Weierstrass-type Representation of Weakly Regular Pseudospherical Surfaces in Euclidean Space

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Abstract

In [To], the author presented a method of constructing all weakly regular pseudospherical surfaces corresponding to given Weierstrass-type data. While the construction itself will appear later as a separate publication, this report contains a complete and detailed description of the Weierstrass representation for weakly regular surfaces with $K = -1$, in terms of moving frames and loop groups.

Mathematics Subject Classification: 53A10, 53C42, 58D10, 49Q05.
Keywords: moving frames, pseudospherical surfaces, loop groups.

1 Moving Frames of Surfaces in E^3

This is a general introduction to the concept of a moving frame for a surface in E^3, in the spirit of [Ei] and [Ch, Te].

In the real Euclidean three-space E^3 endowed with the inner product $\langle \cdot, \cdot \rangle$, a *frame* is an ordered quadruple $F = \{x, e_1, e_2, e_3\}$, where $x \in E^3$ and e_1, e_2, e_3 are orthonormal vectors of positive orientation, i.e., $e_3 = e_1 \times e_2$. Let $\mathcal{F}$ denote the set of all frames. We will mostly be interested in families of frames along certain submanifolds. Such a family is usually called an orthonormal moving frame. Throughout the text, we refer to it briefly as (moving) frame. A Frenet frame is an example of a moving frame.

Example [Frenet frames along a curve]. Let $\alpha = \alpha(t)$ be a curve in E^3. The Frenet frame $\{x, e_1, e_2, e_3\}$ along the curve α, as described in classical differential geometry, consists of the unit tangent vector field e_1, the unit normal vector field e_2 and the unit binormal vector field e_3. These vectors satisfy the Frenet equations

$$
\begin{cases}
\mathrm{d}x = \mathrm{d}s \cdot e_1 \\
\mathrm{d}e_1 = \mathrm{d}s \cdot k(t) \cdot e_2 \\
\mathrm{d}e_2 = \mathrm{d}s \cdot (-k(t) \cdot e_1 + \tau(t) e_3) \\
\mathrm{d}e_3 = -\mathrm{d}s \cdot \tau(t) \cdot e_2
\end{cases} \tag{1.1.1}
$$

Balkan Journal of Geometry and Its Applications, Vol.7, No.2, 2002, pp. 87-136.

Here $ds = s'(t)dt$ represents the arc length differential, while k and τ denote the curvature and torsion, respectively. Conversely, given arbitrary differential forms $ds \neq 0$, $k(t)dt$, $\tau(t)dt$, one can reconstruct the curve uniquely up to Euclidean motions.

For moving frames of surfaces, there exist differential forms generalizing ds, $k(t)dt$, $\tau(t)dt$, satisfying some integrability conditions, the Gauss-Codazzi equations. Cartan showed that these equations can be derived from the integrability conditions satisfied by the so-called Cartan forms (see (1.1.10–11) below).

We will see that the space of all frames $\mathcal{F}$ forms a 6-dimensional manifold. This manifold can be identified with the group of Euclidean motions defined below.

Consider the groups

1.1.2 $$\mathrm{O}(3) = \{A : E^3 \to E^3 \text{ linear; } \langle Ax, Ay\rangle = \langle x, y\rangle,\ x, y \in E^3\}$$

1.1.3 $$\mathrm{SL}(3, \mathbf{R}) = \{A : E^3 \to E^3 \text{ linear; } \det A = 1\}$$

1.1.4 $$\mathrm{SO}(3) = \{A \in \mathrm{O}(3);\ \det A > 0\}.$$

Note, $\mathrm{SO}(3) = \mathrm{SL}(3, \mathbf{R}) \cap \mathrm{O}(3)$.

We define the group of orientation-preserving rigid motions

$$G = \{w \mapsto x + Aw;\ x \in E^3,\ A \in \mathrm{SO}(3)\}. \tag{1.1.5}$$

Note that the groups (1.1.2)–(1.1.5) are real Lie groups.

To identify G with $\mathcal{F}$, we fix a frame $F_0 = \{0, \check{e}_1, \check{e}_2, \check{e}_3\}$ in $\mathcal{F}$. Then if $F = \{x, e_1, e_2, e_3\}$ is an arbitrary frame in $\mathcal{F}$, the map

$$w \mapsto x + \sum_{i=1}^{3} \langle \check{e}_i, w\rangle e_i, \qquad w \in E^3 \tag{1.1.6}$$

is an element of the group G.

Fixing F_0 means fixing an origin and an orthonormal basis. Expressing the entries of an arbitrary frame F in terms of this basis, via (1.1.6), realizes F as a pair consisting of a translation vector and an orientation-preserving matrix.

Conversely, given $g \in G$ we set

$$x = g(0) \quad \text{and} \quad e_i = g(\check{e}_i) - x. \tag{1.1.7}$$

The resulting $F = \{x, e_1, e_2, e_3\}$ is a frame and it is easy to see that the operations (1.1.6) and (1.1.7) are inverse to each other.

The bijection presented above gives an isomorphism between G and $\mathcal{F}$, and thus $\mathcal{F}$ is endowed with a manifold structure.

We consider the maps

1.1.8 $$x^f : \mathcal{F} \to E^3, \quad x^f(\{y, u_1, u_2, u_3\}) = y,$$

1.1.9 $$e_j^f : \mathcal{F} \to E^3, \quad e_j^f(\{y, u_1, u_2, u_3\}) = u_j, \qquad j = 1, 2, 3$$

The differentials $\mathrm{d}x^f$, $\mathrm{d}e_1^f$, $\mathrm{d}e_2^f$ and $\mathrm{d}e_3^f$ estimated at F are linear maps from the tangent space $T_F\mathcal{F}$ of $\mathcal{F}$ to E^3. Therefore they can be written as linear combinations relative to the basis $e_1^f(F), e_2^f(F), e_3^f(F)$.

Thus, at a "point" $F \in \mathcal{F}$, we define the scalar differential forms $\omega_1, \omega_2, \omega_3$, ω_{ij}, $1 \le i, j \le 3$ via

$$\mathrm{d}_F x^f = \omega_1 e_1^f(F) + \omega_2 e_2^f(F) + \omega_3 e_3^f(F) \tag{1.1.10}$$

and

$$\mathrm{d}_F e_j^f = \sum_{k=1}^{3} \omega_{jk} e_k^f(F). \tag{1.1.11}$$

Since $\langle e_j, e_j \rangle = 1$, we have $\langle e_j^f(F), e_j^f(F) \rangle = 1$, for every $F \in \mathcal{F}$. Therefore,

$$\langle \mathrm{d}_F e_j^f(\mathcal{U}), e_j^f(F) \rangle = 0, \qquad \text{for all } \mathcal{U} \in T_F\mathcal{F}. \tag{1.1.12}$$

This last relation implies

$$\omega_{jj} = 0, \qquad j = 1, 2, 3 \tag{1.1.13}$$

Moreover, since $\langle e_j^f(F), e_k^f(F) \rangle = 0$ for all $j \neq k$, differentiation yields

$$\langle \mathrm{d}_F e_j^f(\mathcal{U}), e_k^f(F) \rangle + \langle e_j^f(F), \mathrm{d}_F e_k^f(\mathcal{U}) \rangle = 0, \tag{1.1.14}$$

that is

$$\omega_{jk} = -\omega_{kj}. \tag{1.1.15}$$

Therefore, equations (1.1.10–11) are completely determined by the six 1-forms $\omega_1, \omega_2, \omega_3, \omega_{12}, \omega_{13}, \omega_{23}$.

It is straightforward to verify the Cartan Structure Equations ([Ch, Te], p.106):

$$\mathrm{d}\omega_i = \sum_{j=1}^{3} \omega_j \wedge \omega_{ji}, \qquad i = 1, 2, 3 \tag{1.1.16}$$

$$\mathrm{d}\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj}, \qquad 1 \le i, j \le 3. \tag{1.1.17}$$

Let now $M = (D, \psi)$ be an immersion of an open connected subset $D \subset \mathbf{R}^2$ into $\mathbf{R}^3$, $\psi : D \to \mathbf{R}^3$. This describes a parametric surface, admitting self-intersections.

All the frames $\{x, e_1, e_2, e_3\}$ with $x \in \psi(D)$ form the *zeroth order frame bundle* of M. The set of zeroth order frames will be denoted by $\mathcal{F}_0^{\mathcal{M}}$.

It is easy to see that the diffeomorphism $\mathcal{F} \cong G \cong \mathbf{R}^3 \times \mathrm{SO}(3)$, induced by fixing a frame F_0, yields $\mathcal{F}_0^{\mathcal{M}} \cong \psi(D) \times \mathrm{SO}(3)$.

Let now $F \in \mathcal{F}_0^{\mathcal{M}}$. Since we identified $\mathcal{F}$ with G, the group of orientation-preserving rigid motions of $\mathbf{R}^3$, the frame F in particular is identified with a pair $F = (\psi(u, v), A)$, $(u, v) \in D$, $A \in \mathrm{SO}(3)$.

On $\mathcal{F}_0^{\mathcal{M}}$ we have natural vector fields: $Y_1 = \partial_u F$, $Y_2 = \partial_v F$ and Y_B, where B is any vector field of $SO(3)$, pulled back to $\mathcal{F}_0^{\mathcal{M}}$. Then $\omega_j(Y_B) = 0$.

Let S^2 denote the unit sphere in $\mathbf{R}^3$. Let $\vec{N} : D \to S^2$ denote a unit normal vector field to M. Then $\langle \vec{N}(u,v), \partial_u \psi(u,v)\rangle = 0$. If $Y_1 = \partial_u F$, $Y_2 = \partial_v F$ denote the standard vector fields along $\mathcal{F}_0^{\mathcal{M}}$ introduced above, then

1.1.18 $$0 = \langle \vec{N}(u,v), \partial_u \psi(u,v)\rangle = \langle \vec{N}, \mathrm{d}_F x^f(Y_1)\rangle = \sum_j \langle \vec{N}, e_j^f(F)\rangle \omega_j(Y_1).$$

Similarly, we obtain

(1.1.19) $$0 = \sum_j \langle \vec{N}, e_j^f(F)\rangle \omega_j(Y_2).$$

Since $\omega_j(Y_B) = 0$ for all Y_B, restricting $\omega_1, \omega_2, \omega_3$ to $\mathcal{F}_0^{\mathcal{M}}$, we obtain

(1.1.20) $$\sum_j \langle \vec{N}, e_j^f(F)\rangle \omega_j = 0.$$

Relation (1.1.20) represents the equation of the tangent plane to M at x relative to the frame $F = \{x, e_1, e_2, e_3\}$. The coefficients $a_j := \langle \vec{N}, e_j^f(F)\rangle$ vary smoothly with the frame. Note that if the frame is such that e_1, e_2 span the tangent plane of M at x, then above linear relation (1.1.20) takes the form $\omega_3 = 0$.

For our goals, it is natural to consider moving frames for which e_1 and e_2 are tangent to M.

Definition 1.0.1. Given an immersion $M = (D, \psi)$ as above, we define

(1.1.21) $$\mathcal{F}_1^{\mathcal{M}} = \{(x, e_1, e_2, e_3) \in \mathcal{F}_0^{\mathcal{M}};\ e_1, e_2 \in T_x M\},$$

where $T_x M$ denotes the tangent plane to M at x.

$\mathcal{F}_1$ is called the *first order frame bundle* of M.

Along $\mathcal{F}_1$, ω_3 vanishes, that is

(1.1.22) $$\omega_3|_{T\mathcal{F}_1} = 0.$$

The above relation also implies

(1.1.23) $$0 = \mathrm{d}\omega_3 = \omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} \qquad \text{on} \quad T\mathcal{F}_1 \times T\mathcal{F}_1.$$

For $\mathcal{F}_1$, Cartan's structure equations (1.1.16–17) are written as

1.1.24*a* $$d\omega_1 = \omega_{12} \wedge \omega_2$$

1.1.24*b* $$d\omega_2 = \omega_1 \wedge \omega_{12}$$

1.1.24*c* $$d\omega_{12} = -\omega_{13} \wedge \omega_{23}$$

1.1.24*d* $$d\omega_{13} = \omega_{12} \wedge \omega_{23}$$

1.1.24*e* $$d\omega_{23} = \omega_{13} \wedge \omega_{12}$$

1.1.24*f* $$\omega_3 = 0$$

1.1.24*g* $$\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} = 0$$

The third equation above is also known as the *Gauss equation*, while the fourth and fifth together are known as the *Codazzi equations*.

By Cartan's lemma [Ca, p.61], the last equation of (1.1.24g) implies:

(1.1.25) $$\begin{cases} \omega_{13} = h_{11}\omega_1 + h_{12}\omega_2, \\ \omega_{23} = h_{12}\omega_1 + h_{22}\omega_2. \end{cases}$$

for some functions h_{ij} defined on D.

By (1.1.25) and the Gauss equation (1.1.24c), we obtain

(1.1.26) $$\mathrm{d}\omega_{12} = -\omega_{13} \wedge \omega_{23} = -[\det(h_{ij})]\omega_1 \wedge \omega_2,$$

where $K \overset{\text{def}}{=} \det h_{ij}$ is called the *Gaussian curvature* of M.

For the immersion $M = (D, \psi)$ and the submanifold $\mathcal{F}_1^{\mathcal{M}} \subset \mathcal{F}$, the first fundamental form becomes

(1.1.27) $$\mathrm{I} = \langle \mathrm{d}x^f, \mathrm{d}x^f \rangle = \omega_1^2 + \omega_2^2,$$

while the second fundamendal form is

1.1.28 $$\begin{aligned} \mathrm{II} &= \langle -\mathrm{d}N, \mathrm{d}x^f \rangle = \langle -\mathrm{d}e_3^f, \mathrm{d}x^f \rangle = \left\langle -\textstyle\sum_{k=1}^3 \omega_{3k} e_k^f, \sum_{i=1}^3 \omega_i e_i^f \right\rangle = - \\ &= -(\omega_{31}\omega_1 + \omega_{32}\omega_2) = \omega_{13}\omega_1 + \omega_{23}\omega_2. \end{aligned}$$

In formula (1.1.28), we chose the normal unit vector $N = e_3$.

Taking into account equations (1.1.25), we obtain

(1.1.29) $$\mathrm{II} = h_{11}\omega_1^2 + 2h_{12}\omega_1\omega_2 + h_{22}\omega_2^2.$$

The two-form $\omega_1 \wedge \omega_2$ is an area element for the surface. Therefore, since $\psi : D \to \mathbf{R}^3$ is an immersion, it follows that

(1.1.30) $$\omega_1 \wedge \omega_2 \neq 0.$$

As a consequence of formulas (1.1.25), the Gaussian curvature $K = h_{11}h_{22} - h_{12}^2$ is given by

(1.1.31) $$\omega_{13} \wedge \omega_{23} = (h_{11}\omega_1 + h_{12}\omega_2) \wedge (h_{12}\omega_1 + h_{22}\omega_2) = K(\omega_1 \wedge \omega_2),$$

while the mean curvature $H = (h_{11} + h_{22})/2$ is given by

(1.1.32) $$\omega_1 \wedge \omega_{23} - \omega_2 \wedge \omega_{13} = h_{22}(\omega_1 \wedge \omega_2) - h_{11}(\omega_2 \wedge \omega_1) = 2H(\omega_1 \wedge \omega_2).$$

All the formulas presented in this section were formulated by Elie Cartan ([Ca]). We followed the presentation of Cartan's structure equations for $\mathbf{R}^3$ in [Ch, Te, eqs. (1.1)–(1.3)] and the one for surfaces in space forms of constant Gaussian curvature from [Te, eqs. (1.1)–(1.14)].

2 Pseudospherical Surfaces and the Sine-Gordon Equation

In this section, we begin our study of surfaces with constant negative Gaussian curvature. Among them, surfaces of Gaussian curvature $K = -1$, called *pseudospherical surfaces*, are of particular interest to us. We show that all surfaces with constant Gaussian curvature are described by a sine-Gordon equation, and we write a corresponding Lax system.

The following two parametrizations are of significant importance for this class of surfaces. We will also specify the relationship between the parametrizations.

2.1 The Asymptotic Line Parametrization

Let us consider an immersion $M = (D, \psi)$ with constant negative Gaussian curvature. In the Euclidean space, every unit free vector represents a *direction*.

For each point of M, there are two directions in which the second fundamental form vanishes, called *asymptotic directions* ([Ei, (46.3)]). An *asymptotic line* on the surface M is a regular connected curve whose tangent unit vector is an asymptotic direction at each point. Consequently, we have two families of asymptotic lines, each tangent to an asymptotic direction everywhere. An *asymptotic line parametrization* is a parametrization such that the coordinate lines are asymptotic lines.

The given immersion $M = (D, \psi)$ can be locally reparametrized, such that the coordinate lines are asymptotic lines. For an open and connected domain D, this reparametrization can be done globally. Therefore, for the rest of this section we will assume $\psi : D \to \mathbf{R}^3$ to be an asymptotic line parametrization of the surface M, where D is a open connected domain in $\mathbf{R}^2$.

Let φ represent the angle between the asymptotic lines, measured counterclockwise from the vector field ψ_x to the vector field ψ_y.

We denote $A = |\psi_x|$, $B = |\psi_y|$.

Then the first fundamental form is ([Ei], [Bo2]):

$$\mathrm{I} = |\mathrm{d}\psi|^2 = A^2(\mathrm{d}x)^2 + 2AB\cos\varphi \mathrm{d}x\mathrm{d}y + B^2(\mathrm{d}y)^2.$$

For every point, via a change of coordinates, we can reparametrize the surface such that the asymptotic lines are parametrized in arc length.

Let us assume that A and B never vanish. An immersion ψ with this property is called *weakly regular*. A weakly regular surface can be always reparametrized such that both asymptotic lines are in arc length ($A = B = 1$).

In this context, let $N : D \to S^2$, $N = \dfrac{\psi_x \times \psi_y}{\|\psi_x \times \psi_y\|}$ define the Gauss map of the immersion ψ. Remark that the unit vector field N is orthogonal to ψ_x, ψ_y, ψ_{xx}, ψ_{yy}.

Definition 2.1.1. A parametrization for which $A = B = 1$ is called a Chebyshev net ([Spi]).

Unless stated otherwise, we will assume for the rest of this work that the immersion ψ corresponds to a Chebyshev net of angle (between asymptotic lines) $\varphi(x, y) \in (0, \pi)$. In this case, the metric becomes:

$$\mathrm{I} = |d\psi|^2 = (\mathrm{d}x)^2 + 2\cos\varphi \mathrm{d}x\mathrm{d}y + (\mathrm{d}y)^2. \tag{2.1.1}$$

[McL] presents a way of constructing a Chebyshev net physically, by "a piece of nonstrech fabric that is loosely woven, so that the angle between the threads can change. Then drape it over the surface so that the warp and weft of the fabric become coordinate lines on the surface". Since the threads cannot stretch, $A = B = 1$, but the angle $\varphi(x, y)$ changes. The second fundamental form in asymptotic parametrization is written as

$$\mathrm{II} = 2AB\sqrt{-K}\sin\varphi\,\mathrm{d}x\,\mathrm{d}y.$$

For a Chebyshev net, it clearly becomes

$$\mathrm{II} = 2\sqrt{-K}\sin\varphi\,\mathrm{d}x\,\mathrm{d}y, \tag{2.1.2}$$

where K represents the (constant, negative) Gaussian curvature,

$$K = \det\mathrm{II}/\det\mathrm{I}.$$

Let us now focus on the case of the pseudospherical surfaces, that is surfaces of Gaussian curvature $K = -1$. It is straightforward to calculate the principal curvatures k_1 and k_2 of the immersion. k_1 and k_2 represent the eigenvalues of the matrix

$$\mathrm{II}\cdot\mathrm{I}^{-1} = \begin{pmatrix} -\cot\varphi & \csc\varphi \\ \csc\varphi & -\cot\varphi \end{pmatrix}, \tag{2.1.3}$$

that is, the roots of the characteristic equation

$$\lambda^2 + 2\cdot\cot\varphi\cdot\lambda - 1 = 0,$$

i.e.,

$$k_1 = \tan\frac{\varphi}{2} \quad \text{and} \quad k_2 = -\cot\frac{\varphi}{2}. \tag{2.1.4}$$

The angle between the asymptotic lines can be written as $\varphi(x, y) = 2\arctan k_1$.

