On the Stationary, Potential, Subsonic Flow

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Abstract

In this communication we are concerned with the problem of the stationary, potential, subsonic flow. Firstly, we formulate the mechanical problem and the associated minimization problem with constraints, in a functional space W endowed with a certain norm.

Section 2 contains the main original results. A density Lemma for the space W is presented, then is introduced a stable, convergent, internal approximation of W. We state the approximate minimization problem and prove a convergence theorem. A minimax problem coresponds to the approximate minimization problem with constraints (P_h). For this one, we deduce the existence of the saddle point, using a separation Hahn-Banach theorem. In the end is presented an iterative algorithm for determining the minimax point.

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1 Theoretical background

Let Ω be an open, bounded set in $R^n\,(n = 2,\,3)$ with the boundary $\partial\Omega$ Lipschitz continuous. The governing equations of the potential, subsonic, stationary flow are [1]

$$div\,\left[\left(k-\frac{1}{2}|\nabla\varphi|^2\right)^{1/\gamma-1}\nabla\varphi\right]=0 \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\left(k-\frac{1}{2}|\nabla\varphi|^2\right)^{1/\gamma-1}\frac{\partial\varphi}{\partial n}=g \quad \text{on} \quad \Gamma, \tag{1.2}$$

$$|\nabla\varphi|<v_{cr}, \tag{1.3}$$

where

$$\rho(\nabla\varphi)=\rho_0\left(1-\frac{\gamma-1}{2}\frac{|\nabla\varphi|^2}{c_0^2}\right)^{1/\gamma-1}=\rho_0\left(\frac{\gamma-1}{c_0^2}\right)^{1/\gamma-1}\left(k-\frac{|\nabla\varphi|^2}{2}\right)^{1/\gamma-1}$$

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is the density,

$k = \dfrac{c_0^2}{\gamma - 1}$,

$p = p_0 \left(\dfrac{\rho}{\rho_0}\right)^{\gamma}$ is the pressure,

$\vec{v} = \nabla\varphi$ is the velocity of the flow,

$g \in H^{1/2}(\Gamma)$,

$n =$ the external unir normal.

Remark 1.1. For $|\nabla\varphi| < \sqrt{2k\dfrac{\gamma-1}{\gamma+1}} = v_{cr}$, the flow is subsonic. We denote $|| \, ||_0 =$ the norm in $L^2(\Omega)$, $|| \, ||_1 =$ the norm in $H^1(\Omega)$. We shall use the following result ([2]):
Lemma 1.1. *Let be P_k the set of polynomials of degree less than or equal to k in the variables $x_1, \dots, x_n$. The seminorm*

$$W^{k+1,p}(\Omega)/_{P_k} \ni \bar{v} \mapsto |\bar{v}|_{k+1,p} = |v|_{k+1,p} = \left(\sum_{|\alpha|=k+1} \int_{\Omega} |D^{\alpha}v|^p \, dx \right)^{1/p}$$

is a norm over the quotient space $W^{k+1,p}(\Omega)/_{P_k}$.

Further, introduce the space $W = \left\{ \psi \in H^1(\Omega) / \int_{\Omega} \psi(x)dx = 0 \right\}$.
Lemma 1.2. *The mapping $W \ni u \mapsto ||u||_W = ||grad\, u||_0$ is a norm on W, equivalent to the norm $|| \, ||_1$.*

We need a result due to Pironneau ([1]),
Theorem 1.1. *Let be $b < v_{cr}$. The problem (1.1)-(1.3) is equivalent to the minimisation problem:*

$$\text{find } \varphi \in K_b = \{\psi \in W / |\nabla\varphi| \leq b\} \text{ so that } J_0(\varphi) \leq J_0(\psi),\ \forall\psi \in K_b,$$

with

$$J_0(\psi) = -\int_{\Omega} \left(k - \frac{1}{2}|\nabla\psi|^2\right)^{\gamma/\gamma-1} dx - \frac{\gamma}{\gamma-1}\int_{\Gamma} g\ \gamma_0(\psi) d\sigma, \tag{1.4}$$

where we denoted by γ_0 the trace application. Moreover,

$$J_0^{''}(\psi)v^2 \geq c||v||_W^2 \quad (\forall) v \in W. \tag{1.5}$$

2 Main results

Let be T_h a family of regular triangulations of the poligonal (polyhedral) domain Ω. This means

$$\sigma(h) = \sup_{S \in T_h} \frac{\rho_S}{\rho_S'} \leq \alpha \quad (\forall) h, \tag{2.1}$$

where

S is a simplex in $\mathbf{R}^n$, $n = 2, 3$ (triangle or tethraedron),

$\rho_S =$ the diameter of the smallest ball containing S,

$\rho'_S =$ the diameter of the largest ball contained in S.

