ij} \in \mathcal{S}_2(E) \tag{2.2}$$

is an equivalence relation on $\mathcal{S}_2(E)$.

Definition 2.1. The equivalence class $\hat{g}$ of $\mathcal{S}_2(E)/_\sim$, to which the metrical d-structure g_{ij} belongs, is called *conformal metrical* d-structure on E.

Definition 2.2. An N-linear connection $D\Gamma(N) = (L^i_{jk}, \underset{(1)}{C}{}^i_{jk}, \underset{(2)}{C}{}^i_{jk})$ on E is said to be *compatible* with the conformal metrical d-structure $\hat{g}$, or a conformal metrical N-linear connection on E, if

$$g_{ij|k} = 2\omega_k g_{ij},\ g_{ij} \overset{(\alpha)}{|}_k = 2\lambda_{(\alpha)k}g_{ij},\ (\alpha = 1, 2), \tag{2.3}$$

where $\omega_k = \omega_k(x, y^{(1)}, y^{(2)})$ and $\lambda_{(\alpha)k} = \lambda_{(\alpha)k}(x, y^{(1)}, y^{(2)}),\ (\alpha = 1, 2)$ are covariant d-vector fields and $|$, $\overset{(\alpha)}{|}$, $(\alpha = 1, 2)$ denote the h-and v_α-covariant derivatives, $(\alpha = 1, 2)$ with respect to $D\Gamma(N)$.

Definition 2.3. An N-linear connection $D\Gamma(N) = (L^i_{jk}, \underset{(1)}{C}{}^i_{jk}, \underset{(2)}{C}{}^i_{jk})$ on E, is called *semi-symmetric* if the torsion d-tensor fields $\underset{(0)}{T}{}^i{}_{jk}$, $\underset{(\alpha)}{S}{}^i{}_{jk}, (\alpha = 1, 2)$ have the form

$$\underset{(0)}{T}{}^i{}_{jk} = \frac{1}{n-1}(T_{(0)j}\delta^i_k - T_{(0)k}\delta^i_j), \underset{(\alpha)}{S}{}^i{}_{jk} = \frac{1}{n-1}(S_{(\alpha)j}\delta^i_k - S_{(\alpha)k}\delta^i_j),\ \alpha = 1, 2), \tag{2.4}$$

where $T_{(0)j} = \underset{(0)}{T}{}^i{}_{ji}$, $S_{(\alpha)j} = \underset{(\alpha)}{S}{}^i{}_{ji}$, $(\alpha = 1, 2)$.

Definition 2.4. An N-linear connection $D\Gamma(N) = (L^i_{jk}, \underset{(1)}{C}{}^i_{jk}, \underset{(2)}{C}{}^i_{jk})$ on E is called a *semi-symmetric conformal* metrical N-linear connection if the relations (2.3) and (2.4) are verified.

If $\sigma_j = \frac{1}{n-1}T_{(0)j}$, $\tau_{(\alpha)j} = \frac{1}{n-1}S_{(\alpha)j}$, $(\alpha = 1, 2)$ and if we apply the Theorem 5.4.3., [8], we obtain:

Theorem 2.1 *The set of all semi-symmetric conformal metrical N-linear connections $D\Gamma(N) = (L^i_{jk}, \underset{(1)}{C}{}^i_{jk}, \underset{(2)}{C}{}^i_{jk})$, which preserve the nonlinear connection N, on E is given by*

$$L_{jk}^{i} = \overset{0}{L}{}_{jk}^{i} + 2\Omega_{kj}^{ir}\sigma_r, C_{(\alpha)jk}^{\ i} = \overset{0}{C}{}_{(\alpha)jk}^{\ i} + 2\Omega_{kj}^{ir}\tau_{(\alpha)r}, \ (\alpha = 1, 2), \tag{2.5}$$

where $D\overset{0}{\Gamma}(N) = (\overset{0}{L}{}_{jk}^{i}, \overset{0}{C}{}_{(1)jk}^{\ i}, \overset{0}{C}{}_{(2)jk}^{\ i})$ *is an arbitrary conformal metrical N-linear connection on E, whose d-tensor fields $T_{(0)}$, $S_{(\alpha)}$, $(\alpha = 1, 2)$ are vanish.*

3 Group of transformations of semi- symmetric conformal metrical N-linear connections on $E = Osc^2M$, which preserve the nonlinear connection N.