Let e_1 and e_2 be the principal directions on M corresponding to k_1 and k_2 respectively, that is the eigenvectors of the matrix $\mathrm{II}\cdot\mathrm{I}^{-1}$ at each point of M. Then the relation between the asymptotic directions on M and the principal directions on M is given by

$$\begin{aligned} \partial_x &= \cos\frac{\varphi}{2}e_1 - \sin\frac{\varphi}{2}e_2, \\ \partial_y &= \cos\frac{\varphi}{2}e_1 + \sin\frac{\varphi}{2}e_2. \end{aligned} \tag{2.1.5}$$

2.2 The Curvature Line Parametrization; Sine-Gordon Equation

Another useful parametrization for a pseudospherical immersion $M = (D, \psi)$ is the one by lines of curvature, i.e., the coordinates u_i in which both the first fundamental form I and the second fundamental form II are diagonalized as

$$\mathrm{I} = (a_1)^2(\mathrm{d}u_1)^2 + (a_2)^2(\mathrm{d}u_2)^2 \tag{2.2.1a}$$

2.2.1b $$\mathrm{II} = b_1 \cdot (a_1)^2 (\mathrm{d}u_1)^2 + b_2 \cdot (a_2)^2 (\mathrm{d}u_2)^2.$$

In general, such a parametrization exists only in the neighborhood of a non-umbilical point. Since the Gaussian curvature is negative, there are no umbilics on M.

In particular, on a weakly regular pseudospherical surface we can find a curvature line parametrization around every point.

More specifically, we set

$$u_1 = x + y, \qquad u_2 = x - y,$$

where (x, y) are the Chebyshev net coordinates from Section 2.1. (i.e. $A = B = 1$).

Then formulas (2.1.1) and (2.1.2), for $K = -1$, become:

2.2.2a $$\mathrm{I} = \cos^2 \frac{\varphi}{2} \cdot (\mathrm{d}u_1)^2 + \sin^2 \frac{\varphi}{2} \cdot (\mathrm{d}u_2)^2$$

2.2.2b $$\mathrm{II} = \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} ((\mathrm{d}u_1)^2 - (\mathrm{d}u_2)^2)$$

respectively.

Comparing with (2.2.1) above, we obtain:

2.2.3a $$a_1 = \cos \frac{\varphi}{2},$$

2.2.3b $$a_2 = \sin \frac{\varphi}{2},$$

2.2.3c $$b_1 = k_1 = \tan \frac{\varphi}{2},$$

2.2.3d $$b_2 = k_2 = -\cot \frac{\varphi}{2},$$

where $\varphi(x, y)$ is the angle between the asymptotic directions and k_1, k_2 represent the principal curvatures.

Note that (2.2.3 a-d) correspond to a choice of a_1, a_2, b_1, b_2 made without loss of generality ([Te], 2.7).

We also note that in asymptotic line parametrization, the principal vectors given by (2.1.5) are generally not orthogonal, so the context is different than the one of orthonormal frames (Section 1). However, in curvature line coordinates, the principal vectors e_1 and e_2 are orthogonal, and that enables us to use the moving frame context from Section 1.

Comparing formulas (2.2.2) to the formulas (1.1.27) and (1.1.29), we deduce:

2.2.4a $$\omega_1 = a_1 \mathrm{d}u_1 = \cos \frac{\varphi}{2} \mathrm{d}u_1,$$

2.2.4*b*
$$\omega_2 = a_2 \mathrm{d}u_2 = \sin\frac{\varphi}{2}\mathrm{d}u_2,$$

2.2.4*c*
$$h_{11} = k_1, \quad h_{12} = 0, \quad h_{22} = k_2.$$

Then (1.1.25) together with (2.2.4c) yield

2.2.5*a*
$$\omega_{13} = k_1 a_1 \mathrm{d}u_1,$$

2.2.5*b*
$$\omega_{23} = k_2 a_2 \mathrm{d}u_2.$$

We also aim at finding an expression for ω_{12}: from equations (2.2.4a) and (2.2.4b), we find:

2.2.6*a*
$$\mathrm{d}\omega_1 = \frac{\partial a_1}{\partial u_2}\mathrm{d}u_2 \wedge \mathrm{d}u_1 = -\frac{1}{a_2}\cdot\frac{\partial a_1}{\partial u_2}\mathrm{d}u_1 \wedge \omega_2,$$

2.2.6*b*
$$\mathrm{d}\omega_2 = \frac{\partial a_2}{\partial u_1}\mathrm{d}u_1 \wedge \mathrm{d}u_2 = \frac{1}{a_1}\cdot\frac{\partial a_2}{\partial u_1}\omega_1 \wedge \mathrm{d}u_2.$$

Comparing equation (2.2.6) to the first two structure equations, (1.1.24a) and (1.1.24b), we obtain

(2.2.7)
$$\omega_{12} = \frac{1}{a_1}\frac{\partial a_2}{\partial u_1}\mathrm{d}u_2 - \frac{1}{a_2}\frac{\partial a_1}{\partial u_2}\mathrm{d}u_1.$$

As a consequence of (2.2.5a,b) and (2.2.7), we deduce

2.2.8*a*
$$\omega_{12} \wedge \omega_{23} = -k_2\frac{\partial a_1}{\partial u_2}\mathrm{d}u_1 \wedge \mathrm{d}u_2$$

2.2.8*b*
$$\mathrm{d}\omega_{13} = \mathrm{d}(k_1\omega_1) = \left(-k_1\frac{\partial a_1}{\partial u_2} - a_1\frac{\partial k_1}{\partial u_2}\right)\mathrm{d}u_1 \wedge \mathrm{d}u_2.$$

Therefore, the first Codazzi equation, (1.1.24d), has the form

$$(k_2 - k_1)\frac{\partial a_1}{\partial u_2} = a_1\frac{\partial k_1}{\partial u_2},$$

which can be rewritten as

(2.2.9a)
$$\frac{1}{k_2 - k_1}\frac{\partial k_1}{\partial u_2} = \frac{\partial(\log a_1)}{\partial u_2}.$$

Similarly, the second Codazzi equation, (1.1.24e), becomes

(2.2.9b)
$$\frac{1}{k_1 - k_2}\frac{\partial k_2}{\partial u_1} = \frac{\partial(\log a_2)}{\partial u_1}.$$

Recall now that ψ is a Chebyshev net parametrization: $A = |\psi_x| = 1$ and $B = |\psi_y| = 1$. In general (see, e.g., [Bo2], p. 114), the Codazzi equation can be written as

(2.2.10) $$A_y = B_x = 0.$$

So the Codazzi equations become trivial for a Chebyshev net.

Let us focus now on the Gauss equation (1.1.24c):

$$\mathrm{d}\omega_{12} = -\omega_{13} \wedge \omega_{23}.$$

Substituting the expressions for a_1 and a_2 from (2.2.3) into (2.2.7), we obtain the following expression for the connection form ω_{12}:

(2.2.11) $$\omega_{12} = \frac{1}{2}\left(\frac{\partial\varphi}{\partial u_1}\mathrm{d}u_2 + \frac{\partial\varphi}{\partial u_2}\mathrm{d}u_1\right).$$

Therefore,

(2.2.12) $$\mathrm{d}\omega_{12} = \frac{1}{2}\left(\frac{\partial^2\varphi}{\partial(u_1)^2} - \frac{\partial^2\varphi}{\partial(u_2)^2}\right)\mathrm{d}u_1 \wedge \mathrm{d}u_2.$$

Further, substituting the expressions (2.2.3) for a_1, a_2, k_1, k_2 into (2.2.5), the Gauss equation (1.1.24c) can be written in curvature coordinates as

(2.2.13) $$\frac{\partial^2\varphi}{\partial(u_1)^2} - \frac{\partial^2\varphi}{\partial(u_2)^2} = \sin\varphi.$$

Via $u_1 = x + y, u_2 = x - y$, (2.2.13) becomes, in asymptotic line parametrization,

(2.2.14) $$\varphi_{xy} = \sin\varphi,$$

Note that (2.2.13) and (2.2.14) are two different forms of the sine-Gordon equation.

Conversely, by the existence and uniqueness theorem of surface theory, given φ, a solution to (2.2.14), there exists an immersion $M = (D, \psi)$, in asymptotic line coordinates, whose angle between asymptotic directions is φ.

Summarizing the discussion above, we can state now the following result, due to Enneper (1845):

Theorem 2.2.1. ([Ch], p. 441, and [Bo2], p. 115). *Up to rigid motion, there is a one-to-one correspondence between solutions φ to the sine-Gordon equation (2.2.14) with $0 < \varphi < \pi$ and the weakly regular pseudospherical surfaces in Chebyshev net parametrization immersed in E^3.*

Note. This one-to-one correspondence between solutions φ to the sine-Gordon equation (2.2.14) and pseudospherical surfaces, whose first and second fundamental forms are given by (2.2.2), is the particular case $K < \bar{K} = 0$ of the following general theorem ([Te], Cor.2.7):

Theorem 2.2.1. *Let $M^2(K)$ be a surface with constant Gaussian curvature K, contained in a Riemannian 3-dimensional space form $\bar{M}^3(\bar{K})$ with constant curvature $\bar{K}$ such that $K \neq \bar{K}$. If $K > \bar{K}$, assume that M has no umbilic points. Then there exist local coordinates x_1, x_2 and a real-valued function $\psi(x_1, x_2)$ which satisfies the differential equation*

$*$ $$\psi_{x_1x_1} - \psi_{x_2x_2} = -K\sin\psi \qquad \textit{if} - K < \bar{K},$$

** $$\psi_{x_1x_1} + \psi_{x_2x_2} = -K \sinh\psi \qquad \textit{if } K > \bar{K}.$$

Conversely, suppose ψ is a solution of $(*)$ (resp. $(**)$). Then there exists a surface of constant Gaussian curvature K in a space form $\bar{M}^3(\bar{K})$, which is unique up to rigid motion of $\bar{M}^3$, whose first and second fundamental forms are given respectively by

$$\mathrm{I} = \begin{cases} \cos^2\frac{\psi}{2}\mathrm{d}x_1^2 + \sin^2\frac{\psi}{2}\mathrm{d}x_2^2 \text{if } \ K < \bar{K}, \\ \cosh^2\frac{\psi}{2}\mathrm{d}x_1^2 + \sinh^2\frac{\psi}{2}\mathrm{d}x_2^2 \text{if } \ K > \bar{K}, \end{cases}$$

$$\mathrm{II} = \begin{cases} \sqrt{|K-\bar{K}|}\sin\frac{\psi}{2}\cos\frac{\psi}{2}(\mathrm{d}x_1^2 - \mathrm{d}x_2^2)\text{if } \ K < \bar{K}, \\ \sqrt{|K-\bar{K}|}\sinh\frac{\psi}{2}\cosh\frac{\psi}{2}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)\text{if } \ K > \bar{K}. \end{cases}$$

2.3 Moving Frame of a Pseudospherical Surface. The Lax System

Let D be a simply connected domain in $\mathbf{R}^2$ and $\psi : D \to \mathbf{R}^3$ an immersion corresponding to a pseudospherical surface $M = (D, \psi)$. Let k_1, k_2 be the principal curvatures, given by formulas (2.2.3c,d) and e_1, e_2 corresponding principal directions on M. Let $F = \{x, e_1, e_2, e_3\} \in \mathcal{F}_1^{\mathcal{M}}$ be a fixed moving frame. Clearly, e_3 represents a chosen normal direction N, along M. Let us focus now on the Frenet equations of the frame. We shall omit the component $x \in E^3$ and will identify $F = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$ for the rest of this section. By (1.1.11), we have the following Frenet system on M:

(2.3.1) $$\mathrm{d}F = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} F.$$

The 1-forms ω_{12}, ω_{13} and ω_{23}, as a consequence of formulas (2.2.4–5) and (2.2.11), can be written as

2.3.2*a* $$\omega_{12} = \frac{1}{2}(\varphi_{u_1}\mathrm{d}u_2 + \varphi_{u_2}\mathrm{d}u_1) = \frac{1}{2}(\varphi_x \mathrm{d}x - \varphi_y \mathrm{d}y)$$

2.3.2*b* $$\omega_{13} = k_1\omega_1 = \sin\frac{\varphi}{2}\cdot \mathrm{d}u_1 = \sin\frac{\varphi}{2}\cdot(\mathrm{d}x + \mathrm{d}y)$$

2.3.2*c* $$\omega_{23} = k_2\omega_2 = -\cos\frac{\varphi}{2}\cdot \mathrm{d}u_2 = -\cos\frac{\varphi}{2}\cdot(\mathrm{d}x - \mathrm{d}y)$$

Let us now consider the moving frame $\tilde{F}_\theta \in \mathcal{F}_1^{\mathcal{M}}$, that is obtained from F via a rotation of angle $\theta(x, y)$ in the tangent plane, around N, namely

(2.3.3) $$\tilde{F}_\theta = \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ N \end{pmatrix},$$

where

$$\begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

In particular for $\theta = \varphi/2$, where $\varphi(x, y)$ is the angle between the asymptotic directions, the resulting frame is denoted $\tilde{F}$ and is called the *normalized frame* associated with the moving frame F (see [Wu1], p.18). Unless stated otherwise, we will denote by F the usual coordinate frame, and by $\tilde{F}$ the rotated frame as stated above. A simple calculation leads us to the system of Frenet equations for $\tilde{F}$:

$$\text{(2.3.4)} \qquad \mathrm{d}\tilde{F} = \begin{pmatrix} 0 & \tilde{\omega}_{12} & \tilde{\omega}_{13} \\ -\tilde{\omega}_{12} & 0 & \tilde{\omega}_{23} \\ -\tilde{\omega}_{13} & -tilde\omega_{23} & 0 \end{pmatrix} \tilde{F},$$

where

$$\text{(2.3.5}a\text{)} \qquad \tilde{\omega}_{12} = \mathrm{d}\theta + \omega_{12},$$

$$\text{(2.3.5}b\text{)} \qquad \tilde{\omega}_{13} = k_1 \cos\theta\, \omega_1 + k_2 \sin\theta\, \omega_2,$$

$$\text{(2.3.5}c\text{)} \qquad \tilde{\omega}_{23} = -k_1 \sin\theta\, \omega_1 + k_2 \cos\theta\, \omega_2.$$

In particular for the normalized frame $\tilde{F}$, $\theta = \varphi/2$ implies:

$$\text{(2.3.6}a\text{)} \qquad \mathrm{d}\theta = \frac{1}{2}(\varphi_x \mathrm{d}x + \varphi_y \mathrm{d}y)$$

$$\text{(2.3.6}b\text{)} \qquad \omega_{12} = \frac{1}{2}(\varphi_x \mathrm{d}x - \varphi_y \mathrm{d}y)$$

$$\text{(2.3.6}c\text{)} \qquad \omega_1 = \cos\frac{\varphi}{2}(\mathrm{d}x + \mathrm{d}y)$$

$$\text{(2.3.6}d\text{)} \qquad \omega_2 = \sin\frac{\varphi}{2}(\mathrm{d}x - \mathrm{d}y)$$

$$\text{(2.3.6}e\text{)} \qquad \tilde{\omega}_{12} = \varphi_x \mathrm{d}x$$

$$\text{(2.3.6}f\text{)} \qquad \tilde{\omega}_{13} = \sin\frac{\varphi}{2}\omega_1 - \cos\frac{\varphi}{2}\omega_2 = \frac{\sin\varphi}{2}(\mathrm{d}u_1 - \mathrm{d}u_2) = \sin\varphi \cdot \mathrm{d}y$$

$$\text{(2.3.6}g\text{)} \qquad \tilde{\omega}_{23} = -\sin^2\frac{\varphi}{2}\mathrm{d}u_1 - \cos^2\frac{\varphi}{2}\mathrm{d}u_2 = -\mathrm{d}x + \cos\varphi \cdot \mathrm{d}y.$$

As a consequence of (2.3.6), the Frenet system (2.3.4) is equivalent to the following differential system (also called *Lax system*):

(2.3.7)
$$\begin{aligned}\partial_x \tilde{F} &= \begin{pmatrix} 0 & \varphi_x & 0 \\ -\varphi_x & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tilde{F} = \tilde{\mathcal{A}}\tilde{F},\\ \partial_y \tilde{F} &= \begin{pmatrix} 0 & 0 & \sin\varphi \\ 0 & 0 & \cos\varphi \\ -\sin\varphi & -\cos\varphi & 0 \end{pmatrix} \tilde{F} = \tilde{\mathcal{B}}\tilde{F}.\end{aligned}$$

Note that $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are skew-symmetric matrices.

The compatibility condition for the system (2.3.7) (i.e. $\tilde{F}_{xy} = \tilde{F}_{yx}$) is

(2.3.8)
$$\tilde{\mathcal{A}}_y - \tilde{\mathcal{B}}_x - [\tilde{\mathcal{A}}, \tilde{\mathcal{B}}] = 0.$$

This is equivalent to the Gauss equation, which for pseudospherical surfaces in a Chebyshev parametrization is the sine-Gordon equation (2.2.14). If, for a pseudospherical surface, we use any asymptotic line parametrization ψ, but not necessarily a Chebyshev net, the Gauss equation takes the more general form ([Bo2], p. 114):

(2.3.9)
$$\varphi_{xy} = AB\sin\varphi,$$

where $A = |\psi_x|$, $B = |\psi_y|$.

It is interesting to remark that this equation remains invariant with respect to the transformation

(2.3.10)
$$A \mapsto \lambda A, \quad B \mapsto B/\lambda, \qquad \lambda \in \mathbf{R}_+,$$

which plays an essential role in the theory of pseudospherical surfaces. The transformation (2.3.10) appears in literature as Lie's transformation or Lorentz transformation in plane. To reconcile the two names, it is sometimes called Lie-Lorentz transformation.

The following obvious result is due to Lie (around the year 1870) and is of crucial importance in our context ([Bo2], p. 114):

Theorem 2.3.1. *Every surface with constant negative Gauss curvature has a one-parameter family of deformations preserving the second fundamental form*

(2.3.11)
$$\mathrm{II} = 2AB\sqrt{-K}\sin\varphi dx\, dy,$$

the Gaussian curvature K and the angle φ between the asymptotic lines. The deformation is generated by the transformation (2.3.10) *above.*

The family of immersions mentioned above is called *associated family of surfaces*. It will be denoted as $\psi^\lambda : D \to \mathbf{R}^3$. Note that all the immersions are defined on the same domain D.