We proceed with the concept of stable, convergent, internal approximation (W_h, p_h, r_h) for a normed space W ([6]).

Definition 2.1. Let be $(W, || \, ||)$ a normed space and (W_h, p_h, r_h) a family of triples so that:

W_h is a normed space,

$p_h : W_h \to W$ is a linear, continuous application,

$r_h : W \to W_h$.

The approximation (W_h, p_h, r_h) of W is called internal, stable and convergent if:

1. $W_h \subset W, \quad (\forall) h$,
2. $||p_h||_{L(W_h, W)} \leq M$ independently of h,
3. $\lim\limits_{\rho(h) \to 0} ||p_h r_h u - u||_W = 0, \quad (\forall) u \in W$ where $\rho(h) = \sup\limits_{S \in T_h} \rho_s$.

We introduce the following notations: $(a_i)_{i=1,n+1}$ are the vertices of S;

E_h the set of vertices of all simplices $S \in T_h$;

$(\lambda_i)_{i=1,n+1} =$ the baricentric coordinates;

$V_h = \left\{ u_h : \Omega \to \mathbf{R} \;\middle|\; u_h|_S = \sum\limits_{i=1}^{n+1} u_h(a_i) \lambda_i \right\}$.

We denote by $(u_{hM})_{M \in E_h}$ a basis in V_h, which verifies $u_{hM}(M) = 1$, $(\forall) M \in E_h$ and

$$u_{hM}(P) = 0, \;\; (\forall) P \neq M, \;\; P \in E_h.$$

Then the set of functions $(w_{hM})_{M \in E_h}$, defined by

$$w_{hM}(x) = u_{hM}(x) - \frac{1}{meas(\Omega)} \int\limits_{\Omega} u_{hM}(x) dx$$

is a basis in $W_h = \left\{ v_h \in V_h \;\middle|\; \int\limits_{\Omega} v_h(x) dx = 0 \right\}$.

Lemma 2.1. *The set* $\tilde{W} = \left\{ u \in C^2(\bar{\Omega}) \;\middle|\; \int\limits_{\Omega} u(x) dx = 0 \right\}$ *is dense in* $(W, || \, ||)$.

Lemma 2.2. *Let be* (T_h) *a regular triangulation of the domain* Ω*,*

$$p_h : W_h \to W, \quad p_h u_h = u_h,$$
(2.3)
$$r_h : W \to W_h, \quad r_h v(x) = \sum_{M \in E_h} w_{hM}(x) v(M).$$

Then the approximation (W_h, p_h, r_h) *of* W *is internal, stable and convergent.*

Now, we are able to approximate the minimisation problem (1.4).

Theorem 2.1. *Let be* $K_{hb} = \{\psi_h \in W_h / |\nabla\psi_h| \le b\}$*. The minimisation problem* (P_h) :

find $\varphi_h \in K_{hb}$ *so that* $J_0(\varphi_h) \le J_0(\psi_h)$, $\forall\psi_h \in K_{hb}$ *admits a unique solution, for any* h*. Moreover,* $\lim\limits_{\rho(h)\to 0} ||\varphi_h - \varphi||_W = 0$ *where* φ *denotes the unique solution of the problem (1.4).*

We denote $X_h = \left\{ \mu_h \in L^2(\Omega) \;\middle|\; \mu_h = \sum\limits_{S \in T_h} \mu_{hS}\chi_S \right\}$, where χ_S is the characteristic function of the set S and $\wedge_h = \{\mu_h \in X_h / \mu_h \ge 0\}$.

We state the following two problems:

Primal problem. Find $\varphi_h \in K_{hb}$, so that

$$J_0(\varphi_h) \le J_0(\psi_h), \ (\forall)\psi_h \in K_{hb}. \tag{2.4}$$

The minimax problem. Find $(\varphi_h, \lambda_h) \in W_h \times \Lambda_h$, such that

$$L(\varphi_h, \mu_h) \le L(\varphi_h, \lambda_h) \le L(\psi_h, \lambda_h), \ (\forall)(\psi_h, \mu_h) \in W_h \times \Lambda_h, \tag{2.5}$$

where

$$L(\psi_h, \mu_h) = J_0(\psi_h) + \int\limits_\Omega \mu_h \left(|\nabla\psi_h|^2 - b^2\right) dx \tag{2.6}$$

is the Lagrangean associated to the primal problem.

