Pseudo-Umbilical Spacelike Submanifolds in the Indefinite Space Form

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Abstract

We prove an integral inequality for compact pseudo-umbilical spacelike submanifolds in the indefinite space form. As an application of the inequality, we give a necessary and sufficient condition for such submanifolds to be totally geodesic.

Mathematics Subject Classification: 53C40, 53C42, 53C50
Key words: pseudo-umbilical spacelike submanifold, indefinite space form

1   Introduction

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of constant curvature c whose index is p, which is called as *indefinite space form of index p*. Let M^n be an n-dimensional Riemannian manifold immersed in $M_p^{n+p}(c)$. As the semi-Riemannian metric of $M_p^{n+p}(c)$ induces the Riemannian metric of M^n, M^n is called a spacelike submanifold. Let h be the second fundamental form of the immersion, and ξ the mean curvature vector. Denote by $\langle \cdot, \cdot \rangle$ the scalar product of $M_p^{n+p}(c)$. If there exists a function λ on M^n such that

$$(1.1) \qquad\qquad \langle h(X,Y), \xi \rangle = -\lambda \langle X, Y \rangle$$

for any tangent vector X, Y on M^n, then M^n is called a *pseudo-umbilical spacelike submanifold* of $M_p^{n+p}(c)$. It is clear that $\lambda \geq 0$. If the mean curvature vector ξ vanishes identically, then M^n is called a maximal spacelike submanifold of $M_p^{n+p}(c)$. Every maximal spacelike submanifold of $M_p^{n+p}(c)$ is itself a pseudo-umbilical spacelike submanifold of $M_p^{n+p}(c)$.

Maximal and pseudo-umbilical spacelike submanifolds have been studied by many researches. For example in 1988 Ishihara [1] proved that if M^n is a complete and maximal spacelike submanifold of $M_p^{n+p}(c)$, then either M^n is totally geodesic (when $c \geq 0$) or $0 \leq S \leq -npc$(when $c < 0$), where S is the square length of the second fundamental form of M^n. In 1995 Sun [2] first proved that the mean curvature H of

the pseudo-umbilical submanifolds M^n in $M_p^{n+p}(c)$ is constant, then he generalized Ishihara's result to pseudo-umbilical submanifolds, obtaining the inequality

$$nH^2 \leq S \leq \frac{1}{2}np\left[H^2 - c - \sqrt{(H^2 - c)^2 + 4H^2c/p}\right].$$

In this paper, we prove an integral inequality for compact pseudo-umbilical spacelike submanifolds in the indefinite space form and as an application of the inequality, we give a necessary and sufficient condition for such submanifolds to be totally geodesic. We will prove the following

Theorem 1. *Let M^n be an n-dimensional compact pseudo-umbilical spacelike submanifold in $M_p^{n+p}(c)$, then*

$$(1.2) \qquad \int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + \sum R_{mj}^2 - ncR + nH^2R \right\} * 1 \leq 0.$$

Theorem 2. *Let M^n be an n-dimensional compact pseudo-umbilical spacelike submanifold in $M_p^{n+p}(c)$, then*
(1.3)

$$\int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + (n-2)cS - n(n-1)c^2 + n^2(n-1)cH^2 + nH^2S \right\} * 1 \leq 0,$$

and equality holds if and only if M^n is totally geodesic.

In the above Theorem, $\sum R_{mijk}^2$ is the square length of the Riemannian curvature tensor of M^n, $\sum R_{mj}^2$ the square length of the Ricci curvature tensor and R the scalar curvature. We shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C, \cdots \leq n+p; \quad 1 \leq i, j, k, \cdots \leq n, \quad n+1 \leq \alpha, \beta, \gamma, \cdots \leq n+p,$$

and we shall agree that repeated indices are summed over the respective ranges.

2   Local formulas

We choose a local field of semi-Riemannian orthonormal frames $e_1, \cdots, e_{n+p}$ in $M_p^{n+p}(c)$ such that, restricted to M^n, $e_1, \cdots, e_n$ are tangent to M^n. Let $\omega_1, \cdots, \omega_{n+p}$ be its dual frame field such that the semi-Riemannian metric of $M_p^{n+p}(c)$ is given by

$$ds^2 = \sum (\omega_i)^2 - \sum (\omega_\alpha)^2 = \sum \varepsilon_A (\omega_A)^2,$$

where $\varepsilon_i = 1$ and $\varepsilon_\alpha = -1$. Then the structure equations of $M_p^{n+p}(c)$ are given by

$$(2.1) \qquad d\omega_A = -\sum \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$(2.2) \qquad d\omega_{AB} = -\sum \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum K_{ABCD}\omega_C \wedge \omega_D,$$

$$(2.3) \qquad K_{ABCD} = c\varepsilon_A\varepsilon_B(\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}).$$