Remark 2.3.1. The Lie-Lorentz transformation (2.3.10) can be naturally induced by replacing x with $\lambda^{-1}x$ and y by λy, $\lambda > 0$, and then

(2.3.12)
$$\begin{aligned}\partial_x &= \lambda\left(\cos\tfrac{\varphi}{2}\cdot e_1 - \sin\tfrac{\varphi}{2}\cdot e_2\right),\\ \partial_y &= \tfrac{1}{\lambda}\left(\cos\tfrac{\varphi}{2}\cdot e_1 + \sin\tfrac{\varphi}{2}\cdot e_2\right).\end{aligned}$$

We note here that the Lie-Lorentz transformation defined above on $M = (D, \psi)$ is equivalent to a Lorentz transformation on a Lorentzian 2-manifold, (D, II).

Also note that if $\varphi(x, y)$ denotes the angle of a certain pseudospherical surface M in Chebyshev net coordinates x, y, then by Lie-Lorentz transformation we create a new pseudospherical surface M^*, in the same associated family with the first one. The coordinates $x^* = \lambda^{-1}x$ and $y^* = \lambda y$ are also asymptotic, and the angle between asymptotic lines on the new surface is given by the same function as before, but this time in variables x^* and y^*. Thought of as a function of the *old* coordinates x, y, the angle $\varphi(x^*, y^*)$, corresponding to the new surface M^*, depends on λ. See also the examples in Section 8, (8.1.3) and (8.2.1).

As a consequence of the coordinate change described above via the parameter λ, starting from a Chebyshev parametrization ψ, we see that $|\psi_x| = 1$ becomes $|\psi_x| = \lambda$, while $|\psi_y| = 1$ becomes $|\psi_y| = \lambda^{-1}$. While via this transformation the sine-Gordon equation remains unmodified, the corresponding differential Lax system (2.3.7) depends on λ. In particular, we obtain an *extended frame* $F = F(x, y, \lambda) = F(\lambda^{-1}x, \lambda y)$. For the normalized frame $\tilde{F}$, we obtain the *extended normalized frame* $\tilde{F}(x, y, \lambda)$.

Corolarry 2.3.1. *The extended normalized frame* $\tilde{F}(x, y, \lambda)$ *satisfies the following Lax differential system:*

$$\partial_x \tilde{F} = \begin{pmatrix} 0 & \varphi_x & 0 \\ -\varphi_x & 0 & -\lambda \\ 0 & \lambda & 0 \end{pmatrix} \tilde{F},$$

2.3.13
$$\partial_y \tilde{F} = \frac{1}{\lambda} \begin{pmatrix} 0 & 0 & \sin\varphi \\ 0 & 0 & \cos\varphi \\ -\sin\varphi & -\cos\varphi & 0 \end{pmatrix} \tilde{F}.$$

This type of linear system is essential for the inverse scattering method in soliton theory. Equation (2.3.13) represents the scattering system of the sine-Gordon equation introduced by Lund (see [Lu]).

Remark 2.3.2. The frame F represents the 3×3 matrix $\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$ of rows e_1, e_2, and e_3, respectively. In the spirit of [Wu2] and [DoHa], instead of the classical frame F, it is more convenient to work with $\mathcal{U} := \tilde{F}^T$, the transposed of the extended normalized frame $\tilde{F}(x, y, \lambda)$. This is especially convenient in view of formulas (2.3.15) below. Unless stated otherwise, the term of normalized coordinate frame will refer to $\mathcal{U}$ above, for the rest of this text.

Consequently, formulas (2.3.7) can be rewritten as

(2.3.14)
$$\partial_x \mathcal{U} = \mathcal{U} \cdot \mathcal{A}^T, \partial_y \mathcal{U} = \mathcal{U} \cdot \mathcal{B}^T,$$

where we denoted by $\mathcal{A}$ and $\mathcal{B}$, respectively, the transpose of $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ from (2.3.7).

That is, equations (2.3.13) above can be rewritten as:

Corollary 2.3.2. *The extended normalized frame* $\mathcal{U}^\lambda$ *satisfies the following Lax differential system*

$$\partial_x \mathcal{U}^\lambda = \mathcal{U}^\lambda \cdot \begin{pmatrix} 0 & -\varphi_x & 0 \\ \varphi_x & 0 & \lambda \\ 0 & -\lambda & 0 \end{pmatrix},$$

2.3.15
$$\partial_y \mathcal{U}^\lambda = \mathcal{U}^\lambda \frac{\infty}{\lambda} \begin{pmatrix} 0 & 0 & -\sin\varphi \\ 0 & 0 & -\cos\varphi \\ \sin\varphi & \cos\varphi & 0 \end{pmatrix}.$$

The Lax system will be written in this form for the rest of this work. It plays a crucial role in the study of pseudospherical surfaces.

3 Associated Families of Pseudospherical Surfaces via Spectral Parameter λ

In this section we study in detail the effects of introducing the real positive parameter λ. We obtain in this way a λ-transformation of the Cartan forms (respectively an extended Maurer-Cartan form ω^λ) corresponding to the associated family of pseudospherical surfaces (respectively the extended normalized frame $\mathcal{U}^\lambda$).

3.1 The λ-Transformation on the 1-Forms ω_i and ω_{ij}

Let us study the effect that the transformation (2.3.10) has on the 1-forms ω_1, ω_2, ω_{12}, ω_{13}, ω_{23}. Replacing x by $x^* := \lambda^{-1}x$ and y by $y^* := \lambda y$ in the system (2.3.2), and taking into account the invariance of φ under this deformation (Thm. 2.3.1), we obtain the "extended" forms:

3.1.1*a* $$\omega_1^\lambda = \cos\frac{\varphi}{2}(\mathrm{d}x^* + \mathrm{d}y^*) = \cos\frac{\varphi}{2}(\lambda^{-1}\mathrm{d}x + \lambda\mathrm{d}y)$$

3.1.1*b* $$\omega_2^\lambda = \sin\frac{\varphi}{2}(\mathrm{d}x^* - \mathrm{d}y^*) = \sin\frac{\varphi}{2}(\lambda^{-1}\mathrm{d}x - \lambda\mathrm{d}y)$$

3.1.1*c* $$\omega_{12}^\lambda = \frac{1}{2}(\varphi_{x^*}\mathrm{d}x^* - \varphi_{y^*}\mathrm{d}y^*) = \frac{1}{2}(\varphi_x\mathrm{d}x - \varphi_y\mathrm{d}y)$$

3.1.1*d* $$\omega_{13}^\lambda = \sin\frac{\varphi}{2}(\mathrm{d}x^* + \mathrm{d}y^*) = \sin\frac{\varphi}{2}(\lambda^{-1}\mathrm{d}x + \lambda\mathrm{d}y)$$

3.1.1*e* $$\omega_{23}^\lambda = -\cos\frac{\varphi}{2}(\mathrm{d}x^* - \mathrm{d}y^*) = -\cos\frac{\varphi}{2}(\lambda^{-1}\mathrm{d}x - \lambda\mathrm{d}y).$$

The system above can be rewritten as

3.1.2*a* $$\omega_1^\lambda = \frac{1}{2}(\lambda + \lambda^{-1})\omega_1 + \frac{1}{2}(\lambda - \lambda^{-1})\omega_{23},$$

3.1.2*b* $$\omega_2^\lambda = \frac{1}{2}(\lambda + \lambda^{-1})\omega_2 - \frac{1}{2}(\lambda - \lambda^{-1})\omega_{13},$$

3.1.2*c* $$\omega_{12}^\lambda = \omega_{12},$$

3.1.2*d* $$\omega_{13}^\lambda = -\frac{1}{2}(\lambda - \lambda^{-1})\omega_2 + \frac{1}{2}(\lambda + \lambda^{-1})\omega_{13},$$

3.1.2e $$\omega_{23}^{\lambda} = \frac{1}{2}(\lambda - \lambda^{-1})\omega_1 + \frac{1}{2}(\lambda + \lambda^{-1})\omega_{23},$$

where $\lambda > 0$. Note that λ occurs rationally, with simple poles at $\lambda = 0$ and at infinity. This will be essential below.

Cartan's structure equations for $\mathcal{F}_1^{\mathcal{M}}$, where ω_3 is identically zero, given by (1.1.24 a-f), together with equation (1.1.31) for $K = -1$, form the set of equations below, called *conditions* (K):

3.1.3.a $$\mathrm{d}\omega_1 = \omega_{12} \wedge \omega_2,$$

3.1.3.b $$\mathrm{d}\omega_2 = \omega_1 \wedge \omega_{12},$$

3.1.3.c $$\mathrm{d}\omega_{12} = -\omega_{13} \wedge \omega_{23},$$

3.1.3.d $$\mathrm{d}\omega_{13} = \omega_{12} \wedge \omega_{23},$$

3.1.3.e $$\mathrm{d}\omega_{23} = \omega_{13} \wedge \omega_{12},$$

3.1.3.f $$\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} = 0,$$

3.1.3.g $$\omega_1 \wedge \omega_2 + \omega_{13} \wedge \omega_{23} = 0.$$

Let $\omega_1, \omega_2, \omega_{12}, \omega_{13}, \omega_{23}$ be differential forms defined by (1.1.10) and let ω_1^{λ}, $\omega_2^{\lambda}, \omega_{12}^{\lambda}, \omega_{13}^{\lambda}, \omega_{23}^{\lambda}$ be given by (3.1.2). Then

Theorem 3.1.1. *The forms $\omega_1, \omega_2, \omega_{12}, \omega_{13}, \omega_{23}$ satisfy the conditions (K) if and only if ω_1^{λ}, $\omega_2^{\lambda}, \omega_{12}^{\lambda}, \omega_{13}^{\lambda}, \omega_{23}^{\lambda}$ satisfy the conditions (K).*

For every pseudospherical surface $M = (D, \psi)$, there exists a family $M_\lambda = (D, \psi_\lambda)$, $\lambda > 0$, of pseudospherical surfaces associated with $\omega_1^{\lambda}, \omega_2^{\lambda}, \omega_{12}^{\lambda}$, $\omega_{13}^{\lambda}, \omega_{23}^{\lambda}$ preserving the angle φ between the asymptotic lines and also preserving the second fundamental form.

Proof. By Theorem 2.3.1, we know that the λ-transformation (2.3.10) preserves the angle φ and the second fundamental form. This means that the forms ω_i, ω_{ij}, and ω_i^{λ}, ω_{ij}^{λ}, $\lambda > 0$, respectively, satisfy the same Gauss equation (3.1.3.c). The Gauss equation is equivalent with $\varphi_{xy} = \sin\varphi$, and so the angle φ is preserved for the family M_λ. We remark that the Codazzi equation is trivially satisfied for M_λ, since for the whole associated family ψ^λ, $A = |\psi_x^\lambda| = \lambda$, $B = |\psi_y^\lambda| = 1/\lambda$, $\lambda > 0$, and the Codazzi equations are $A_y = B_x = 0$.

In order to finish the proof of the theorem, it is enough to show that if the Gauss and Codazzi equations are satisfied for every real positive λ, then the rest of conditions (K) are also satisfied for every real positive λ. This is stated in the following:

Lemma 3.1.1. *If ω_i^{λ} and ω_{ij}^{λ} are given by the equations* (3.1.2)*, and if the following conditions are satisfied for all $\lambda > 0$:*

3.1.4.i $$d\omega_{12}^{\lambda} = -\omega_{13}^{\lambda} \wedge \omega_{23}^{\lambda},$$

3.1.4.*ii* $$d\omega_{13}^{\lambda} = \omega_{12}^{\lambda} \wedge \omega_{23}^{\lambda},$$

3.1.4.*iii* $$d\omega_{23}^{\lambda} = \omega_{13}^{\lambda} \wedge \omega_{12}^{\lambda},$$

then all the conditions (K) are satisfied for ω_i^{λ}, ω_{ij}^{λ}.

Proof. Assume that (3.1.4.i-iii) are satisfied. Then, by (3.1.2), after a few simplifications, we obtain

3.1.5.*i* $$\mathrm{d}\omega_{12}^{\lambda} + \omega_{13}^{\lambda} \wedge \omega_{23}^{\lambda} = \frac{\lambda^2 - \lambda^{-2}}{4}(-\omega_1 \wedge \omega_{13} - \omega_2 \wedge \omega_{23}) + \\ + \frac{\lambda^2 + \lambda^{-2}}{4}(\omega_1 \wedge \omega_2 + \omega_{13} \wedge \omega_{23}) + \frac{1}{2}(-\omega_1 \wedge \omega_2 - \omega_{13} \wedge \omega_{23}) = 0,$$

3.1.5.*ii* $$\mathrm{d}\omega_{13}^{\lambda} - \omega_{12}^{\lambda} \wedge \omega_{23}^{\lambda} = \frac{\lambda - \lambda^{-1}}{2}(\mathrm{d}\omega_2 - \omega_1 \wedge \omega_{12}) + \frac{\lambda + \lambda^{-1}}{2}(\mathrm{d}\omega_{13} - \omega_{12} \wedge \omega_{23}) = 0,$$

3.1.5.*iii* $$\mathrm{d}\omega_{23}^{\lambda} - \omega_{13}^{\lambda} \wedge \omega_{12}^{\lambda} = -\frac{\lambda - \lambda^{-1}}{2}(\mathrm{d}\omega_1 - \omega_{12} \wedge \omega_2) + \frac{\lambda + \lambda^{-1}}{2}(\mathrm{d}\omega_{23} - \omega_{13} \wedge \omega_{12}) = 0.$$

Comparing the coefficients of the corresponding λ^2 and λ^{-2} powers, we obtain

$$\begin{cases} \omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} = 0 \\ \omega_1 \wedge \omega_2 + \omega_{13} \wedge \omega_{23} = 0, \end{cases}$$

that is equations (3.1.3.f) and (3.1.3.g).

Equations (3.1.3.c-e) represent a particularization for $\lambda = 1$ of equations (3.1.4.i-iii). The coefficients of λ and equations (3.1.5.ii,iii) determine the expressions of $\mathrm{d}\omega_1$ and $\mathrm{d}\omega_2$, that is the remaining conditions (K). □

This also completes the proof of the Theorem 3.1.1.

Remark 3.1.1. As frequently observed in soliton theory, the introduction of a parameter reduces the number of defining equations.

Theorem 3.1.1, that we just proved, is of central importance for the present study. In section 2.3, we analyzed the pseudospherical surfaces in detail and described a λ-transformation, $\lambda > 0$, that preserves the second fundamental form, the Gaussian curvature and the angle between asymptotic lines. We also presented the extended normalized frame $\mathcal{U}^{\lambda}$ (2.3.15) associated with this transformation. In this section we studied in more detail the effects of introducing the real positive parameter λ by the Lie-Lorentz transformation. We obtained a λ-family of 1-forms ω_i^{λ}, ω_{ij}^{λ}, $i < j$, which characterizes the above-mentioned λ-family $M = (D, \psi^{\lambda})$ of associated surfaces via the λ-transformation.

3.2 The Extended Maurer-Cartan Form ω^{λ} of an Associated Family of Pseudospherical Surfaces and the Extended Normalized Frame $\mathcal{U}^{\lambda}$

In section 1.1, we identified the set $\mathcal{F}$ of all frames with G, the group of orientation-preserving rigid motions, via a map $g^f : \mathcal{F} \to G$, $g^f(x, e_1, e_2, e_3) = (x, A)$, with $x \in \mathbf{R}^3$, $A \in \mathrm{SO}(3)$, such that $e_i = A\check{e}_i$, where $F_0 = \{0, \check{e}_1, \check{e}_2, \check{e}_3\}$ was a fixed frame.

E_i, E_{ij} are, by definition, the *six vector fields dual* to the 1-forms ω_i, ω_{ij}, $i,j = 1,2,3$, $i<j$, i.e. the vector fields satisfying

$$E_j(x^f)(F) = \mathrm{d}_F x^f(E_j) = \sum_i \omega_i(E_j) e_i^f(F) = e_j^f(F)$$

respectively

$$\begin{aligned} E_{ij}(e_m^f)(F) &= \mathrm{d}_F e_m^f(E_{ij}) = \left(\sum_n \omega_{mn}(E_{ij}) e_n^f(F)\right) = \left(\delta_i^m \delta_j^n e_n^f - \delta_i^n \delta_j^m e_n^f\right) = \\ &= \left(\delta_i^m e_j^f - \delta_j^m e_i^f\right) = S_{ij}(e_m^f). \end{aligned}$$

Here S_{ij} represents the 3×3 matrix with (i,j)-entry equal 1, (j,i)-entry equal to -1 and zero elsewhere, $i<j$. According to the way E_{ij} acts on the frame F, it can be identified with the matrix S_{ij}.

We note that the vector fields E_i, E_{ij}, $i,j=1,2,3$, $i<j$ are invariant with respect to the particular choice of the fixed frame F_0.

Remark 3.2.1. Reviewing, we obtained above the formulas

$$E_j(x^f)(F) = \mathrm{d}_F x^f(E_j) = e_j^f(F), \tag{3.2.1}$$

$$(E_{ij}(e_m^f))_{m=1,2,3} = \left(\mathrm{d}_F e_m^f(E_{ij})\right)_{m=1,2,3} = S_{ij} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{3.2.2}$$

These equations are satisfied for every frame $F = \{x, e_1, e_2, e_3\}$.

Definition 3.2.1. Consider the $so(3)$-valued 1-form ω given by

$$\omega = \tilde{\omega}_{12} E_{12} + \tilde{\omega}_{13} E_{13} + \tilde{\omega}_{23} E_{23}, \tag{3.2.3}$$

where $\tilde{\omega}_{12}$, $\tilde{\omega}_{13}$ and $\tilde{\omega}_{23}$ are given by formulas (2.3.6 e,f,g). We will call ω the *Maurer-Cartan form* of the group of Euclidean motions.

As a linear combination of matrices E_{12}, E_{13}, E_{23}, the form ω becomes an $so(3)$-valued 1-form on G. For a vector field Y on G, we have

$$\omega(Y) = \sum_{i<j} \tilde{\omega}_{ij}(Y) E_{ij} \tag{3.2.4}$$

Remark 3.2.2. (a) Writing $\tilde{\omega}_{12}$, $\tilde{\omega}_{13}$ and $\tilde{\omega}_{23}$ explicitely as in (2.3.6), the Maurer-Cartan form of the group of Euclidean motions restricted to $\mathcal{F}_1^{\mathcal{M}}$ for a pseudospherical surface can be written as

$$\omega = -\mathcal{U}^{-1}\mathrm{d}\mathcal{U} = \begin{pmatrix} 0 & \varphi_x \mathrm{d}x & \sin\varphi\, \mathrm{d}y \\ -\varphi_x \mathrm{d}x & 0 & -\mathrm{d}x + \cos\varphi\, \mathrm{d}y \\ -\sin\varphi\, \mathrm{d}y & \mathrm{d}x - \cos\varphi\, \mathrm{d}y & 0 \end{pmatrix}. \tag{3.2.5}$$

(b) Let us recall briefly the results from the previous section. We have proved in Lemma 3.1.1 that if ω_i^λ, ω_{ij}^λ are given by formulas (3.1.2) and conditions (3.1.4iii–v) are satisfied, then all the conditions (K) are satisfied. We have also seen that

ω_i^λ, ω_{ij}^λ, $\lambda > 0$, given in (3.1.2) correspond to an associated family of surfaces that preserve the angle φ between asymptotic lines, the Gaussian curvature and the second fundamental form (Theorem 2.3.1) and that ω_i^λ and ω_{ij}^λ can be naturally induced by a transformation $x \mapsto \lambda^{-1}x$, $y \mapsto \lambda y$, $\lambda > 0$ of the asymptotic line parametrization.