Remark 2.1. According to [4], if (φ_h, λ_h) is a saddle point for the Lagrangean L, then φ_h is solution for the primal problem.

Theorem 2.2. *The minimax problem (2.5) has a solution.*

For the proof, is used the following separation theorem of Hahn-Banach type [5]:

Let be V *a topological vector space. Suppose* T *and* S *are two convex sets in* V *such that* T *has at least one interior point and* S *does not contain any interior point of* T *. Then there exists a functional* $F \in V^*, F \ne 0$ *and* $\alpha \in \boldsymbol{R}$ *such that*

$$F(x) \le \alpha \le F(y), \quad (\forall)x \in T, \ (\forall)y \in S.$$

Remark 2.2. The relation $L(\varphi_h, \mu_h) \le L(\varphi_h, \lambda_h)$, $(\forall)\mu_h \in \wedge_h$ can be rewritten as:

$$\int\limits_\Omega (\mu_h - \lambda_h) J_1(\varphi_h) dx \le 0, \quad (\forall)\mu_h \in \wedge_h,$$

where $J_1(\varphi_n) = |\nabla\varphi_n|^2 - b^2$.

Taking in consideration the variational characterization of the projection on a convex set in a Hilbert space, we infer

$$\lambda_h = P_{\wedge_h}(\lambda_h + \rho J_1(\varphi_h), \quad (\forall)\rho > 0,$$

where $P_{\wedge_h}$ projection operator on $\wedge_h$. We proceed with an iterative algorithm for determining the saddle point $(\varphi_h, \lambda_h) \in W_h \times \wedge_h$.

Theorem 2.3. *Let be* $(\varphi_{hn})_n \subset W_h$, $(\lambda_{hn}) \subset \wedge_h$ *the sequences computed by the following steps:* $\lambda_{h0} \in \wedge_h$ *is arbitrary,*

$$L(\varphi_{hn}, \lambda_{hn}) \leq L(\psi_h, \lambda_{hn}), \quad (\forall)\psi_h \in W_h, \tag{2.7}$$

$$\lambda_{h,n+1} = P_{\wedge_h}(\lambda_{hn} + \rho_n J_1(\varphi_{hn})). \tag{2.8}$$

Then, for $\rho_h > 0$ suitable chosen, $\varphi_{hn} \overset{n\to\infty}{\longrightarrow} \varphi_h$ in W_h.

Remark 2.3. The inequality (2.7) is equivalent to the variational equation

$$\frac{\gamma}{\gamma-1}\int_\Omega \left(k - \frac{1}{2}|\nabla\varphi_{hn}|^2\right)^{\frac{1}{\gamma-1}} \nabla\varphi_{hn}\nabla\psi_h dx + 2\int_\Omega \lambda_{hn}\nabla\varphi_{hn}\nabla\psi_h dx =$$

$$= \frac{\gamma}{\gamma-1}\int_\Gamma g\psi_h d\sigma, \ \forall\psi_h \in W_h$$

Remark 2.4. By the variational characterization of the projection, from eq. (2.8) infer

$$< \lambda_{h,n+1}, \mu_h - \lambda_{h,n+1} >_{L^2(\Omega)} \geq < \lambda_{hn} + \rho_n J_1(\varphi_{hn}), \mu_h - \lambda_{h,n+1} >_{L^2(\Omega)}, \quad (\forall)\mu_h \in \wedge_h,$$

which is a variational inequality of the form

$$a\,(u, v-u) \leq < f, v-u >, (\forall)\, v \in K,$$

where $a : X_h \times X_h \to \mathbf{R}$ is a bilinear, simetric, coercive form and $K = \wedge_h$ is a convex set. The variational inequation has an unique solution and is equivalent to the minimization problem:

$$\text{find } u \in K, \text{ so that } J\,(u) \leq J\,(v)\,, \quad (\forall)\, v \in K,$$

$$\text{where } J\,(v) = \frac{1}{2}a\,(v,v) - < f, v >.$$

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