Let us consider the transformations $D\Gamma(N) \longrightarrow D\bar{\Gamma}(N)$ of semi-symmetric conformal metrical N-linear connections on E, which preserve the nonlinear connection N.

Theorem 3.1 *Two semi-symmetric conformal metrical N-linear connections on E, $D\Gamma(N) = (L_{jk}^{i}, C_{(1)jk}^{\ i}, C_{(2)jk}^{\ i})$, $D\bar{\Gamma}(N) = (\bar{L}_{jk}^{i}, \bar{C}_{(1)jk}^{\ i}, \bar{C}_{(2)jk}^{\ i})$, are related as follows:*

$$\bar{L}_{jk}^{i} = L_{jk}^{i} - \delta_j^i\omega_k + 2\Omega_{kj}^{ir}\theta_r, \bar{C}_{(\alpha)jk}^{\ i} = C_{(\alpha)jk}^{\ i} - \delta_j^i\lambda_{(\alpha)k} + 2\Omega_{kj}^{ir}\gamma_{(\alpha)r}, \quad (\alpha = 1, 2), \tag{3.1}$$

where we have $\theta_r = \sigma_r - \omega_r$, $\gamma_{(\alpha)r} = \tau_{(\alpha)r} - \lambda_{(\alpha)r}$, $(\alpha = 1, 2)$.

Theorem 3.2 *The set $\mathcal{C}^s{}_N$ of all the transformations given by* (3.1) *is a transformation group of the set of all conformal metrical N-linear connections on E, which preserve the nonlinear connection N, together with the mapping product.*

The transformation $t : D\Gamma \longrightarrow D\bar{\Gamma}$ given by (3.1) is expresed by the product of the following two transformations:

$$\bar{L}_{jk}^{i} = L_{jk}^{i} - \delta_j^i\omega_k, \ \bar{C}_{(\alpha)jk}^{\ i} = C_{(\alpha)jk}^{\ i} - \delta_j^i\lambda_{(\alpha)k}, \ (\alpha = 1, 2), \tag{3.2}$$

$$\bar{L}_{jk}^{i} = L_{jk}^{i} + 2\Omega_{kj}^{ir}\theta_r, \ \bar{C}_{(\alpha)jk}^{\ i} = C_{(\alpha)jk}^{\ i} + 2\Omega_{kj}^{ir}\gamma_{(\alpha)r}, \ (\alpha = 1, 2). \tag{3.3}$$

Theorem 3.3 *The group $\mathcal{C}^s{}_N$ is the direct product of the group $\mathcal{C}^p{}_N$ (of all transformations* (3.2)*) and the group $\mathcal{C}^m{}_N$ (of all transformations* (3.3)*).*

One can notice that the invariants of the group $\mathcal{C}^s{}_N$ will be invariants of each of these subgroups, and reciprocally.

In our previous paper [16], starting from the tensor fields $K_h{}^i{}_{jk}$, $\mathcal{P}_{(\alpha)h}{}^i{}_{jk}$, $(\alpha = 1, 2)$, $S_{(22)h}{}^i{}_{jk}$, where