Let us now recall from Corollary 2.3.2 that the *extended normalized moving frame* $\mathcal{U}^\lambda : D \to \mathrm{SO}(3)$ of this family of one-forms ω_i^λ, ω_{ij}^λ, $\lambda > 0$ satisfies the equations

$$\text{(3.2.6)} \qquad \begin{cases} (\mathcal{U}^\lambda)^{-1} \cdot \partial_x \mathcal{U}^\lambda = \begin{pmatrix} 0 & -\varphi_x & 0 \\ \varphi_x & 0 & \lambda \\ 0 & -\lambda & 0 \end{pmatrix} \\ (\mathcal{U}^\lambda)^{-1} \cdot \partial_y \mathcal{U}^\lambda = \dfrac{1}{\lambda} \begin{pmatrix} 0 & 0 & -\sin\varphi \\ 0 & 0 & -\cos\varphi \\ \sin\varphi & \cos\varphi & 0 \end{pmatrix} \end{cases}.$$

Comparing (3.2.5) to (3.2.6), we formulate

Definition 3.2.2. The so(3)-valued family of 1-forms

$$\omega^\lambda = -(\mathcal{U}^\lambda)^{-1} \mathrm{d}\mathcal{U}^\lambda$$

$$\text{(3.2.7)} \qquad = \begin{pmatrix} 0 & \varphi_x \,\mathrm{d}x & \lambda^{-1}\sin\varphi\,\mathrm{d}y \\ -\varphi_x\,\mathrm{d}x & 0 & -\lambda \mathrm{d}x + \lambda^{-1}\cos\varphi\,\mathrm{d}y \\ -\lambda^{-1}\sin\varphi\,\mathrm{d}y & \lambda\mathrm{d}x - \lambda^{-1}\cos\varphi\,\mathrm{d}y & 0 \end{pmatrix},$$

is called *extended Maurer-Cartan form.*

Proposition 3.2.1. *The system of equations* (3.1.4.i-iii) *is equivalent to*

$$\text{(3.2.8)} \qquad d\omega^\lambda + \omega^\lambda \wedge \omega^\lambda = 0,$$

for every $\lambda > 0$.

Proof. Assume the equations (3.1.4.i–iii) are satisfied. Then, by Lemma 3.1.1, the system of equations (3.1.4.i–iii) is equivalent to the conditions (K), defined in (3.1.3). On the other hand, (3.1.4.i–iii) are by definition the Gauss-Codazzi equations for a pseudospherical surface. On the other hand, (3.2.8) can be checked directly, and it reduces to the Gauss-Codazzi equations: e.g., the sine-Gordon equation is recovered immediately from the (1,2) entry of the matrix-valued form $\mathrm{d}\omega^\lambda + \omega^\lambda \wedge \omega^\lambda$. □

We will call formula (3.2.8) the *flatness condition*, or the *zero-curvature condition* for the extended Maurer-Cartan form ω^λ.

Remark 3.2.3. From equation (3.2.7), we see that the extended Maurer-Cartan form ω^λ can be written in the form

$$\text{(3.2.9)} \qquad \omega^\lambda := \lambda^{-1} \cdot \alpha_{-1} + \alpha_0 + \lambda \cdot \alpha_1,$$

where $\alpha_0 \in \underline{k} = \mathbf{R}E_{12}$ and $\alpha_{-1}, \alpha_1 \in \underline{p} = \mathbf{R}E_{13} + \mathbf{R}E_{23}$.

More precisely, we have

$$\text{(3.2.10)} \qquad \alpha_0 = \varphi_x E_{12} \mathrm{d}x,$$

while

$$\text{(3.2.11)} \qquad \alpha_{-1} = (\sin\varphi \cdot E_{13} + \cos\varphi \cdot E_{23})\mathrm{d}y,$$

and

$$\text{(3.2.12)} \qquad \alpha_1 = -E_{23}\mathrm{d}x.$$

4 Loop Algebras and Groups Corresponding to Pseudospherical Surfaces

We now examine the system (3.1.4) in the context of the loop algebra $\mathrm{so}(3,\mathbf{R}) \otimes \mathbf{R}[\lambda^{-1},\lambda]$. This will lead to interpreting the extended moving frame equations in terms of loop groups, which opens some completely new possibilities. E.g., the extended frame $\mathcal{U}^\lambda$ can be decomposed in the form $\mathcal{U} = \mathcal{U}_+ \cdot V_- = \mathcal{U}_- \cdot V_+$. Here $\mathcal{U}_-$ is an element of the form $\mathcal{U}_- = I + \lambda^{-1}\mathcal{U}_{-1} + \lambda^{-2}\mathcal{U}_{-2} + \cdots$, while V_+ is an element of the form $V_+ = V_0 + \lambda V_1 + \lambda^2 V_2 + \cdots$, respectively. Eventually, this will allow us to find unconstrained data, "potentials" from which all pseudospherical surfaces can be constructed.

4.1 Loop Algebras and Structure Equations. Introduction

Let a be a Lie algebra over $\mathbf{R}$ with a finite basis $X_1, X_2, \ldots, X_m$; i.e. every $X \in a$ is expressed uniquely as a linear combination

$$X = a_1X_1 + a_2X_2 + \cdots + a_mX_m, \tag{4.1.1}$$

where $a_j \in \mathbf{R}$.

The structure of the Lie algebra a is given by *Lie's equations*

$$[X_i, X_j] = C^k_{ij} X_k, \tag{4.1.2}$$

where for convenience we used the Einstein summation convention for the index k, which will be used from now on.

An immediate consequence of the skew-symmetry of the Poisson bracket is the skew-symmetry of the structural constants C^k_{ij} with respect to the indices i, j. Also, as a consequence of the Jacobi identity, the structural constants satisfy the following identity:

$$C^k_{sj}C^s_{ir} + C^k_{si}C^s_{rj} + C^k_{sr}C^s_{ji} = 0.$$

This identity appears in literature as Lie's quadratic identity.

Let a^* be the dual space of a. By definition, the dual basis of a^* is $\{\eta^1, \eta^2, \ldots, \eta^m\}$ such that $\eta^i(X_j) = \delta^i_j$. Also, for every $\eta \in a^*$, there is a unique linear combination

$$\eta = \beta_1\eta^1 + \beta_2\eta^2 + \cdots + \beta_m\eta^m. \tag{4.1.3}$$

Let $\Lambda^p a^*$ denote all the p-forms on a. Clearly,

$$\Lambda^1 a^* = a^*. \tag{4.1.4}$$

Definition 4.1.1. The exterior differential $\mathrm{d}\eta \in \Lambda^2 a^*$ of a 1-form $\eta \in a^*$ is defined by the equation

$$\mathrm{d}\eta(X,Y) = -\eta([X,Y]), \tag{4.1.5}$$

where $X, Y \in a$.

Equation (4.1.5) is equivalent to *Cartan's structure equations*:

$$\mathrm{d}\eta^k + \frac{1}{2} C^k_{ij} \eta^i \wedge \eta^j = 0. \tag{4.1.6a}$$

This equivalence is straightforward and is presented in classical texts (e.g., [Ca], p.45). In (4.1.6a), $\eta^i \wedge \eta^j$ represents the exterior product of the 1-forms η^i and η^j.

It is easy to see that (4.1.6a) can be rewritten as

$$\mathrm{d}\eta^k + C^k_{ij} \eta^i \wedge \eta^j = 0, \tag{4.1.6b}$$

where $i < j$.

Multiplying equation (4.1.6b) by X_k and taking into account Lie's equations (4.1.2), we obtain

$$X_k \cdot d\eta^k + [X_i, X_j] \eta^i \wedge \eta^j = 0, \qquad i < j,$$

which can be rewritten as

$$\mathrm{d}\eta + \frac{1}{2} [\eta \wedge \eta] = 0, \tag{4.1.7}$$

where

$$\eta = X_1 \eta^1 + X_2 \eta^2 + \cdots + X_m \eta^m. \tag{4.1.8}$$

Remark 4.1.1. If the basis $\{\eta^1, \eta^2, \ldots, \eta^m\}$ of a^* is divided into two groups distinguished by indices $i, j, k \in N_1$ and $\alpha, \beta, \gamma \in N_2$ respectively, then the structure equations become

$$\begin{cases} \mathrm{d}\eta^k + C^k_{ij}\eta^i \wedge \eta^j + C^k_{i\beta}\eta^i \wedge \eta^\beta + C^k_{\alpha\beta}\eta^\alpha \wedge \eta^\beta = 0, i < j, \alpha < \beta \\ \mathrm{d}\eta^\gamma + C^\gamma_{ij}\eta^i \wedge \eta^j + C^\gamma_{i\beta}\eta^i \wedge \eta^\beta + C^\gamma_{\alpha\beta}\eta^\alpha \wedge \eta^\beta = 0, i < j, \alpha < \beta. \end{cases} \tag{4.1.9}$$

Note that the restriction $\eta^\gamma = 0$, for every $\gamma \in N_2$, defines a linear subspace of a.

Example 4.1.1. Consider the group of Euclidean motions T given by the structure equations (1.1.17) and introduce the restrictions $\omega_{12} = \omega_{13} = \omega_{23} = 0$, which define the normal subgroup of all translations. The groups of indices specified in Remark 4.1.1 are $1, 2, 3 \in N_1$ and $12, 13, 23 \in N_2$ respectively, where we replaced η by ω.

Let us consider the quotient group $G/T = \mathrm{O}(3, \mathbf{R})$ of the Euclidean motion group modulo the group of translations. Thus, in the second group of equations of the system (4.1.9), the terms containing $\omega_j, j = 1, 2, 3$ disappear, and the equations become

$$\mathrm{d}\omega_{ij} = \omega_{ik} \wedge \omega_{kj},$$

with Einstein summation with respect to k and $i, j = 1, 2, 3, i < j$.

This gives a concrete illustration of the structure equations (1.1.24 c,d,e).

The form (4.1.8) for the Euclidean motion group is written here as

$$\hat{\omega} = \omega_1 E_1 + \omega_2 E_2 + \omega_3 E_3 + \omega_{12} E_{12} + \omega_{13} E_{13} + \omega_{23} E_{23}, \tag{4.1.10}$$

The form $\hat{\omega}$ is sometimes called the *total Maurer-Cartan form.*

4.2 The Loop Algebra Setting

Let now a represent a Lie algebra with basis $X_1, X_2, \dots, X_m$ satisfying $[X_i, X_j] = C_{ij}^k X_k$. This is equivalent to the structure equations (4.1.6).

Definition 4.2.1. The *polynomial loop algebra* $a = b \otimes \mathbf{R}[\lambda^{-1}, \lambda]$ is the Lie algebra with basis $X_{k,t} = X_k \lambda^t$, $k = 1, 2, \dots, m$, $t = 0, \pm 1, \pm 2, \dots$, where λ is a formal parameter.

This basis satisfies the Lie equations

$$[X_{i,r}, X_{j,s}] = C_{ij}^k X_{k,r+s}. \tag{4.2.1}$$

The notation $\mathbf{R}[\lambda^{-1}, \lambda]$ used above represents the ring of Laurent polynomials in the variable λ over the field $\mathbf{R}$. Let $\{\eta^{i,r}\}$ represent the basis of 1-forms dual to the basis $\{X_{i,r}\}$. Then, analogous to the derivation of (4.1.6b) we obtain, as a consequence of (4.2.1), the structure equations of the loop algebra a

$$\mathrm{d}\eta^{k,t} + \sum_{r+s=t, i<j} C_{ij}^k \eta^{i,r} \wedge \eta^{j,s} = 0. \tag{4.2.2}$$

Multiplying these equations by $\lambda^t = \lambda^{r+s}$, we obtain

$$\mathrm{d}\eta^{k,t}\lambda^t + \sum_{r+s=t, i<j} C_{ij}^k \eta^{i,r}\lambda^r \wedge \eta^{j,s}\lambda^s = 0. \tag{4.2.3}$$

That is, the structure equations of the form (4.1.6), where

$$\eta^k = \sum_{t=-\infty}^{\infty} \eta^{k,t}\lambda^t$$

represent infinite Laurent series in the variable λ with 1-forms $\eta^{k,t}$ as coefficients.

Let us now consider the particular case of $a = \mathrm{so}(3, \mathbf{R})$, so that $a = \mathrm{so}(3, \mathbf{R}) \otimes \mathbf{R}[\lambda^{-1}, \lambda]$. The main reason why we focus on this loop algebra is provided by the extended Maurer-Cartan form ω^λ of a pseudospherical surface, introduced in (3.2.7). Moreover, we shall introduce the twisted loop algebra

$$\Lambda\mathrm{so}(3)_P^{\mathrm{alg}} = \{X \in \mathrm{so}(3) \otimes br[\lambda, \lambda^{-1}];\ X(-\lambda) = PX(\lambda)P^{-1}\}, \tag{4.2.4}$$

where

$$P = \mathrm{diag}\{1, 1, -1\}.$$

Note that $P^{-1} = P$ and

$$PE_{12}P = E_{12}, PE_{13}P = -E_{13}, PE_{23}P = -E_{23}. \tag{4.2.5}$$

From (3.2.7), it is easy to see that $\omega^\lambda(-\lambda) = P \cdot \omega^\lambda(\lambda) \cdot P^{-1}$ holds. Hence, $\omega^\lambda \in \Lambda\mathrm{so}(3)_P^{\mathrm{alg}}$.

It will be convenient to use certain Banach completions of the Lie algebra (4.2.4). For this purpose, for a matrix $A \in \mathrm{so}(3, \mathbf{R})$ independent of λ, we introduce the norm

$$\|A\| = \max_i \{\sum_{j=1}^{3} |A_{ij}|\}, \tag{4.2.6}$$

where A_{ij} denotes the (i,j)-coefficient of A.
It can be checked by a direct computation that

$$\|AB\| \leq \|A\| \cdot \|B\|. \qquad \|I\| = 1.$$

Further, if $X(\lambda) = \sum_{k\in\mathbf{Z}} X_k \cdot \lambda^k$, we define its norm as follows:

$$\|X(\lambda)\| = \sum_{k\in\mathbf{Z}} \|X_k\| < \infty. \tag{4.2.7}$$

Remark 4.2.1. The norm defined by (4.2.7) can be also introduced as follows:
We start by defining the norm of a real-valued function in λ,

$$\|h\| := \sum_{k\in\mathbf{Z}} |h_k| < \infty, \qquad h(\lambda) = \sum_{k\in\mathbf{Z}} h_k\lambda^k.$$

Then we define the norm of the matrix-valued function $X(\lambda)$ as

$$\|X\| = \max_i\{\sum_{j=1}^{3}\|X_{ij}(\lambda)\|\}.$$

It is easy to see that we obtain this way the same norm as in (4.2.7).

Note that in (4.2.6) and (4.2.7), by abuse of notation, we use the same symbol $\|\cdot\|$ for the following three different items: norm of a function, norm of a λ-independent matrix and norm of $X(\lambda)$. It will always be clear from the context which norm we mean.

We set

$$\Lambda\mathrm{so}(3)_P := \text{completion of } \Lambda\mathrm{so}(3)_P^{\mathrm{alg}} \text{ relative to } \|\cdot\|. \tag{4.2.8}$$

Proposition 4.2.1. $\Lambda\mathrm{so}(3)_P$ *is a Banach Lie algebra.*
Proof. We can define the norm (4.2.7) for arbitrary matrices in $\mathrm{gl}(3) \otimes \mathbf{R}[\lambda, \lambda^{-1}]$.

The fixed point algebra of the automorphism $X(\lambda) \mapsto P{\cdot}X(-\lambda){\cdot}P^{-1}$ of $\Lambda\mathrm{GL}(3, \mathbf{R})$ is an associative Banach subalgebra. Inside the connected component of the Banach Lie group of invertible elements of this fixed point algebra, we consider the connected component of the group

$$\Lambda\mathrm{SO}(3)_P = \{g \in \Lambda\mathrm{SO}(3, \mathbf{R});\ Pg(\lambda)P^{-1} = g(-\lambda)\}. \tag{4.2.9}$$

From [Ha,Ka], it follows that $\Lambda\mathrm{SO}(3)_P$ is a Banach Lie group with Lie algebra

$$\mathrm{Lie}\,\Lambda\mathrm{SO}(3)_P = \Lambda\mathrm{so}(3)_P. \tag{4.2.10}$$

Remark 4.2.2. If $M = (D, \psi)$ is, as usual, a pseudospherical surface given by the Chebyshev immersion $\psi : D \to \mathbf{R}^3$, where D is a simply connected domain, then there exists a normal $N : D \to S^2$ along ψ and a frame $\mathcal{U} : D \to \mathrm{SO}(3)$ along ψ such that $e_3 = N$ denotes the Gauss map of ψ:
π above denotes the canonical projection relative to the base point e_3. Thus, $S^2 \cong \mathrm{SO}(3)/K$. Note that the Lie algebra of the group $K \simeq SO(2)$ is Lie $K = \underline{k} = \mathbf{R}E_{12}$.

Remark 4.2.3. As we pointed out, giving an extended Maurer-Cartan form ω^λ satisfying the flatness condition is equivalent to giving the forms ω_i^λ, ω_{ij}^λ, $i<j$ satisfying the conditions (K), which is also equivalent to giving a family of surfaces M_λ of constant negative Gaussian curvature $K=-1$. To such an associated family of surfaces, we attached (see (3.2.7)) the extended frame $\mathcal{U}^\lambda : D \times \mathbf{R}_+ \to \Lambda\mathrm{SO}(3)_P$ satisfying $(\mathcal{U}^\lambda)^{-1}\mathrm{d}\mathcal{U}^\lambda + \omega^\lambda = 0$, where $\mathbf{R}_+$ represents the set of strictly positive real numbers λ. It will be convenient for our purposes to fix a base point $x_0 \in D$, e.g. $x_0 = (0,0)$, and require that the frame satisfies the "initial condition"

$$\mathcal{U}(x_0, \lambda) = I, \tag{4.2.11}$$

for every λ. We will use this assumption from now on.

Remark 4.2.4. The subalgebra $\Lambda\mathrm{so}(3)_P^{\mathrm{alg}}$ of $\mathrm{so}(3) \otimes \mathbf{R}[\lambda, \lambda^{-1}]$ defined by (4.2.4) can also be characterized as the subalgebra consisting of elements with the following

- Property $\mathcal{P}$: In a representation relative to the basis E_{12}, E_{13}, E_{23}, the coefficient of E_{12} is an even function of λ, while the coefficients of E_{13} and E_{23} are odd functions of λ.