We restrict these forms to M^n, then

$$(2.4) \qquad \omega_\alpha = 0, \quad \omega_{\alpha i} = \sum h^\alpha_{ij} \omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}$$

$$(2.5) \qquad d\omega_{ij} = -\sum \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l,$$

$$(2.6) \qquad R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) - \sum (h^\alpha_{il} h^\alpha_{jk} - h^\alpha_{ik} h^\alpha_{jl}),$$

$$(2.7) \qquad R_{jk} = \sum R_{ljkl} = c(n-1)\delta_{jk} + \sum h^\alpha_{lk} h^\alpha_{jl},$$

$$(2.8) \qquad R = \sum R_{jj} = n(n-1)c + S,$$

$$(2.9) \qquad d\omega_{\alpha\beta} = -\sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\alpha} - \frac{1}{2} \sum R_{\alpha\beta ij} \omega_i \wedge \omega_j,$$

$$(2.10) \qquad R_{\alpha\beta ij} = -\sum (h^\alpha_{il} h^\beta_{jl} - h^\alpha_{jl} h^\beta_{il}).$$

For indefinite Riemannian manifolds, refer to O'Neill [3].

We call $h = \sum h^\alpha_{ij} \omega_i \omega_j e_\alpha$ the second fundamental form of the immersed manifold M^n. Denote by $S = \sum (h^\alpha_{ij})^2$ the square length of h , $\xi = \frac{1}{n} \sum \mathrm{tr} H_\alpha e_\alpha$ the mean curvature vector and

$$H = \frac{1}{n} \sqrt{\sum (\mathrm{tr} H_\alpha)^2}$$

the mean curvature of M^n respectively. Here tr is the trace of the matrix $H_\alpha = (h^\alpha_{ij})$. Now let e_{n+1} be parallel to ξ. Then we have

$$(2.11) \qquad \mathrm{tr} H_{n+1} = nH, \quad \mathrm{tr} H_\alpha = 0, \quad \alpha \neq n+1.$$

Let h^α_{ijk} and h^α_{ijkl} denote the covariant derivative and the second covariant derivative of h^α_{ij} respectively, defined by

$$(2.12) \qquad \sum h^\alpha_{ijk} \omega_k = dh^\alpha_{ij} - \sum h^\alpha_{ik} \omega_{kj} - \sum h^\alpha_{jk} \omega_{ki} - \sum h^\beta_{ij} \omega_{\beta\alpha},$$

$$(2.13) \qquad \sum h^\alpha_{ijkl} \omega_l = dh^\alpha_{ijk} - \sum h^\alpha_{ijl} \omega_{lk} - \sum h^\alpha_{ilk} \omega_{lj} - \sum h^\alpha_{ljk} \omega_{li} - \sum h^\beta_{ijk} \omega_{\beta\alpha},$$

then we have

$$(2.14) \qquad h^\alpha_{ijk} - h^\alpha_{ikj} = 0,$$

$$(2.15) \qquad h^\alpha_{ijkl} - h^\alpha_{ijlk} = -\sum h^\alpha_{im} R_{mjkl} - \sum h^\alpha_{jm} R_{mikl} - \sum h^\beta_{ij} R_{\alpha\beta kl}.$$

The Laplacian $\triangle h^\alpha_{ij}$ of h^α_{ij} is defined by $\triangle h^\alpha_{ij} = \sum h^\alpha_{ijkk}$. By a direct calculation we have (cf. [4])

$$(2.16) \qquad \begin{aligned} \sum h^\alpha_{ij} \triangle h^\alpha_{ij} &= \sum h^\alpha_{ij} h^\alpha_{kkij} - \sum h^\alpha_{ij} h^\alpha_{km} R_{mijk} - \\ &\quad - \sum h^\alpha_{ij} h^\alpha_{mi} R_{mkjk} - \sum h^\alpha_{ij} h^\beta_{ki} R_{\alpha\beta jk}. \end{aligned}$$

3   Proof of Theorem

Proof of Theorem 1. From (1.1) and (2.11)

(3.1) $$\langle h(e_i, e_j), He_{n+1} \rangle = -H^2 \delta_{ij} \quad \text{i.e.} \quad h_{ij}^{n+1} = -H \delta_{ij},$$

therefore

(3.2) $$\sum h_{ij}^{\alpha} h_{kkij}^{\alpha} = -nH \triangle H.$$

Since H is constant (see [2]),

(3.3) $$\sum h_{ij}^{\alpha} h_{kkij}^{\alpha} = 0.$$

On the other hand, from (2.6)