$$(3.4)\quad \begin{cases} K_{h\ jk}^{\ i} = R_{h\ jk}^{\ i} - C_{(1)\ hs}^{\ \ i} R_{(01)\ jk}^{\ \ s} - C_{(2)\ hs}^{\ \ i} R_{(02)\ jk}^{\ \ s}, \\ \mathcal{P}_{(1)h\ jk}^{\ \ i} = \mathcal{A}_{jk}\{P_{(1)h\ jk}^{\ \ i} - C_{(1)\ hs}^{\ \ i} \frac{\delta N_{(1)\ j}^{\ \ s}}{\delta y^{(1)k}} - C_{(2)\ hs}^{\ \ i}(N_{(1)\ m}^{\ \ s} \frac{\delta N_{(1)\ j}^{\ \ m}}{\delta y^{(1)k}} + \\ + \frac{\delta N_{(2)\ j}^{\ \ s}}{\delta y^{(1)k}} - \frac{\delta N_{(2)\ k}^{\ \ s}}{\delta y^{(1)j}})\}, \\ \mathcal{P}_{(2)h\ jk}^{\ \ i} = \mathcal{A}_{jk}\{P_{(2)h\ jk}^{\ \ i} - C_{(1)\ hs}^{\ \ i} \frac{\partial N_{(1)\ j}^{\ \ s}}{\delta y^{(2)k}} - C_{(2)\ hs}^{\ \ i}(N_{(1)\ m}^{\ \ s} \frac{\partial N_{(1)\ j}^{\ \ m}}{\delta y^{(2)k}} - \\ - \frac{\partial N_{(2)\ j}^{\ \ s}}{\partial y^{(2)k}})\}, \end{cases}$$

we obtained the following important invariants of the group of transformations of semi-symmetric metrical N-linear connections, which preserve the nonlinear connection N, on E:

$$(3.5)\quad \begin{cases} H_{h\ jk}^{\ i} = K_{h\ jk}^{\ i} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega_{jh}^{ir}(K_{rk} - \frac{K g_{rk}}{2(n-1)})\}, \\ N_{(\alpha)h\ jk}^{\ \ i} = \mathcal{P}_{(\alpha)h\ jk}^{\ \ i} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega_{jh}^{ir}(\mathcal{P}_{(\alpha)rk} - \frac{\mathcal{P}_{(\alpha)} g_{rk}}{2(n-1)})\},\ (\alpha = 1,2), \\ M_{(22)h\ jk}^{\ \ i} = S_{(22)h\ jk}^{\ \ i} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega_{jh}^{ir}(S_{(22)rk} - \frac{S_{(22)} g_{rk}}{2(n-1)})\}, \end{cases}$$

where

$$K_{hj} = K_{h\ ji}^{\ i},\ \mathcal{P}_{(\alpha)hj} = \mathcal{P}_{(\alpha)h\ ji}^{\ \ i},\ S_{(22)hj} = S_{(22)h\ ji}^{\ \ i},$$

$$K = g^{hj}K_{hj},\ \mathcal{P}_{(\alpha)} = g^{hj}\mathcal{P}_{(\alpha)hj},\ S_{(22)} = g^{hj}S_{(22)hj},\ (\alpha = 1,2).$$

If we replace $K_{h\ jk}^{\ i}$, $\mathcal{P}_{(\alpha)h\ jk}^{\ \ i}$, $(\alpha = 1,2)$, $S_{(22)h\ jk}^{\ \ i}$ by the tensor fields $K^{*\ i}_{\ h\ jk}$, $\mathcal{P}^{*\ \ i}_{(\alpha)h\ jk}$, $(\alpha = 1,2)$, $S^{*\ \ i}_{(22)h\ jk}$ defined by:

$$(3.6)\quad \begin{cases} K^{*\ i}_{\ h\ jk} = K_{h\ jk}^{\ i} - \frac{1}{n}\delta_h^i K_{s\ jk}^{\ s}, \\ \mathcal{P}^{*\ \ i}_{(\alpha)h\ jk} = \mathcal{P}_{(\alpha)h\ jk}^{\ \ i} - \frac{1}{n}\delta_h^i \mathcal{P}_{(\alpha)s\ jk}^{\ \ s},\ (\alpha = 1,2), \\ S^{*\ \ i}_{(22)h\ jk} = S_{(22)h\ jk}^{\ \ i} - \frac{1}{n}\delta_h^i S_{(22)s\ jk}^{\ \ s}, \end{cases}$$

we can obtain the invariants of the group of transformations of semi-symmetric conformal metrical N-linear connections on E, which preserve the nonlinear connection N