4.3 Loop Groups and Group Splittings Used for Pseudospherical Surfaces

In order to carry out the DPW method in the context of pseudospherical surfaces, we introduce the following subalgebras of $\Lambda\mathrm{so}(3)_P$:

4.3.1 $\Lambda^+\mathrm{so}(3)_P = \{X(\lambda) \in \Lambda\mathrm{so}(3)_P;\ X(\lambda)\text{contains only non-negativepowers of } \lambda\}$

4.3.2 $\Lambda^-\mathrm{so}(3)_P = \{X(\lambda) \in \Lambda\mathrm{so}(3)_P;\ X(\lambda)\text{contains only non-positivepowers of } \lambda\}$

4.3.3 $$\Lambda_*^-\mathrm{so}(3)_P = \{X(\lambda) \in \Lambda^-\mathrm{so}(3)_P;\ X(\infty) = 0\}$$

The connected Banach loop groups whose Lie algebras are described by definitions (4.3.1–4.3.3) are denoted, respectively, $\Lambda^+SO(3)_P$, $\Lambda^-SO(3)_P$ and $\Lambda_*^-SO(3)_P$.

A first question arises when we aim to split à la Birkhoff elements from $\Lambda SO(3)_P$ with $\lambda \in \mathbf{R}_+$ instead of $\lambda \in S^1$. The classical factorization theorem is stated and proved in [Pr, Se] for smooth loops on S^1 and reformulated in [DPW], [DGS] for a complexified Banach loop group G^C.

For our applications, the relevant part is

Theorem 4.3.1. [DPW; Thm. 2.2.], [Pr, Se; Thm. 8.1.1–8.1.2]: *Let G be a compact Lie group. Then the multiplication $\Lambda_*^-G^C \times \Lambda^+G^C \to \Lambda G^C$ is an analytic diffeomorphism onto the open and dense subset $\Lambda_*^-G^C \cdot \Lambda^+G^C$, called the "big cell". In particular, if $g \in \Lambda G^C$ is contained in the big cell, then g has a unique decomposition*

$$g = g_- g_+, \tag{4.3.4}$$

where $g_- \in \Lambda_*^-G^C$ and $g_+ \in \Lambda^+G^C$. The analogous result holds for the multiplication map $\Lambda_*^+G^C \times \Lambda^-G^C \to \Lambda G^C$.

The results stated above hold in particular for $G = SO(3)$. The splitting (4.3.4) is called the Birkhoff factorization of ΛG^C.

Remark A. Regarding the $\lambda \in S^1$ versus $\lambda \in \mathbf{R}_+$ issue, our Appendix contains the proof of the fact that the splitting works also for some specific "loop" group with real, positive λ.

Let $\tilde{\Lambda} SO(3)_P$ be the subset of $\Lambda SO(3)_P$ whose elements, as maps defined on $\mathbf{R}_+$, admit an analytic extension to $\mathbf{C}_*$. It is easy to see that $\tilde{\Lambda} SO(3)_P$ is a subgroup of $\Lambda SO(3)_P$. We have the following result:

Theorem 4.3.2. *$\tilde{\Lambda}^-_* SO(3)_P \times \tilde{\Lambda}^+ SO(3)_P \to \tilde{\Lambda} SO(3)_P$ is a diffeomorphism onto the open and dense subset $\tilde{\Lambda}^-_* SO(3)_P \cdot \tilde{\Lambda}^+ SO(3)_P$, called the "big cell". In particular, if $g \in \tilde{\Lambda} SO(3)_P$ is contained in the big cell, then g has a unique decomposition*

$$g = g_- g_+, \tag{4.3.5}$$

where $g_- \in \tilde{\Lambda}^-_* SO(3)_P$ and $g_+ \in \tilde{\Lambda}^+ SO(3)_P$. The analogous result holds for the multiplication map $\tilde{\Lambda}^+_* SO(3)_P \times \tilde{\Lambda}^- SO(3)_P \to \tilde{\Lambda} SO(3)_P$.

Proof. See Appendix. □

Remark that any extended frame $\mathcal{U}^\lambda$, as a function of the real positive parameter λ, admits an analytic extension to C_*. This is straight-forward and is stated and proved in Lemma A.1.

Hence, any extended frame $\mathcal{U}(x, y, \lambda)$ from the "big cell" of $\tilde{\Lambda} SO(3)_P$ can be split as

$$\mathcal{U} = \mathcal{U}_+ \cdot V_- = \mathcal{U}_- \cdot V_+. \tag{4.3.6}$$

Here $\mathcal{U}_-$ is an element of the form $\mathcal{U}_- = I + \lambda^{-1}\mathcal{U}_{-1} + \lambda^{-2}\mathcal{U}_{-2} + \cdots$, while V_+ is an element of the form $V_+ = V_0 + \lambda V_1 + \lambda^2 V_2 + \cdots$, respectively. Analogous expressions can be written for $\mathcal{U}_+$ and V_-, respectively. Namely, $\mathcal{U}_+$ is an element of the form $\mathcal{U}_+ = I + \lambda \mathcal{U}_1 + \lambda^2 \mathcal{U}_2 + \cdots$, while V_- is an element of the form $V_- = V_0 + \lambda^{-1} V_{-1} + \lambda^{-2} V_{-2} + \cdots$.

5 Harmonic Maps and Generalized Weierstrass Data

In this section we present the notion of harmonic map from a pseudospherical surface M to S^2. This is a particular case of a harmonic map from a pseudo-Riemannian manifold to another pseudo-Riemannian manifold, i.e. a differentiable map whose tension field vanishes (see [EL]). The Gauss maps of certain classes of surfaces (e.g. constant mean curvature, minimal, constant Gaussian curvature) are harmonic with respect to some suitable (pseudo)metrics. It was proved that the harmonic maps from these classes of surfaces to S^2 are in one-to-one correspondence with the equivalence classes of flat extended forms ω^λ (3.2.8) under the action of a gauge group. In connection with Sections 3 and 4, this is a strong motivation for studying such harmonic maps.

5.1 Harmonic Maps

Definition 5.1.1. Let (M, g) and $(\tilde{M}, \tilde{g})$ be pseudo-Riemannian manifolds. A harmonic map $f : M \to \tilde{M}$ is a differentiable map such that its *tension field* $\tau(f)$ vanishes:

$$\tau(f) := \mathrm{Trace}(\nabla \mathrm{d}f) = 0, \tag{5.1.1}$$

where ∇ is the Levi-Civita connection on the vector bundle $T^*(M) \otimes f^*(T\tilde{M})$, provided with the natural pseudo-metric induced by g and $\tilde{g}$.

For Riemannian manifolds, the system (5.1.1) is elliptic. This property is not maintained on pseudo-Riemannian manifolds. In this case, harmonic maps are sometimes called *pseudo-harmonic.*

The notion of harmonic map was first introduced by Eells and Sampson for Riemannian manifolds, then generalized to pseudo-Riemannian manifolds by Eells and Lemaire ([EL]) and then studied by several authors (e.g., [GU], [Me, St, 1]).

If (M, g) and $(\tilde{M}, \tilde{g})$ are two Riemannian manifolds, $df(x)$ represents the differential of f (linear map from TM to $T\tilde{M}$ at a point x of M), while its *tension field* is

$$\tau(f) = \mathrm{div}(df) = g^{ij}(\nabla(df))_{ij}. \tag{5.1.2}$$

Here we used again the Einstein summation convention with respect to both indices i, j. g^{ij} are the entries of the inverse g^{-1} of the matrix g.

The integral over M of the energy density $|df|^2$ with respect to the area element on M is frequently called *energy functional*. Equation (5.1.1) arises as the Euler-Lagrange equation for the variational problem of the energy integral. Harmonic maps f represent critical points of the energy functional.

We shall now introduce a concept which is actually equivalent to the one of extended Maurer-Cartan form ω^λ.

Remark 5.1.1. The following represents a necessary and sufficient condition for a map to be harmonic ([UR]):

Lemma: *Let f be a smooth map from a pseudo-Riemannian manifold to the sphere S^n. Then f is harmonic iff*

$$\delta f = \rho \cdot f, \tag{5.1.3}$$

for some function ρ, where δ represents the Lorentz-Laplace operator.

In this case, $\rho = e(f) = |df|^2$ is the energy density of f.

For the case $f : M \to S^2$, where M is a 2-dimensional manifold, see also [Me, St, 1], Prop. 1.1. Moreover, harmonicity is invariant under conformal transformations.

Remark 5.1.2. A classically known fact is the following:

If M is a weakly regular surface with $K < 0$, then M, endowed with its second fundamental form II (2.1.2) in asymptotic coordinates, is a *Lorentzian* 2-manifold (M, II).

Moreover, the Gauss map $N : (M, \mathrm{II}) \to S^2$ is harmonic *iff* $K =$ constant. With respect to the second fundamental form, (5.1.3) is written as

$$N_{xy} = \rho \cdot N. \tag{5.1.4}$$

In this sense, the Gauss map of every pseudospherical surface is harmonic.

This property of pseudospherical surfaces is sometimes called Lorentz-harmonicity.

Definition 5.1.2. Let us consider an $so(3)$-valued form ω.

Recall from the previous section the Lie algebras $\underline{k} = \mathbf{R}E_{12}$ and $\underline{p} = \mathbf{R}E_{13} + \mathbf{R}E_{23}$.

Let $\eta = \eta_0 + \eta_1$ be the Cartan decomposition of η into its $\underline{k}$-part η_0, respectively its $\underline{p}$- part, η_1. Then η is called an *admissible connection* if it satisfies the following pair of equations (sometimes called *Yang-Mills-Higgs* equations):

(5.1.5) $$\mathrm{d}\eta + \eta \wedge \eta = \mathrm{d}\eta + \frac{1}{2}[\eta \wedge \eta] = 0,$$

(5.1.6) $$\mathrm{d}(*\eta_1) + [\eta_0 \wedge *\eta_1] = 0.$$

For (5.1.5) and (5.1.6), see [Gu, Oh].

From the Remark 4.2.1, the smooth Gauss map N has the frame $\mathcal{U}$ as a lift. It follows (e.g, [Bo 2]) that the maps N and $\mathcal{U}$ are related by the identification

(5.1.7) $$N \equiv \mathcal{U} \cdot E_{12} \cdot \mathcal{U}^{-1}.$$

Note: In (5.1.7), [Bo2] uses $-i\sigma_3$ instead of our E_{12}. σ_3 is the third Pauli matrix (6.4.1). This fact is explained by the (spinor representation) isomorphim between $su(2)$ and $so(3)$, which is presented in Section 6.4.

A very important result obtained by A. Sym ([Sy]) allows us to obtain the immersion once we have the expression of the extended frame. This is presented in several papers, including for the particular case of pseudospherical surfaces (e.g. [1, Me, St], [Bo, Pi]) and can be stated as follows:

Theorem 5.1.1. *Starting from a given $\varphi(x, y)$, a solution to the sine-Gordon equation, let us consider the initial value problem consisting of the Lax system (2.3.15) together with the initial condition $\mathcal{U}(0, 0, \lambda) = I$. Let $\mathcal{U}(\lambda)$ be the solution to this initial value problem. Then $\mathcal{U}(\lambda)$ represents the extended frame corresponding to the Chebyshev immersion*

(5.1.8) $$\psi^\lambda = \frac{d}{dt}\mathcal{U}^\lambda \cdot (\mathcal{U}^\lambda)^{-1},$$

where $\lambda = e^t$.

By Theorem 5.1.1, once we have the extended frame, we can reconstruct the surface. Also, the relationship between the extended frame $\mathcal{U}$ and the Gauss map N is clear, via (5.1.7). So in a sense we could reconstruct everything starting from the Gauss map. However, there is a freedom in the frame given by a gauge action.

Definition 5.1.3. Let us consider a rotation of angle θ around e_3,

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The rotation R, thought of as an element of $SO(2)$, acts on the frame $\mathcal{U}$, and produces the so called *gauged frame* $\hat{\mathcal{U}}$ of the pseudospherical surface M, via the rule

(5.1.9) $$\hat{\mathcal{U}} = \mathcal{U} \cdot R^{-1}.$$

As a consequence of this action by a rotation matrix on the frame, the Maurer-Cartan form ω changes accordingly, to a $\hat{\omega}$. On the other hand, the Gauss map $N = \mathcal{U} \cdot E_{12} \cdot \mathcal{U}^{-1}$ from equation (5.1.7) is obviously invariant under such a gauge transformation.

The following very important result is a particular case of [Me, St, 1], Prop.1.4.

Proposition 5.1.1. *There is a one-to-one correspondence between the space of harmonic maps from the Lorentzian surface M to S^2 and the equivalence classes of admissible connections, under the action of the gauge group introduced by (5.1.9).*

Remark 5.1.3. On the other hand, every admissible connection "ω corresponds to its associated loop ω^λ satisfying the flatness condition (3.2.8):

$$\mathrm{d}\omega^\lambda + \omega^\lambda \wedge \omega^\lambda = 0.$$

Recall that we called ω^λ extended Maurer-Cartan form.

The result above provides a strong interest in harmonic maps. Summarizing, the Gauss map of a pseudospherical surface has the following properties:

Theorem 5.1.2. [Bo2, Prop. 7] *The Gauss map $N : M \to S^2$ of a surface with $K = -1$ is* Lorentz-harmonic*, i.e.,*

$$N_{xy} = qN, \qquad q : M \to \mathbf{R}. \tag{5.1.10}$$

Moreover, N forms in S^2 the same kind of Chebyshev net as the immersion function does in $\mathbf{R}^3$:

$$|N_x| = A, \quad |N_y| = B, \qquad \text{where } A = |\psi_x|,\ B = |\psi_y|. \tag{5.1.11}$$

Proof. A lengthy but straight-forward calculation using formulas (5.1.7) and (5.1.8) leads to formulas (5.1.10, 5.1.11). □

Via Proposition 5.1.1 and Theorem 5.1.2, we state the following:

Remark 5.1.4. As a consequence of the previous results and remarks, we conclude:

A smooth map $N : D \to S^2$ is Lorentz-harmonic if and only if there is an extended frame $\mathcal{U} : D \to \Lambda\mathrm{SO}(3)_P$ such that $\pi \circ \mathcal{U}^\lambda|_{\lambda=1} = N$, and such that

$$\omega^\lambda := -(\mathcal{U}^\lambda)^{-1}\mathrm{d}\mathcal{U}^\lambda \tag{5.1.12}$$

satisfies the flatness condition (3.2.8).

Here we denoted by $\pi : \mathrm{SO}(3) \to \mathrm{SO}(3)/K$ the canonical projection, and K a Lie subgroup isomorphic to SO(2), which is the isotropy group of the action of SO(3) on the vector e_3 in $\mathbf{R}^3$.

Let O be the point corresponding to $x = 0$, $y = 0$ in M. We consider the extended frame corresponding to the frame $\mathcal{U}$ the solution $\mathcal{U}^\lambda$ of equation (5.2.6) that satisfies the additional initial condition

$$\mathcal{U}^\lambda(0, 0, \lambda) = \mathcal{U}(0, 0) = I, \tag{5.1.13}$$

where $\mathcal{U}$ is the frame of $N : D \to S^2$, $N(0, 0) = eK$, such that Lie $K = \underline{k} = \mathbf{R}E_{12}$. Clearly, $\mathcal{U}^\lambda(x, y, 1) = \mathcal{U}(x, y)$.

Let us now consider the Cartan decomposition $\underline{g} = \underline{k} + \underline{p}$ where $\underline{k} = \mathbf{R}E_{12}$ and $\underline{p} = \mathbf{R}E_{13} + \mathbf{R}E_{23}$. Let ω^λ be a 1-form that satisfies the flatness condition (3.2.8).

Via the Cartan decomposition above, ω^λ can be written in the form

$$\omega^\lambda := \alpha_0 + \omega_1^\lambda, \tag{5.1.14}$$

where $\alpha_0 \in \underline{k}$ and $\omega_1^\lambda = \lambda^{-1} \cdot \alpha_- + \lambda \cdot \alpha_1 \in \underline{p}$.

As a consequence of Theorem 5.1.1, we obtain:

Proposition 5.1.2. *Let $\mathcal{U}^\lambda : D \to \Lambda\mathrm{SO}(3)_P$ be any map such that $(\mathcal{U}^\lambda)^{-1} d\mathcal{U}^\lambda$ is of the form (5.1.14) and satisfies the flatness condition (3.2.8). Then $\mathcal{U}^\lambda$ represents an extended normalized frame corresponding to the associated family of Chebyshev immersions*

$$\psi^\lambda = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}^\lambda(\mathcal{U}^\lambda)^{-1}, \tag{5.1.15}$$

5.2 The Weierstrass-type Representation

A.*Generalized Weierstrass Representation of Constant Mean Curvature Surfaces.*

In [DPW], the authors have introduced a Weierstrass type representation through which every harmonic map from a Riemann surface M to an arbitrary compact symmetric space G/K is described by a Lie G^C - valued meromorphic differential on the universal covering of M. In [Do, Ha], the authors present the case of a constant mean curvature surface M in $\mathbf{R}^3$, parametrized in conformal coordinates, obtaining the above-mentioned differential explicitely.

For the case of $G = \mathrm{SO}(3)$ and $K = \mathrm{SO}(2)$, $G/K \cong S^2$, this procedure is based on introducing the extended normalized frame $\mathcal{U}^\lambda : D \to \Lambda\mathrm{SO}(3)_P$, which for $\lambda = 1$ represents the normalized moving frame. In this case, the $\mathrm{so}(3, \mathbf{C})$-valued meromorphic differential is characterized by two different meromorphic functions. The poles of the above mentioned meromorphic functions are situated at points where the Birkhoff loop group factorization $\mathcal{U} = \mathcal{U}_- V_+$ fails to exist.

The Weierstrass-type data is expressed via a Lie algebra-valued differential form

$$\xi = \mathcal{U}_-^{-1}\mathrm{d}\mathcal{U}_- = \lambda^{-1}\eta. \tag{5.2.1}$$

Definition 5.2.1. The forms η and ξ given by equation (5.2.1) are called (see also [Wu2] and [DoHa]) *normalized*, and respectively *meromorphic* potentials.

Starting from the normalized potential, we can construct the associated family of CMC surfaces $M_\lambda = (D, \psi_\lambda)$.

An analogous result is presented in [DPT] for minimal surfaces in $\mathbf{R}^3$, parametrized in conformal coordinates.

B.*Generalized Weierstrass Representation of Pseudospherical Surfaces*

The aim of Sections 5 and 6 is to present the analogue of the DPW method explained above for the case of pseudospherical surfaces. The main result of the Section 6 is the Weierstrass-type data for pseudospherical surfaces. In Section 6 we define the generalized Weierstrass representation as a pair of Lie algebra-valued differential forms

$$\xi^x = -\mathcal{U}_+^{-1}\mathrm{d}\mathcal{U}_+ = \lambda\eta^x, \tag{5.2.2a}$$

$$\xi^y = -\mathcal{U}_-^{-1}\mathrm{d}\mathcal{U}_- = \lambda^{-1}\eta^y. \tag{5.2.2b}$$

Definition 5.2.2. The forms η^x and η^y given by equations (5.2.2a, 5.2.2b) are called *normalized* x-potential and y-potential, respectively.

Starting from such a pair of normalized potentials, we can construct the associated family of pseudospherical surfaces $M_\lambda = (D, \psi_\lambda)$.

6 Explicit Forms of the Normalized Potentials of Pseudospherical Surfaces

6.1 Normalized Potential for CMC Surfaces Revisited

For constant mean curvature surfaces $M = (D, \psi)$ parametrized in conformal coordinates with metric $\mathrm{d}s^2 = 4e^{2\omega(z,\bar{z})}\mathrm{d}z\mathrm{d}\bar{z}$, Theorem 2.1, [Wu2], offers a simple method to calculate the normalized potential.