(3.4)
$$-\sum h_{ij}^{\alpha} h_{km}^{\alpha} R_{mijk} = -\frac{1}{2} \sum (h_{ij}^{\alpha} h_{km}^{\alpha} - h_{mj}^{\alpha} h_{ik}^{\alpha}) R_{mijk}$$
$$= \frac{1}{2} \sum \{R_{imkj} - c(\delta_{ij}\delta_{mk} - \delta_{mj}\delta_{ik})\} R_{mijk} = \frac{1}{2} \sum R_{mijk}^2 - cR,$$

(3.5)
$$-\sum h_{ij}^{\alpha} h_{mi}^{\alpha} R_{mkjk} = \sum (h_{ij}^{\alpha} h_{mi}^{\alpha} - h_{ii}^{\alpha} h_{mj}^{\alpha} + nH^2 \delta_{mj}) R_{mj} =$$
$$= \sum \{c(\delta_{ij}\delta_{mi} - \delta_{ii}\delta_{mj}) - R_{imij} + nH^2 \delta_{mj}\} R_{mj} =$$
$$= \sum R_{mj}^2 - (n-1)cR + nH^2 R.$$

From (2.10)

(3.6) $$-\sum h_{ij}^{\alpha} h_{ki}^{\beta} R_{\alpha\beta jk} = \sum h_{ij}^{\alpha} h_{ki}^{\beta} h_{jl}^{\alpha} h_{kl}^{\beta} - \sum h_{ij}^{\alpha} h_{ki}^{\beta} h_{kl}^{\alpha} h_{jl}^{\beta}.$$

Since

(3.7)
$$\sum h_{ij}^{\alpha} h_{jl}^{\beta} h_{lk}^{\alpha} h_{ki}^{\beta} = \sum \text{tr}(H^{\alpha} H^{\beta})^2 \le$$
$$\le \sum \text{tr}((H^{\alpha})^2 (H^{\beta})^2) = \sum h_{ij}^{\alpha} h_{jl}^{\alpha} h_{lk}^{\beta} h_{ki}^{\beta},$$

therefore

(3.8) $$-\sum h_{ij}^{\alpha} h_{ki}^{\beta} R_{\alpha\beta jk} \ge 0.$$

From (2.14), (3.3-3.8)

(3.9) $$\sum h_{ij}^{\alpha} \triangle h_{ij}^{\alpha} \ge \frac{1}{2} \sum R_{mijk}^2 + \sum R_{mj}^2 - ncR + nH^2 R.$$

Since $\int_{M^n} \left\{ \sum h_{ij}^{\alpha} \triangle h_{ij}^{\alpha} \right\} * 1 \le 0$ (see [4]), we have

(3.10) $$\int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + \sum R_{mj}^2 - ncR + nH^2 R \right\} * 1 \le 0.$$

Theorem 1 is proved.
Proof of Theorem 2. From (2.7)

$$(3.11) \qquad \sum R_{mj}^2 = n(n-1)^2 c^2 + 2(n-1)cS + \sum_{m,j} \left(\sum_{i,\alpha} h_{im}^\alpha h_{ij}^\alpha \right)^2,$$

$$(3.12) \qquad \sum_{m,j} \left(\sum_{i,\alpha} h_{im}^\alpha h_{ij}^\alpha \right)^2 \geq \sum_j \left(\sum_{i,\alpha} (h_{ij}^\alpha)^2 \right)^2 \geq \frac{1}{n} \left(\sum (h_{ij}^\alpha)^2 \right)^2 = \frac{1}{n} S^2,$$

therefore from (3.10),
(3.13)
$$\int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + (n-2)cS - n(n-1)c^2 + n^2(n-1)cH^2 + nH^2 S \right\} * 1 \leq 0.$$

If M^n is totally geodesic, i.e., $S = 0$, $h_{ij}^\alpha = 0$, then from (2.6)

$$(3.14) \qquad \frac{1}{2} \sum R_{mijk}^2 = n(n-1)c^2,$$

in this case, (3.13) becomes an equality; Inversely, if (3.13) becomes an equality, then from (3.12) $S = 0$, i.e. M^n is totally geodesic. Theorem 2 is proved.

References

[1] T. Ishihara, *Maximal spacelike submanifolds of a pseudo-Riemannian space of constant curvature*, Michigan Math. J., 35(1988), 345-352.

[2] H. Sun, *On spacelike submanifolds of a pseudo-Riemannian space form*, Note Mat., 15(1995), 215-224.

[3] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

[4] S.S. Chern, M. do Carmo and S. Kobayashi, *Minimal submanifolds of sphere with second fundamental form of constant length*, Functional Analysis and Related Fields, Springer, 1970, 59-75.

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