Theorem 3.4 *Let*

$K^*_{\ hj} = K^{*\ i}_{\ h\ ji}$, $\mathcal{P}^*_{(\alpha)hj} = \mathcal{P}^{*\ \ i}_{(\alpha)h\ ji}$, $(\alpha = 1,2)$, $S^*_{(22)hj} = S_{(22)h\ ji}^{\ \ i}$, $K^* = g^{hj}K^*_{\ hj}$
$\mathcal{P}^*_{(\alpha)} = g^{hj}\mathcal{P}^*_{(\alpha)hj}$, $(\alpha = 1,2)$, $S^*_{(22)} = g^{hj}S^*_{(22)hj}$.
For $n > 2$ the following tensor fields

$$
(3.7)\quad \begin{cases} H^*{}_h{}^i{}_{jk} = K^*{}_h{}^i{}_{jk} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(K^*{}_{rk} - \frac{K^* g_{rk}}{2(n-1)})\}, \\[2ex] N^*_{(\alpha)h}{}^i{}_{jk} = \mathcal{P}^*_{(\alpha)h}{}^i{}_{jk} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(\mathcal{P}^*_{(\alpha)rk} - \frac{\mathcal{P}^*_{(\alpha)} g_{rk}}{2(n-1)})\},\ (\alpha = 1,2), \\[2ex] M^*_{(22)h}{}^i{}_{jk} = S^*_{(22)h}{}^i{}_{jk} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(S^*_{(22)rk} - \frac{S^*_{(22)} g_{rk}}{2(n-1)})\}, \end{cases}
$$

determined by of semi-symmetric conformal metrical N-linear connections, which preserve the nonlinear connection N, on E, are invariants of the group $\mathcal{C}^s{}_N$:

Proof. We can easily show that $H^*{}_h{}^i{}_{jk}$, $N^*_{(\alpha)h}{}^i{}_{jk}$, $(\alpha = 1,2)$, $M^*_{(22)h}{}^i{}_{jk}$ are invariants of $\mathcal{C}^p_N$. Owing to Theorem 3.3, it suffices to prove the theorem for $\mathcal{C}^m{}_N$.

From Theorem 2.3 of [16], the tensor fields $K_h{}^i{}_{jk}$, $\mathcal{P}_{(\alpha)h}{}^i{}_{jk}$, $(\alpha = 1,2)$, $S_{(22)h}{}^i{}_{jk}$ of a semi-symmetric metrical N-linear connection $D\Gamma(N)$, are transformed on the basis of the relations (3.3) of $D\Gamma(N)$ to $D\bar{\Gamma}(N)$ as follows:

$$
(3.8)\quad \begin{cases} \bar{K}_h{}^i{}_{jk} = K_h{}^i{}_{jk} + 2\mathcal{A}_{jk}\{\Omega^{ir}_{jh}\sigma_{rk}\}, \\[2ex] \bar{\mathcal{P}}_{(\alpha)h}{}^i{}_{jk} = \mathcal{P}_{(\alpha)h}{}^i{}_{jk} + 2\mathcal{A}_{jk}\{\Omega^{ir}_{jh}\rho_{(\alpha)rk}\},\ (\alpha = 1,2), \\[2ex] \bar{S}_{(22)h}{}^i{}_{jk} = S_{(22)h}{}^i{}_{jk} + 2\mathcal{A}_{jk}\{\Omega^{ir}_{jh}\tau_{(2)rk}\}, \end{cases}
$$

where σ_{rk}, $\rho_{(\alpha)rk}$, $\tau_{(2)rk}$, $(\alpha = 1,2)$ are some d-tensor fields determined from $D\Gamma(N)$. Since $\Omega^{sr}_{ks} = 0$, the tensor fields $K^*{}_h{}^i{}_{jk}$, $\mathcal{P}^*_{(\alpha)h}{}^i{}_{jk}$, $(\alpha = 1,2)$, $S^*_{(22)h}{}^i{}_{jk}$ obey the same transformation laws as (3.8), Hence, (3.7) follows from the well-known elimination method used in Theorem 2.4 of [16].

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