Namely, if the Maurer-Cartan form is

6.1.1
$$\mathcal{U}^{-1}\mathrm{d}\mathcal{U} = \alpha_{-1}\lambda^{-1} + \alpha_0 + \alpha_1 \cdot \lambda$$
$$\alpha_0 = \alpha_0'\mathrm{d}z + \alpha_0''\mathrm{d}\bar{z},$$

we denote by $\beta_0(z)$ and $\beta_1(z)$, respectively, the holomorphic part $\alpha_0'(z,0)\mathrm{d}z$ of $\alpha_0'\,\mathrm{d}z$ and the holomorphic part $\alpha_{-1}(z,0)$ of α_{-1}. Recall that the holomorphic part of a function $f(z,\bar{z}) = \sum_{k,l} a_{kl} z^k \bar{z}^l$ is $f(z,0)$.

Then the following theorem will provide the normalized potential η: **Theorem 6.1.1** (2.1, [Wu2]) *The normalized potential η of the surface, with the origin $z = 0$ as the reference point, is given by*

(6.1.2)
$$\eta(z) = \psi_0(z) \cdot \beta_1(z) \cdot \psi_0(z)^{-1},$$

where ψ_0 is the solution to

(6.1.3)
$$\psi_0(z)^{-1}\,d\psi_0(z) = \beta_0(z), \quad \psi_0(0) = \mathcal{U}(0),$$

and $\mathcal{U}$ is the normalized frame at the origin.

For CMC surfaces (see, for example [Wu2], formula (3.18)) the normalized potential is of the form

(6.1.4)
$$P(z) = \begin{pmatrix} 0 & 0 & -b(z) \\ 0 & 0 & -c(z) \\ b(z) & c(z) & 0 \end{pmatrix} \mathrm{d}z,$$

where

6.1.5
$$b(z) = \frac{1}{2}\left(e^{2\xi(z)-\xi(0)} + Q(z)e^{\xi(0)-2\xi(z)}\right),$$
$$c(z) = \frac{i}{2}\left(-e^{2\xi(z)-\xi(0)} + Q(z)e^{\xi(0)-2\xi(z)}\right),$$

and ξ represents the *holomorphic part* $\omega(z,0)$ of $\omega(z,\bar{z})$, where

$$\mathrm{d}s^2 = 4e^{2\omega(z,\bar{z})}\mathrm{d}z\mathrm{d}\bar{z}$$

represents the metric of the surface, while $Q(z) = (N, \psi_{zz})$ is the (holomorphic) coefficient of the Hopf differential $Q(z)(\mathrm{d}z)^2$.

Equivalently, under the adjoint map Ad : SU(2) → SO(3) (see [Wu2], Remark 3.22, and [DoHa]), via a lifting to SU(2), the normalized potential can be written as

(6.1.6)
$$\eta(z) = \frac{1}{2}\begin{pmatrix} 0 & e^{2\xi(z)-\xi(0)} \\ -Q(z)e^{\xi(0)-2\xi(z)} & 0 \end{pmatrix} \mathrm{d}z.$$

In the following subsection we shall state and prove a similar result for pseudospherical surfaces parametrized in asymptotic line coordinates.

6.2 Normalized x- and y- Potentials for Pseudospherical Surfaces. Ordinary Differential Systems Associated with Normalized Potentials

By analogy with the normalized potential introduced for constant mean curvature surfaces, it becomes natural to consider a normalized potential for other classes of surfaces whose Gauss map is harmonic, as a map between pseudo-Riemannian surfaces, in particular for the class of pseudospherical surfaces.

We will introduce the generalized Weierstrass representation for pseudospherical surfaces in a Chebyshev parametrization, as two normalized potentials:

η^x and η^y, where η^x does not depend on y, and η^y does not depend on x.

Theorem 6.2.1 below will make this explicit.

Theorems 6.3.1 and 6.3.2 in the next section will give explicit formulas for the normalized potentials. They are consequences of Theorem 6.2.1.

In our case, the group K represents the group of rotations around e_3, isomorphic to $SO(2)$,

$$K = \left\{ \begin{pmatrix} \cos r & -\sin r & 0 \\ \sin r & \cos r & 0 \\ 0 & 0 & 1 \end{pmatrix} ;\ r \in [0; 2\pi) \right\}. \tag{6.2.1}$$

Its Lie algebra Lie$K$ is

$$\underline{k} = \left\{ \begin{pmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ;\ a \in \mathbf{R} \right\} \tag{6.2.2}$$

while its complement in $so(3)$ is

$$\underline{p} = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ -b & -c & 0 \end{pmatrix} ;\ b, c \in \mathbf{R} \right\}. \tag{6.2.3}$$

For the extended frame $\mathcal{U}^\lambda : M \to \Lambda\mathrm{SO}(3)_P$, with

$$\mathcal{U}^\lambda(0,0,\lambda) = \mathcal{U}(0,0) = I, \tag{6.2.4}$$

we have the Lax system ((3.2.6), restated).

$$\begin{cases} (\mathcal{U}^\lambda)^{-1} \cdot (\mathcal{U}^\lambda)_x = \begin{pmatrix} 0 & -\varphi_x & 0 \\ \varphi_x & 0 & \lambda \\ 0 & -\lambda & 0 \end{pmatrix} = \mathcal{A}, \\[2ex] (\mathcal{U}^\lambda)^{-1} \cdot (\mathcal{U}^\lambda)_y = \begin{pmatrix} 0 & 0 & -\lambda^{-1}\sin\varphi \\ 0 & 0 & -\lambda^{-1}\cos\varphi \\ \lambda^{-1}\sin\varphi & \lambda^{-1}\cos\varphi & 0 \end{pmatrix} = \mathcal{B}. \end{cases} \tag{6.2.5}$$

Consequently, the Maurer-Cartan form is written as

$$\omega^\lambda = -(\mathcal{U}^\lambda)^{-1} \cdot \mathrm{d}\mathcal{U}^\lambda = -\mathcal{A} \cdot \mathrm{d}x - \mathcal{B} \cdot \mathrm{d}y = \alpha_{-1} \cdot \lambda^{-1} + \underbrace{(\alpha_0' \mathrm{d}x + \alpha_0'' \mathrm{d}y)}_{\alpha_0} + \alpha_1 \cdot \lambda,$$

where, obviously,

6.2.6.a
$$\alpha_{-1} = \begin{pmatrix} 0 & 0 & \sin\varphi \\ 0 & 0 & \cos\varphi \\ -\sin\varphi & -\cos\varphi & 0 \end{pmatrix} \mathrm{d}y,$$

6.2.6.b
$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \mathrm{d}x,$$

6.2.6.c
$$\alpha_0' = \begin{pmatrix} 0 & \varphi_x & 0 \\ -\varphi_x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \alpha_0'' = 0.$$

Definition 6.2.1. For any real smooth function $f(x,y)$ defined on a sufficiently small neighborhood of $(0,0)$ in D, we shall call $f(x,0)$ the *x-part* (of f), respectively $f(0,y)$ the *y-part*.

We also set

6.2.7
$$f^x := f(x,0) \quad f^y := f(0,y)$$

We call $f(x,0)dx$ the *x-part* of the form $f(x,y)dx$. Analogously, we call $f(0,y)dy$ the *y-part* of the form $f(x,y)dy$.

Let $N : D \to S^2$ be the Gauss map of a weakly regular pseudospherical surface M. Thus, N is real and smooth, and Lorentz harmonic. By Remark 5.1.4, there is a λ- family of frames $\mathcal{U}^\lambda : D \to \Lambda\mathrm{SO}(3)_P$ such that $\pi \circ \mathcal{U}^\lambda|_{\lambda=1} = N$ and such that $-(\mathcal{U}^\lambda)^{-1}\mathrm{d}\mathcal{U}^\lambda$ is the corresponding Maurer-Cartan form ω^λ.

Consequently, the 1-forms α_0 and α_1 defined by $\omega^\lambda = \alpha_{-1}\lambda^{-1} + \alpha_0 + \alpha_1\lambda$ are also smooth in x and y.

In $\omega^\lambda = -(\mathcal{U}^\lambda)^{-1}\mathrm{d}\mathcal{U}^\lambda = -\mathcal{A}\mathrm{d}x - \mathcal{B}\mathrm{d}y$, where $\mathcal{A} = (\mathcal{U}^\lambda)^{-1}\mathcal{U}^\lambda_x$ and $\mathcal{B} = (\mathcal{U}^\lambda)^{-1}\mathcal{U}^\lambda_y$, we denote by

6.2.8a
$$\begin{aligned} \beta : \ &= \text{ the } x\text{-part of the form } -\mathcal{A}\mathrm{d}x \text{ at } \lambda = 1 \\ &= \text{ the } x\text{-part of the form } \alpha_0'\mathrm{d}x + \alpha_1. \end{aligned}$$

6.2.8b
$$\beta_0 := \text{the } x\text{-part of the form } \alpha_0'\,\mathrm{d}x, \text{ where}$$
$$\alpha_0(x,y) = \alpha_0'(x,y)\mathrm{d}x + \alpha_0''(x,y)\mathrm{d}y.$$

6.2.8c
$$\beta_1 := \beta - \beta_0 = \text{the } x\text{-part of } \alpha_1\ .$$

γ_0 and γ_1 above are the analogs of β_0 and β_1, with respect to y. That is

6.2.9a
$$\begin{aligned} \gamma : \ &= \text{ the } y\text{-part of the form } -\mathcal{B}\mathrm{d}y \text{ at } \lambda = 1 \\ &= \text{ the } y\text{-part of } \alpha_0''\mathrm{d}y + \alpha_{-1}. \end{aligned}$$

6.2.9b
$$\gamma_0 := \text{the } y\text{-part of the form } \alpha_0''\,\mathrm{d}y, \text{ where}$$
$$\alpha_0(x,y) = \alpha_0'(x,y)\mathrm{d}x + \alpha_0''(x,y)\mathrm{d}y.$$

6.2.9c $$\gamma_1 := \gamma - \gamma_0 = \text{the } y\text{-part of } \alpha_{-1} \ .$$

Remark that formulas like the ones above may be also useful in the study of other types of surfaces parametrized in some real coordinates x, y. That is why we shall state some results (like Theorems 6.2.3 and 6.2.4) using generic β's and γ's.

For our purposes, it is important to make β and γ explicit for pseudospherical surfaces in Chebyshev parametrization. We obtain:

(6.2.10a) $$\beta = \beta_1 + \beta_0$$

(6.2.10b) $$\beta_1 = \alpha_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \mathrm{d}x,$$

(6.2.10c) $$\beta_0 = \alpha_0'(x,0)\mathrm{d} = \begin{pmatrix} 0 & \varphi_x(x,0) & 0 \\ -\varphi_x(x,0) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathrm{d}x.$$

(6.2.11a) $$\gamma = \gamma_1,$$

(6.2.11b) $$\gamma_1 = \begin{pmatrix} 0 & 0 & \sin\varphi(0,y) \\ 0 & 0 & \cos\varphi(0,y) \\ -\sin\varphi(0,y) & -\cos\varphi(0,y) & 0 \end{pmatrix} \mathrm{d}y,$$

(6.2.11c) $$\gamma_0 = 0.$$

Let us now recall the two Birkhoff-type factorizations presented in Theorem 4.3.2.

The first type of Birkhoff factorization from Theorem 4.3.2 is performed on the "big cell" $\tilde{\Lambda}_*^- \mathrm{SO}(3)_P \cdot \tilde{\Lambda}^+ \mathrm{SO}(3)_P$. That is, away from a singular set $S_1 \subset D$, we can split the extended moving frame $\mathcal{U}^\lambda : D \to \mathrm{SO}(3)$ into two parts. Recall that the first factor of this splitting is of the form $g_- = I + \lambda^{-1} g_{-1} + \lambda^{-2} g_{-2} + \cdots$, while the second factor of the splitting is of the form $g_+ = g_0 + \lambda g_1 + \lambda^2 g_2 + \cdots$, respectively.

Since the "big cell" is open and $\mathcal{U}^\lambda : D \to \mathrm{SO}(3)$ is continuous, the set

$$\tilde{D}_1 = \{(x,y)\ ;\ \mathcal{U}^\lambda(x,y) \text{belongs to the "big cell"}\}$$

is open. Note that $(0,0) \in \tilde{D}_1$.

Let $S_1 = D - \tilde{D}_1$ denote the "singular" set. We have just shown that S_1 is closed and $(0,0)$ is not an element of the set S_1. Similarly, we have S_2 and $\tilde{D}_2$ for the second splitting.

The second type of Birkhoff splitting is the analogous splitting in the "big cell" $\tilde{\Lambda}_*^+ \mathrm{SO}(3)_P \times \tilde{\Lambda}^- \mathrm{SO}(3)_P$. The goal of this section is to show that the first factor of each type of splitting is an essential one, and can be viewed as an integral of the unconstrained data that we call normalized potential.

We can perform the two splittings on the extended frame $\mathcal{U}^\lambda$. *Let* $\mathcal{U} = \mathcal{U}^\lambda$ *be the extended normalized moving frame of a pseudospherical surface and let* $(x,y) \in$

$D \setminus (S_1 \cup S_2)$. Then, for some uniquely determined $V_+ \in \Lambda^+\mathrm{SO}(3)_P$, $V_- \in \Lambda^-\mathrm{SO}(3)_P$ and $\mathcal{U}_- \in \Lambda^-_*\mathrm{SO}(3)_P$, $\mathcal{U}_+ \in \Lambda^+_*\mathrm{SO}(3)_P$, $\mathcal{U}$ can be written as

$$\mathcal{U} = \mathcal{U}_+ \cdot V_- = \mathcal{U}_- \cdot V_+. \tag{6.2.12}$$

The factors carrying the "genetic material" to recreate the frame and then the surface are $\mathcal{U}_+$ and $\mathcal{U}_-$. They can be obtained, starting from two normalized potentials η^x and η^y respectively, by solving the two ordinary differential equations presented in Theorem 6.2.1.

From $\mathcal{U}_-$ and $\mathcal{U}_+$, one can reproduce the frame $\mathcal{U}$ and then construct the corresponding pseudospherical immersion via the Sym-Bobenko formula (5.1.5).

Theorem 6.2.1. *Let $\mathcal{U} = \mathcal{U}^\lambda$, $\mathcal{U}_+$ and $\mathcal{U}_-$ be as above. Then the following systems of differential equations are satisfied:*

1)
$$(\mathcal{U}_+)^{-1}\frac{\partial \mathcal{U}_+}{\partial x}\,dx = -\lambda \cdot V_0 \cdot \beta_1 \cdot V_0^{-1}, \tag{6.2.13}$$

with initial condition $\mathcal{U}_+(x=0) = I$, where V_0 is some matrix $V_0(x) \in \mathrm{SO}(3)$.

2)
$$(\mathcal{U}_-)^{-1}\frac{\partial \mathcal{U}_-}{\partial y}\,dy = -\lambda^{-1} \cdot W_0 \cdot \gamma_1 \cdot W_0^{-1}, \tag{6.2.14}$$

with initial condition $\mathcal{U}_-(y=0) = I$, where W_0 is some matrix $W_0(y) \in \mathrm{SO}(3)$.

Moreover, $\mathcal{U}_+$ does not depend on y and $\mathcal{U}_-$ does not depend on x.

In some other words, $\mathcal{U}_+$ and $\mathcal{U}_-$ are solutions of some first order systems of differential equations in x and y, respectively.

Proof. of 2) From equation (6.2.12), we know

$$\mathcal{U}_- = \mathcal{U} \cdot V_+^{-1}. \tag{6.2.15}$$

Differentiating (6.2.15), we obtain

$$\mathrm{d}\mathcal{U}_- = \mathrm{d}\mathcal{U} \cdot V_+^{-1} - \mathcal{U} \cdot V_+^{-1} \cdot \mathrm{d}V_+ \cdot V_+^{-1}, \tag{6.2.16}$$

which can be rewritten as

$$\mathcal{U}_-^{-1} \cdot d\mathcal{U}_- = V_+ \cdot (\mathcal{U}^{-1} \cdot \mathrm{d}\mathcal{U}) \cdot V_+^{-1} - \mathrm{d}V_+ \cdot V_+^{-1} \tag{6.2.17}$$

after left multiplication by $\mathcal{U}_-^{-1}$.

The coefficient of $\mathrm{d}x$ on the left-hand side of (6.2.17) contains only negative powers of λ, while the coefficient of $\mathrm{d}x$ on the right-hand side of (6.2.17), in view of (6.2.5), contains only non-negative powers of λ. Therefore, $\partial_x \mathcal{U}_- = 0$, so $\mathcal{U}_-$ depends on y only.

To determine (6.2.14), we consider the coefficient of $\mathrm{d}y$ in (6.2.17). The left-hand side of (6.2.17) contains only negative powers of λ, while the one on the right-hand side, due to (6.2.5),

$$\mathcal{B} = \mathcal{U}^{-1}\partial_y \mathcal{U} = \lambda^{-1} \cdot \begin{pmatrix} 0 & 0 & -\sin\varphi \\ 0 & 0 & -\cos\varphi \\ \sin\varphi & \cos\varphi & 0 \end{pmatrix},$$

contains only one term in λ^{-1}, and no terms in λ^k, $k < -1$.

On the other hand, let

(6.2.18) $$V_+ = \tilde{W}_0 + \lambda\tilde{W}_1 + \lambda^2\tilde{W}_2 + \cdots = \tilde{W}_0 \cdot T_+,$$

with $T_+ \in \Lambda_*^+\mathrm{SO}(3)_P$.

Therefore, $\mathcal{U}_-^{-1}\partial_y\mathcal{U}_- = \tilde{W}_0\mathcal{B}\tilde{W}_0^{-1}$, where the left-hand side only depends on y. Since $\mathcal{U}$, V_+ and $\tilde{W}_0$ are all defined on D, a neighborhood of $(0,0)$, we can specialize to the points of the form $(0,y)$ for a sufficiently small interval on the line $x=0$, containing the origin.

Thus,

(6.2.19) $$\mathcal{U}_-^{-1}\partial_y\mathcal{U}_- = \tilde{W}_0(0,y)\cdot\mathcal{B}(0,y)\cdot\tilde{W}_0(0,y)^{-1}.$$

We observe that

$$\mathcal{B}(0,y) = \lambda^{-1}\cdot\begin{pmatrix} 0 & 0 & -\sin\varphi(0,y) \\ 0 & 0 & -\cos\varphi(0,y) \\ \sin\varphi(0,y) & \cos\varphi(0,y) & 0 \end{pmatrix}$$

From formulas (6.2.5-6), we note that $\mathcal{B} = -\lambda^{-1}\cdot\alpha_{-1}$, and restricting to the y-parts, we obtain

$$\mathcal{B}(0,y)\mathrm{d}y = -\lambda^{-1}\cdot\gamma_1,$$

where the form γ_1 is the one given in formulas (6.2.11.c).

In (5.2.2), we defined the normalized y-potential by

(6.2.20) $$\eta^y = -\lambda\cdot\mathcal{U}_-^{-1}\partial_y\mathcal{U}_-\mathrm{d}y.$$

and the meromorphic y-potential as

$$\xi^y = -\mathcal{U}_-^{-1}\partial_y\mathcal{U}_-\mathrm{d}y = \lambda^{-1}\eta^y,$$

Denoting $W_0(y) := \tilde{W}_0(0,y)$, we obtain

$$\mathcal{U}_-^{-1}\partial_y\mathcal{U}_-\mathrm{d}y = -\lambda^{-1}\cdot W_0(y)\cdot\gamma_1\cdot[W_0(y)]^{-1}$$

and therefore (6.2.14).

Proof of 1)

From equation (6.2.12), we obtain

(6.2.21) $$\mathcal{U}_+ = \mathcal{U}\cdot V_-^{-1}, \quad \mathcal{U}_+ \in \Lambda_*^+\mathrm{SO}(3)_P, \quad V_- \in \Lambda^-\mathrm{SO}(3)_P,$$

which by differentiation leads to

(6.2.22) $$\mathrm{d}\mathcal{U}_+ = \mathrm{d}\mathcal{U}\cdot V_-^{-1} - \mathcal{U}\cdot V_-^{-1}\cdot\mathrm{d}V_-\cdot V_-^{-1},$$

and then

(6.2.23) $$\mathcal{U}_+^{-1}\mathrm{d}\mathcal{U}_+ = V_-(\mathcal{U}^{-1}\mathrm{d}\mathcal{U})V_-^{-1} - \mathrm{d}V_-\cdot V_-^{-1}.$$

We compare the coefficient of $\mathrm{d}y$ on the left-hand side of (6.2.23) with the coefficient of $\mathrm{d}y$ on the right-hand side of (6.2.23), via formula (6.2.5). The left-hand side of (6.2.23) clearly contains only positive powers of λ, while the coefficient of $\mathrm{d}y$ on the right-hand side of (6.2.23), in view of (6.2.5), contains non-positive powers of λ only. Thus, $\mathcal{U}_+$ depends exclusively on x.

In order to obtain (6.2.13), we consider the coefficient of dx in (6.2.23). The left-hand side of (6.2.23) contains only positive powers of λ, while the one on the right-hand side, due to

$$\mathcal{A} = \mathcal{U}^{-1}\partial_x\mathcal{U} = \begin{pmatrix} 0 & -\varphi_x & 0 \\ \varphi_x & 0 & \lambda \\ 0 & -\lambda & 0 \end{pmatrix},$$

contains one term in λ and no terms in λ^k, with $k > 1$.

Like we did before in case 2), we can restrict to a sufficiently small interval around $(0, 0)$ on the line $y = 0$.

Let now

$$V_- = \tilde{V}_0 + \lambda^{-1}\tilde{V}_1 + \lambda^{-2}\tilde{V}_2 + \cdots = \tilde{V}_0 \cdot T_-, \tag{6.2.24}$$

with $T_- \in \Lambda_*^- \mathrm{SO}(3)_P$.

Then we note

$$\mathcal{U}_+^{-1}(x) \cdot \partial_x\mathcal{U}_+ = \tilde{V}_0(x, 0) \cdot \mathcal{A}(x, 0) \cdot \tilde{V}_0(x, 0)^{-1}. \tag{6.2.25}$$

Moreover, since the left-hand side of (6.2.23) contains only positive powers of λ, we conclude that

$$\mathcal{U}_+^{-1}(x) \cdot \partial_x\mathcal{U}_+ \mathrm{d}x = -\tilde{V}_0(x, 0) \cdot \lambda \cdot \beta_1 \cdot \tilde{V}_0(x, 0)^{-1}, \tag{6.2.26}$$

where according to formula (6.2.10.b), $\beta_1 = \alpha_1 = -E_{23}$. This is exactly the claim of the equation (6.2.13) stated in the theorem, if we denote $\tilde{V}_0(x, 0) := V_0$. □

6.3 Normalized Potentials for Pseudospherical Surfaces

In this section we find the explicit expressions of the two normalized potentials. Theorems 6.3.1. and 6.3.2 can be thought of as corollaries to Theorem 6.2.1. Basically, we construct the normalized potentials from the solutions to the ordinary differential systems introduced in Theorem 6.2.1. Theorems 6.3.1. and 6.3.2 are phrased analogously to Wu's Theorem 6.1.1 for the normalized potential of the constant mean curvature surfaces.

Theorem 6.3.1. (x-potential) *The normalized potential η^x with the origin as the reference point is given by*

$$\eta^x = V_0(x) \cdot \beta_1(x) \cdot V_0(x)^{-1}, \tag{6.3.1}$$

where V_0 is the solution of

$$\begin{cases} V_0(x)^{-1} dV_0(x) = -\beta_0(x), \\ V_0(0) = \mathcal{U}(0, 0). \end{cases} \tag{6.3.2}$$

where β_0 and β_1 are given by formulas (6.2.10).

Similarly, we have the following result:

Theorem 6.3.2. (y-potential) *The normalized potential η^y with the origin as the reference point is given by*

$$\eta^y = W_0(y) \cdot \gamma_1(y) \cdot W_0(y)^{-1}, \tag{6.3.3}$$

where W_0 is the solution of

$$\begin{cases} W_0(y)^{-1} dW_0(y) = -\gamma_0(y), \\ W_0(0) = \mathcal{U}(0,0). \end{cases} \tag{6.3.4}$$

where γ_0 and γ_1 are given by formulas (6.2.11).

Proof of Theorem 6.3.1.

Relation (6.3.3) is a rephrasing of (6.2.13):

$$\mathcal{U}_+^{-1} \partial_x \mathcal{U}_+ \mathrm{d}x = -\lambda \cdot V_0(x) \cdot \beta_1 \cdot V_0(x)^{-1},$$

where we substitute $\eta^x = -\lambda^{-1} \cdot (\mathcal{U}_+)^{-1} \cdot \mathrm{d}\mathcal{U}_+$, that is the definition of the normalized x-potential.

Let us now consider again equation (6.2.23) from the proof of Theorem 6.2.1, namely

$$\mathcal{U}_+^{-1} \mathrm{d}\mathcal{U}_+ = V_-(\mathcal{U}^{-1}\mathrm{d}\mathcal{U})V_-^{-1} - \mathrm{d}V_- \cdot V_-^{-1}$$

We proved that both sides depend on x only. Now let us take a look at the coefficient of λ^0 in this equation.

The left-hand side has positive powers of λ only, while the x-part of right-hand side only has $-V_0 \cdot \beta_0 \cdot V_0^{-1} - \mathrm{d}V_0 \cdot V_0^{-1}$ as a term that does not depend on λ.

Consequently, we obtain $V_0(x)^{-1}\mathrm{d}V_0 = -\beta_0(x)$. Formula (6.2.10.c) shows that

$$\beta_0 = \alpha_0'(x,0) = \begin{pmatrix} 0 & \varphi_x(x,0) & 0 \\ -\varphi_x(x,0) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here it was taken into account that $\varphi_x(x,0) = (\varphi(x,0))_x$, where $\xi(x) := \varphi(x,0)$ is the part in x of the smooth angle function $\varphi(x,y)$. If we consider the matrix

$$\theta = \begin{pmatrix} 0 & \xi & 0 \\ -\xi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.3.5}$$

then

$$\beta_0 = \theta' \mathrm{d}x = \begin{pmatrix} 0 & \xi'(x) & 0 \\ -\xi'(x) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathrm{d}x. \tag{6.3.6}$$

The solution V_0 of the system (6.3.2) must take into account that $\mathcal{U}(0,0,\lambda) = I$, so the solution is

$$V_0(x) = e^{\theta(0) - \theta(x)}. \tag{6.3.7}$$

Using also the expression of the form β_1, the normalized x-potential η^x can be written as

$$\eta^x = V_0(x)\beta_1(x)V_0^{-1}(x) = e^{\theta(0)-\theta(x)}(-E_{23})e^{\theta(x)-\theta(0)}\mathrm{d}x.$$

Since

$$e^{\theta(0)-\theta(x)} = \begin{pmatrix} \cos(\xi(0)-\xi(x)) & \sin(\xi(0)-\xi(x)) & 0 \\ -\sin(\xi(0)-\xi(x)) & \cos(\xi(0)-\xi(x)) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6.3.8}$$

the formula above leads to the final expression of the x-potential, as

$$\eta^x = \begin{pmatrix} 0 & 0 & -\sin(\xi(0)-\xi(x)) \\ 0 & 0 & -\cos(\xi(0)-\xi(x)) \\ \sin(\xi(0)-\xi(x)) & \cos(\xi(0)-\xi(x)) & 0 \end{pmatrix} \mathrm{d}x, \tag{6.3.9}$$

where $\xi(x) := \varphi(x, 0)$. □

Proof of Theorem 6.3.2.

Relation (6.3.3) is a rephrasing of (6.2.14):

$$\mathcal{U}_-^{-1}(y) \cdot \partial_y \mathcal{U}_- = -\lambda^{-1} \cdot W_0 \cdot \gamma_1 \cdot W_0^{-1}$$

where we substitute (6.2.20)

$$\eta^y = -\lambda \cdot \mathcal{U}_-^{-1} \partial_y \mathcal{U}_-,$$

that is the definition of the normalized y-potential.

Let us now consider equation (6.2.17) from the proof of Theorem 6.2.1, namely

$$\mathcal{U}_-^{-1} \cdot d\mathcal{U}_- = V_+ \cdot (\mathcal{U}^{-1} \cdot \mathrm{d}\mathcal{U}) \cdot V_+^{-1} - \mathrm{d}V_+ \cdot V_+^{-1}.$$

We proved that both sides depend on y only. Now let us take a look at the coefficient of λ^0 in this equation.

The left-hand side has negative powers of λ only, while the y-part of right-hand side only has $-W_0 \cdot \gamma_0 \cdot W_0^{-1} - \mathrm{d}W_0 \cdot W_0^{-1}$ as a term that does not depend on λ.

Consequently, we obtain $W_0(y)^{-1}\mathrm{d}W_0 = -\gamma_0(x)$. Formula (6.2.11.c) tells us that $\gamma_0 = 0$. From this we conclude that $W_0(y)$ is actually a constant matrix, and from the initial condition on the frame $\mathcal{U}$, together with the initial condition of (6.3.4), it follows that for every y,

$$W_0(y) = \mathcal{U}(0, 0) = I.$$

It follows that

$$\eta^y = W_0(y) \cdot \gamma_1(y) \cdot W_0(y)^{-1} = \gamma_1(y).$$

Therefore,

$$\eta^y = \begin{pmatrix} 0 & 0 & \sin\varphi(0, y) \\ 0 & 0 & \cos\varphi(0, y) \\ -\sin\varphi(0, y) & -\cos\varphi(0, y) & 0 \end{pmatrix} \mathrm{d}y. \tag{6.3.10}$$

□

Remark 6.3.1. Let us review the expressions (6.3.9) and (6.3.10) for the two normalized potentials η^x and η^y, that is

$$\begin{pmatrix} 0 & 0 & -\sin(\varphi(0,0)-\varphi(x,0)) \\ 0 & 0 & -\cos(\varphi(0,0)-\varphi(x,0)) \\ \sin(\varphi(0,0)-\varphi(x,0)) & \cos(\varphi(0,0)-\varphi(x,0)) & 0 \end{pmatrix} \mathrm{d}x,$$

and

$$\begin{pmatrix} 0 & 0 & \sin(\varphi(0,y)) \\ 0 & 0 & \cos(\varphi(0,y)) \\ -\sin(\varphi(0,y)) & -\cos(\varphi(0,y)) & 0 \end{pmatrix} \mathrm{d}y,$$

respectively.

Note that the normalized potentials depend exclusively on the angle $\varphi(x,y)$ between the asymptotic lines.

6.4 Another Method for Normalized Potentials: Passage to 2×2 Matrices

We introduce the matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.4.1}$$

called *Pauli matrices*.

We can rewrite in terms of 2×2 matrices the potentials η^x and η^y, given by formulas (6.3.9) and (6.3.10). The Pauli matrices above will allow us to do this.

This passage from 3×3 to 2×2 matrices can be done via the well-known isomorphism between $\mathrm{sl}(2,\mathbf{C})$ and $\mathrm{so}(3,\mathbf{C})$, which induces an isomorphism between su(2) and $\mathrm{so}(3,\mathbf{R})$. This isomorphism is defined by the correspondence

$$E_{12} \longleftrightarrow (-i/2)\sigma_3, \tag{6.4.2a}$$

$$E_{13} \longleftrightarrow (-i/2)\sigma_2, \tag{6.4.2b}$$

$$E_{23} \longleftrightarrow (-i/2)\sigma_1. \tag{6.4.2c}$$

Via this passage to 2×2 matrices, the two potentials become

6.4.3*a*, $$\eta^x = \frac{i}{2} \begin{pmatrix} 0 & e^{i(\varphi(x,0)-\varphi(0,0))} \\ e^{i(\varphi(0,0)-\varphi(x,0))} & 0 \end{pmatrix} \mathrm{d}x$$

6.4.3*b* $$\eta^y = -\frac{i}{2} \begin{pmatrix} 0 & e^{-i(\varphi(0,y))} \\ e^{i(\varphi(0,y))} & 0 \end{pmatrix} \mathrm{d}y,$$

respectively.

Remark 6.4.1. The fact that we chose to work with Chebyshev nets ($A = B = 1$, where $A = |\psi_x|$, $B = |\psi_y|$) allows this form of the normalized pair of potentials.

Had we chosen to work with arbitrary A and B (that is, not necessarily a Chebyshev net), a more tedious but straightforward calculation would lead us to the Weierstrass data

6.4.4*a* $$\eta^x = \frac{iA}{2}\begin{pmatrix} 0 & e^{i(\varphi(x,0)-\varphi(0,0))} \\ e^{i(\varphi(0,0)-\varphi(x,0))} & 0 \end{pmatrix}\mathrm{d}x,$$

6.4.4*b* $$\eta^y = -\frac{iB}{2}\begin{pmatrix} 0 & e^{-i(\varphi(0,y))} \\ e^{i(\varphi(0,y))} & 0 \end{pmatrix}\mathrm{d}y.$$

The asymmetry of the two potential comes from the definition of a normalized extended frame. Although the two potentials look asymmetric, one can make the two expressions look similar by gauging with a certain rotation. This will be shown in Section 7.2. The corresponding symmetric potentials are given in (7.2.19 a,b).

Remark 6.4.2. The product of the off-diagonal elements is A^2 and B^2 respectively for η^x and η^y (with a factor of $-1/4$). This is similar to the CMC case, where the meromorphic (normalized) potential has the form

(6.4.5) $$\eta = \begin{pmatrix} 0 & f(z) \\ g(z) & 0 \end{pmatrix}\mathrm{d}z,$$

with $f \cdot g\mathrm{d}z^2 = -Q\mathrm{d}z^2$ (Hopf differential).

For the CMC case, the λ-transformation was given by

(6.4.6) $$\begin{cases} Q \mapsto e^{2it}Q = \lambda^2 Q, \\ \bar{Q} \mapsto e^{-2it}\bar{Q} = \lambda^{-2}Q, \end{cases}$$

while here it is $A \mapsto \lambda A$, $B \mapsto \lambda^{-1}B$, $\lambda = e^t$. So the role played in the case of CMC surfaces by the Hopf differential Q is taken for the case of pseudospherical surfaces by the pair A, B.

The globally defined differential forms $(A^2)\mathrm{d}x^2$ and $(B^2)\mathrm{d}y^2$ are sometimes called Klotz differentials.

Remark 6.4.3. The isomorphism described above in (6.4.2 a,b,c), between su(2) and so(3), is provided by the spinor representation J defined as follows:

$$J : \mathbf{R}^3 \to \mathrm{su}(2),$$

(6.4.7) $$J(x,y,z) = \frac{1}{2}\begin{pmatrix} -iz & -ix-y \\ -ix+y & iz \end{pmatrix},$$

which identifies $\mathbf{R}^3$ and su(2) via

(6.4.8) $$J(\mathbf{r}) = -\frac{i}{2}\mathbf{r}\sigma,$$

where $\mathbf{r}\sigma = r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3$, and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices defined by (6.4.1). Then

(6.4.9) $$J(\mathbf{r}_1 \times \mathbf{r}_2) = [J\mathbf{r}_1, J\mathbf{r}_2].$$

If $\mathcal{U} = (e_1, e_2, e_3)$ is the normalized moving frame of the surface M in asymptotic line parametrization, we define the "2×2" *frame* $P : D \to \mathrm{SU}(2)$ with the initial condition $P(0,0) = I$, via

6.4.10
$$J(e_1) = -\tfrac{i}{2} P\sigma_1 P^{-1},\quad J(e_2) = -\tfrac{i}{2} P\sigma_2 P^{-1},\quad J(e_3) = -\tfrac{i}{2} P\sigma_3 P^{-1}.$$

We have this way a correspondence between all the frames $\mathcal{U}$ in SO(3) and frames P in SU(2).

A tedious but straightforward computation completely similar to the one in [DoHa], Appendix A.4, transfers the 3×3 matrices $\mathcal{A}$ and $\mathcal{B}$ from (6.2.5) to the 2×2 matrices U and V given by the corresponding Lax system:

(6.4.11)
$$U = P^{-1}P_x = \frac{-i}{2}\begin{pmatrix} -\varphi_x & \lambda \\ \lambda & \varphi_x \end{pmatrix}$$

(6.4.12)
$$V = P^{-1}P_y = \frac{i}{2}\lambda^{-1}\begin{pmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}$$

with $P(0,0) = I$.

This Lax system can be also obtained directly from (2.3.15) through the isomorphism between su(2) and so$(3, \mathbf{R})$ defined by (6.4.2 a,b,c), that is

$$E_{12} \longleftrightarrow (-i/2)\sigma_3, \quad E_{13} \longleftrightarrow (-i/2)\sigma_2, \quad E_{23} \longleftrightarrow (-i/2)\sigma_1.$$

7 Examples of Pseudospherical Surfaces

A variety of surfaces of constant negative Gaussian curvature, including the pseudosphere, were presented for the first time in a paper by Ferdinand Minding (*Crelles's Journal*, 1939). Other examples, coming from non-trivial solutions of the sine-Gordon equation were published in the fifties: e.g., one and two solitons, periodic solutions in elliptic functions and some wave packets, described by A. Seeger, H. Douth, and A. Kochendörfer, Theorie der Versetzungen in eindimensionalen Atomreihen, *Z. Phys.* **134**, 1953, 173–193. There are remarkably many such examples, exactly because the sine-Gordon equation is a soliton equation which is completely integrable. This equation was solved by Ablowitz et al. in 1973 via the method of inverse scattering ([ABW]).

In this section, I will only present a few well-known examples of infinite-type pseudospherical surfaces and the normalized potentials associated to them.

1.**The Pseudosphere.** Consider the curve

(7.1.1)
$$y(x) = \sqrt{1 - x^2} - \cosh^{-1}(1/x)$$

and rotate it around the y-axis, as in [Spi], or [Ei]. The surface obtained this way is called *pseudosphere* and its area is 2π.

In the Chebyshev net parametrization, the pseudosphere corresponds to a u_2-independent solution of the sine-Gordon equation

(7.1.2)
$$\frac{\partial^2\varphi}{\partial(u_1)^2} - \frac{\partial^2\varphi}{\partial(u_2)^2} = \sin\varphi,$$

where, as before, $u_1 = x + y$ and $u_2 = x - y$ represent the curvature line coordinates, while x, y represent the asymptotic line coordinates. The angle φ between the asymptotic lines is given by

$$\varphi = 4\tan^{-1} e^{x+y}. \tag{7.1.3}$$

The pseudosphere has a so-called $y = \pm x^{3/2}$-type cusp at the singularity.

The angle φ above corresponds to a Chebyshev net, i.e., parametrized by arc length.

The angle $\varphi(x, y) = \varphi(u_1)$ is a solution to the equation

$$\frac{\partial^2 \varphi}{\partial (u_1)^2} = \sin\varphi, \tag{7.1.4}$$

a particular case of a sine-Gordon. This solution passes through π at the cusp, that is, where $u_1 = x + y = 0$. The two infinite spikes correspond to $\varphi \to 0$ and $\varphi \to 2\pi$, i.e., $x + y \mapsto \pm\infty$, where the two coordinate lines are asymptotically tangent to one another, but the singularity isn't reached, and the spike goes to infinity. Only one spike will show up if φ takes values in $(0, \pi)$, which was the general assumption in all the previous sections of this text.

Proposition 7.1

The symmetric x-potential and y-potential associated with the modified frame $\tilde{\mathcal{U}}$ *of the pseudosphere are, respectively*

$$\tilde{\eta^x} = \begin{pmatrix} 0 & 0 & \alpha(x) \\ 0 & 0 & \beta(x) \\ -\alpha(x) & -\beta(x) & 0 \end{pmatrix} \mathrm{d}x, \tag{7.1.5a}$$

$$\tilde{\eta^y} = \begin{pmatrix} 0 & 0 & a(y) \\ 0 & 0 & b(y) \\ -a(y) & -b(y) & 0 \end{pmatrix} \mathrm{d}y, \tag{7.1.5b}$$

where

$$\alpha(x) = 1 - \frac{8e^{2x}}{(1+e^{2x})^2}, \tag{7.1.6a}$$

$$\beta(x) = \frac{4e^x(e^{2x}-1)}{(1+e^{2x})^2}, \tag{7.1.6b}$$

$$a(y) = \frac{8e^{2y}}{(1+e^{2y})^2} - 1, \tag{7.1.6c}$$

$$b(y) = \frac{4e^y(1-e^{2y})}{(1+e^{2y})^2}. \tag{7.1.6d}$$

Note that formally $\alpha(y) = -a(y)$ and $\beta(y) = -b(y)$.

Proof. A straight-forward computation using elementary trigonometric formulas yields all desired expresions.

2. **1-Soliton Traveling Wave**

The surfaces occuring in the associated family of the pseudosphere are called "1-soliton traveling waves". The pseudosphere is the particular case $\lambda = 1$. For this family of pseudospherical surfaces, the angle $\varphi(x, y)$ between the asymptotic lines is given by

$$\varphi = 4 \tan^{-1} e^{\lambda x + \lambda^{-1} y}, \qquad \lambda > 0. \tag{7.2.1}$$

The 1-soliton travelling waves can be obtained from the pseudosphere via the Lie-Lorentz transformation $x \mapsto \lambda x$, $y \mapsto \lambda^{-1} y$, $\lambda > 0$.

As λ changes away from 1, the surface changes from a pseudosphere into a surface that looks almost like a helicoid. The surface has a self-intersection on the central axis, and the hidden part corresponds to the spike of the pseudosphere.

[Has] specifies the parametrization of this surface as

$$\psi(x, y) = (-2\alpha \operatorname{sech} u \sin \beta u, 2\alpha \operatorname{sech} u \cos \beta u, x - 2\alpha \tanh u),$$

where $\beta = 1/\lambda$ and $\alpha = 2/(\lambda + \lambda^{-1})$.

Here the cusp corresponds to $u = 0$. Any coordinate patch that doesn't intersect the cusp line $u = 0$ must have area less than or equal to 2π. Diagonal strips $-c < u < c$ in the coordinate plane result in infinite areas.

Finally, we note that the x- and y- symmetric potentials of the travelling wave are given by

$$\tilde{\xi^x} = \lambda \cdot \tilde{\eta^x}, \tag{7.2.2}$$

$$\tilde{\xi^y} = \lambda^{-1} \cdot \tilde{\eta^y}, \tag{7.2.3}$$

where $\tilde{\eta^x}$ and $\tilde{\eta^y}$ represent the symmetric potentials of the pseudosphere, given by (7.1.5 a,b) with (7.1.6 a,b).

3. **Amsler's Surface.**

In Chebyshev net parametrization, this surface corresponds to an angle $\varphi(x, y)$ that is constant on both x- and y-axes, which will be expressed by (7.3.2).

While $\varphi(x, y)$ cannot be written explicitely, we can write the sine-Gordon equation in a very simple form ([Me, St, 2]):

Let $t := xy$ with $(x, y) \in D = \mathbf{R}^2$. If we express $\varphi(x, y) = h(xy)$, with $h : \mathbf{R} \to (0, \pi)$ a differentiable function, then

$$\frac{d}{dt}(t \cdot \frac{dh}{dt}) = \sin h(t) \tag{7.3.1}$$

represents the sine-Gordon equation.

Since $\varphi(x, y)$ is smooth, a straight-forward calculation yields

$$\varphi(0, 0) = \varphi(x, 0) = \varphi(0, y) := \varphi_0 \tag{7.3.2}$$

for every pair $(x, y) \in D$.

Amsler ([Ams]) investigated this surface for values $\varphi \in [0, \pi]$. He showed that the solution $\varphi(x, y) = h(xy)$ oscillates near π when $t > 0$ and near 0 when $t < 0$. He also proved that the surface has two cuspidal edges corresponding to $\varphi = 0$ and $\varphi = \pi$, respectively.

We also note the two straight-lines contained in the Amsler surface, corresponding to $x = 0$ and $y = 0$.

As an obvious consequence of the angle being constant along the axes, the *normalized potentials of the pseudosphere are*:

$$\eta^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \mathrm{d}x, \tag{7.3.3a}$$

$$\eta^y = \begin{pmatrix} 0 & 0 & \sin\varphi_0 \\ 0 & 0 & \cos\varphi_0 \\ -\sin\varphi_0 & -\cos\varphi_0 & 0 \end{pmatrix} \mathrm{d}y. \tag{7.3.3b}$$

Here we must note that formulas representing symmetric potentials are written in this case

$$\tilde{\eta^x} = \begin{pmatrix} 0 & 0 & \sin\frac{\varphi_0}{2} \\ 0 & 0 & -\cos\frac{\varphi_0}{2} \\ -\sin\frac{\varphi_0}{2} & \cos\frac{\varphi_0}{2} & 0 \end{pmatrix} \mathrm{d}x, \tag{7.3.4a}$$

$$\tilde{\eta^y} = \begin{pmatrix} 0 & 0 & \sin\frac{\varphi_0}{2} \\ 0 & 0 & \cos\frac{\varphi_0}{2} \\ -\sin\frac{\varphi_0}{2} & -\cos\frac{\varphi_0}{2} & 0 \end{pmatrix} \mathrm{d}y. \tag{7.3.4b}$$

In the 2×2 matrix approach, they appear as:

$$\tilde{\eta^x} = \frac{i}{2} \begin{pmatrix} 0 & e^{i\frac{\varphi_0}{2}} \\ e^{-i\frac{\varphi_0}{2}} & 0 \end{pmatrix} \mathrm{d}x \tag{7.3.5a}$$

$$\tilde{\eta^y} = -\frac{i}{2} \begin{pmatrix} 0 & e^{-i\frac{\varphi_0}{2}} \\ e^{i\frac{\varphi_0}{2}} & 0 \end{pmatrix} \mathrm{d}y. \tag{7.3.5b}$$

Remark 7.0.4. For Amsler surfaces, the sine-Gordon equation is written as the second order differential equation

$$th''(t) + h'(t) = \sin(h(t)). \tag{7.3.6}$$

Note that a change of function $w = e^{i\psi}$ will transform equation (7.3.6) into the so-called third Painleve equation.

Appendix

By definition, the deformation parameter λ that generates an associated family of pseudospherical surfaces is real and positive, $\lambda = e^t$. In general, λ can be real and negative as well. Our choice is motivated by the convenience of working within the connected Banach loop group

$$(\Lambda \mathrm{SO}(3)_P, \|\cdot\|) = \{g : \mathbf{R}_+ \to \mathrm{SO}(3, \mathbf{R}) \mid Pg(\lambda)P^{-1} = g(-\lambda)\},$$

endowed with the norm $\|\cdot\|$ defined by (4.1.16).

The goal of this appendix is to show that we can split à la Birkhoff any extended frame $\mathcal{U}^\lambda$ which admits an analytic extension on $\mathbf{C}_*$.

For our purpose, it is useful to extend the real positive parameter λ such that we can apply the Birkhoff splitting to complex loop groups with loop parameter in S^1.

Let us first consider the Lax system

$$\text{(A.1)} \qquad \begin{cases} \mathcal{U}^{-1} \cdot \partial_x \mathcal{U} = -\varphi_x \cdot E_{12} + \mu \cdot E_{23} \\ \mathcal{U}^{-1} \cdot \partial_y \mathcal{U} = \mu^{-1} \cdot (-\sin\varphi \cdot E_{13} - \cos\varphi \cdot E_{23}), \end{cases}$$

where $\mu \in \mathbf{C}_* = \mathbf{C} \setminus \{0\}$.

Clearly, the Lax system (2.3.15) is the same as (A.1) if we restrict μ to $\mathbf{R}_+$. **Lemma A.1** *Every solution* $\mathcal{U}$ *to* (A.1) *with initial condition* $\mathcal{U}(0,0,\mu) = I$ *is analytic in* $\mu \in \mathbf{C}_*$ *and*

$$\text{(A.2)} \qquad \overline{\mathcal{U}(x, y, \bar{\mu})} = \mathcal{U}(x, y, \mu).$$

Proof. Note that the right-hand side of $A.1$ is analytic in $\mu \in \mathbf{C}_*$. Since the initial condition is analytic in $\mu \in \mathbf{C}_*$, it follows that, for every x and y arbitrarily fixed, the solution $\mathcal{U}$ of (A.1) is also analytic in $\mu \in \mathbf{C}_*$. Relation (A.2) is straight-forward, as a consequence of the reality of φ. □

In order to use the classical loop group factorization, let us choose $\lambda \in S^1$. We consider the restriction to S^1 of the extended frame $\mathcal{U}$ satisfying the Lax system (A.1), and will denote it $\mathcal{U}^\lambda$.

Taking into consideration the property (A.2) of $\mathcal{U}^\lambda$, we introduce the following group of continuous maps:

$$\text{A.3} \qquad \begin{aligned} H_P = \{A : S^1 \to \mathrm{SO}(3, \mathbf{C}) \mid \ & A \text{ continuous}, \overline{A(\bar{\lambda})} = A(\lambda), \\ & P \cdot A(\lambda) \cdot P^{-1} = A(-\lambda)\}, \end{aligned}$$

with the supplementary condition

$$\text{(A.4)} \qquad \|A\| = \sum_{k \in \mathbf{Z}} \|A_k\| < \infty,$$

where

$$A(\lambda) = \sum_{k \in \mathbf{Z}} A_k \cdot \lambda^k,$$

and

(A.5) $$\|B\| = \max_i\{\sum_{j=1}^{3}|B_{ij}|\},$$

for every λ-independent 3×3 matrix B.

H_P is a Banach Lie group with respect to the norm $\|\cdot\|$.

(Note that we have used the same symbol $\|\cdot\|$ for different entities. We hope this will not lead to any confusion).

Clearly, (A.3) expresses the reality of the coefficient matrices in the Fourier expansion.

Proposition A.1 *For the group H_P we define $(H_P^-)_*$ and H_P^+ as in Section 4.3. The multiplication $(H_P^-)_* \times H_P^+ \to (H_P^-)_* \cdot H_P^+$ is an analytic diffeomorphism onto the open and dense subset $(H_P^-)_* \cdot H_P^+$, called the "big cell". In particular, if $A \in H_P$ is contained in the big cell, then A has a unique decomposition*

(A.6) $$A = A_- A_+,$$

where $A_- \in (H_P^-)_*$ and $A_+ \in H_P^+$. The analogous result holds for the multiplication map $(H_P^+)_* \times H_P^- \to (H_P^+)_* \cdot H_P^-$.

Proof. Let $A : S^1 \to \mathrm{SO}(3,\mathbf{C})$ be an element of H_P. By the definition of H_P, we have

$$A(\lambda) = \overline{(A(\bar\lambda)}, \qquad \text{for every } \lambda \in S^1.$$

On the other hand, by Theorem 4.3.1, $A(\lambda)$ can be decomposed à la Birkhoff in a big cell of $\Lambda\mathrm{SO}(3,\mathbf{C})$, as

(A.7), $$A(\lambda) = A_-(\lambda)A_+(\lambda), \qquad \text{for every } \lambda \in S^1,$$

$A_-(\lambda) \in \Lambda_*^-\mathrm{SO}(3,\mathbf{C})$, $A_+(\lambda) \in \Lambda^+\mathrm{SO}(3,\mathbf{C})$.

As a consequence of (A.6) and (A.7), we obtain

(A.8) $$A(\lambda) = \overline{A_-(\bar\lambda)} \cdot \overline{A_+(\bar\lambda)}.$$

Also, (A.7) and (A.8) yield

$$A_-(\lambda)^{-1} \cdot \overline{A_-(\bar\lambda)} = A_+(\lambda) \cdot \overline{A_+(\bar\lambda)}^{-1}.$$

The left-hand side is an element of $\Lambda_*^-\mathrm{SO}(3,\mathbf{C})$, while the right-hand side is an element of $\Lambda^+\mathrm{SO}(3,\mathbf{C})$, and hence both sides are equal to the identity matrix. Therefore,

A.9 $$\begin{aligned}\overline{A_-(\bar\lambda)} &= A_-(\lambda)\\ \overline{A_+(\bar\lambda)} &= A_+(\lambda), \qquad \text{for every } \lambda \in S^1.\end{aligned}$$

Hence, A_+ and A_- satisfy the first condition (A.6) from the definition of the group H_P, meaning that their coefficient matrices are real.

On the other hand, the symmetry condition

$$A(-\lambda) = P \cdot A(\lambda) \cdot P^{-1} = P \cdot A_-(\lambda) \cdot P^{-1} \cdot P \cdot A_+(\lambda) \cdot P^{-1},$$

together with the uniqueness of the Birkhoff splitting

$$A(-\lambda) = A_-(-\lambda) \cdot A_+(-\lambda),$$

yield the symmetry condition for A_-, A_+:

$$A_-(-\lambda) = P \cdot A_-(\lambda) \cdot P^{-1}$$
$$A_+(-\lambda) = P \cdot A_+(\lambda) \cdot P^{-1}.$$

Thus, the Birkhoff factorization holds for H_P. The analytic diffeomorphism $(H_P^-)_* \times H_P^+ \to (H_P^-)_* \cdot H_P^+$ is a particularization of the analytic diffeomorphism analyzed in Theorem 4.2.1. □

Theorem 4.3.2. states the Birkhoff splitting for arbitrary elements g in the Banach loop group $\Lambda\mathrm{SO}(3)_P$ which admit an analytic extension to $\mathbf{C}_*$.

Now we are ready to present its proof.

Proof of Theorem 4.3.2. Let $g : \mathbf{R}_+ \to \mathrm{SO}(3)$, $g(-\lambda) = P \cdot g(\lambda) \cdot P^{-1}$, be an element of the Banach loop group $\Lambda\mathrm{SO}(3)_p$ that has an analytic extension $\tilde{g}$ to $\mathbf{C}_*$.

Set $A := \tilde{g}|_{S^1}$.

Since $g \in \Lambda\mathrm{SO}(3)_P$, the matrix coefficients of g are real, that is

$$A(\lambda) = \sum_{k \in \mathbf{Z}} A_k \lambda^k, \qquad \lambda \in S^1,$$

Note that $A \in H_P$, where H_P denotes the loop group defined by (A.3). The algebraic conditions are obviously satisfied. Also, by [GO], Theorem 1.4, analytic functions satisfy the finite norm condition.

Here we are only interested in elements A belonging to the big cell of H_P.

The previous proposition shows that the Birkhoff splitting holds for the big cell of H_P.

Then, let $A_- \in (H_P^-)_*$ and $A_+ \in H_P^+$ be such that

$$A = A_- A_+,$$

where

$$A_- = I + \sum_{k<0} A_k \lambda^k,$$
$$A_+ = \sum_{k \geq 0} A_k \lambda^k, \qquad \lambda \in S^1.$$

We need to show that A_- and A_+ admit analytic extensions to $\mathbf{C}_*$.

By our hypothesis, A has an analytic extension to $\mathbf{C}_*$. The element A_- admits an analytic extension to the exterior of the unit circle S^1. Therefore, $(A_-)^{-1} A = A_+$ can be extended analytically outside of the unit disk.

On the other hand, A_+ admits an analytic extension inside the unit disk. Thus, by analytic prolongation, A_+ admits an analytic extension to $\mathbf{C}_*$.

From $A(A_+)^{-1} = A_-$, it follows next that A_- also admits an analytic extension to $\mathbf{C}_*$.

Let $\tilde{A}_-$ and $\tilde{A}_+$ be the analytic extensions of A_- and A_+ to $\mathbf{C}_*$, respectively.

Next, let g_- and g_+ denote their restrictions to $\mathbf{R}_+$:

$$g_- = \tilde{A}_-|_{\mathbf{R}_+}, \quad g_+ = \tilde{A}_+|_{\mathbf{R}_+}.$$

Clearly, g, g_- and g_+ have analytic extensions to $\mathbf{C}_*$, respectively: $\tilde{g}$, $\tilde{A}_-$ and $\tilde{A}_+$ such that

$$\tilde{g}|_{S^1} = A = A_- \cdot A_+ = \tilde{A}_-|_{S^1} \cdot \tilde{A}_+|_{S^1},$$

that is $\tilde{g}$ and $\tilde{A}_-\tilde{A}_+$ coincide on S^1. Therefore, $\tilde{g}$ and $\tilde{A}_-\tilde{A}_+$ will coincide on $\mathbf{R}_+$ as well, and $g = g_-g_+$ is a unique factorization.

This proves the splitting.

It remains to prove that $\tilde{\Lambda}^-_* SO(3)_P \times \tilde{\Lambda}^+ SO(3)_P \to \tilde{\Lambda} SO(3)_P$ is a diffeomorphism onto the open and dense subset $\tilde{\Lambda}^-_* SO(3)_P \cdot \tilde{\Lambda}^+ SO(3)_P$.

Note that $\tilde{\Lambda} SO(3)_P$ is a subgroup of $\Lambda SO(3)_P$ with the induced topology. On the other hand, it is natural to view the diffeomorphism $\Lambda^-_* SO(3)_P \times \Lambda^+ SO(3)_P \to \Lambda SO(3)_P$ as a restriction of the analytic diffeomorphism $(H^-_P)_* \times H^+_P \to (H^-_P)_* \cdot H^+_P$ from Proposition A.1.

Consequently, we have the the induced diffeomorphism $\tilde{\Lambda}^-_* SO(3)_P \times \tilde{\Lambda}^+ SO(3)_P \to \tilde{\Lambda} SO(3)_P$. □

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