Homogeneous Geodesics in Five-Dimensional Generalized Symmetric Spaces

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

O.Kowalski and J.Szenthe [7] proved that every homogeneous Riemannian manifold admits a homogeneous geodesic, that is, a geodesic which is an orbit of a one-parameter group of isometries. Then, several authors investigated the set of all homogeneous geodesics of some homogeneous spaces.

In this paper, we study the set of homogeneous geodesics of five-dimen- sional generalized symmetric spaces and we find several interesting behaviours.

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1 Introduction

Let (M, g) be a (connected) homogeneous space, that is, a Riemannian manifold admitting a connected group of isometries G, acting transitively and effectively on M. Then, M can be identified with $(G/H, g)$, where H is the isotropy group at a fixed point o of M. The Lie algebra $\underline{\mathbf{g}}$ of G has a reductive decomposition $\underline{\mathbf{g}} = \mathbf{m} \oplus \underline{\mathbf{h}}$, where $\mathbf{m} \subset \underline{\mathbf{g}}$ is a subspace of $\underline{\mathbf{g}}$ isomorphic to the tangent space $T_o(M)$ and $\underline{\mathbf{h}}$ is the Lie algebra of H. In general, such decomposition is not unique. A geodesic $\gamma(t)$ through the origin o of $M = G/H$ is called *homogeneous* if it is an orbit of a one-parameter subgroup of G, that is

$$\gamma(t) = \exp(tZ)(o), \quad t \in \mathbb{R}, \tag{1.1}$$

where Z is a nonzero vector of $\underline{\mathbf{g}}$.

A homogeneous Riemannian manifold is called a *g.o. space* if all geodesics are homogeneous with respect to the largest connected group of isometries. All naturally reductive spaces are g.o.spaces, but the converse does not hold. In fact, A. Kaplan [3] proved the existence of g.o. spaces which are in no way naturally reductive; the examples of A.Kaplan are generalized Heisenberg groups with two-dimensional center. In [8], O.Kowalski and L.Vanhecke gave a classification of all g.o.spaces (which are in no way naturally reductive) up to dimension six.

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About the existence of homogeneous geodesics in a general homogeneous Riemannian manifold, V.V.Kajzer [2] proved that a Lie group endowed with a left-invariant metric admits at least one homogeneous geodesic. More recently, O.Kowalski and J.Szenthe [7] proved that *any homogeneous Riemannian space $M = G/H$ admits at least one homogeneous geodesic through the origin.* They also proved that if G is semi-simple, then M admits $n = \dim M$ mutually orthogonal homogeneous geodesics through the origin.

A natural problem is then to study the set of all homogeneous geodesics of a homogeneous Riemannian manifold. Several authors investigated the sets of homogeneous geodesics on some types of homogeneous spaces (see [6], [9], [10]).

In this paper, we consider five-dimensional generalized symmetric spaces. The study of homogeneous geodesics in these homogeneous spaces is of particular interest, because their Lie groups in general are not semi-simple. For the ones of order 6 (type 9, see [4]), the Lie group is solvable. In section 2 we shall recall some basic facts about homogeneous geodesics in homogeneous Riemannian manifolds. In sections 3,4,5,6,7 and 8 we study the sets of all homogeneous geodesics of five-dimensional generalized symmetric spaces of type $2, 3, 4, 7, 8$ and 9, respectively, which are all the examples of five-dimensional generalized symmetric spaces which are not g.o. spaces. The most interesting results we found concern homogeneous geodesics of five-dimensional generalized symmetric spaces of type $7a$ and 9. Some other cases, in particular types 3 and 8, are interesting because they present quite complicated sets of geodesic vectors.

2 Preliminaries on homogeneous geodesics and generalized symmetric spaces

Let $(M = G/H, g)$ be a homogeneous Riemannian manifold with a fixed origin o, $\underline{\mathbf{g}}$ and $\underline{\mathbf{h}}$ the Lie algebras of G and H respectively and

$$\underline{\mathbf{g}} = \mathbf{m} \oplus \underline{\mathbf{h}} \tag{2.1}$$

a reductive decomposition. The canonical projection $p : G \to G/H$ induces an isomorphism between the subspace $\mathbf{m}$ and the tangent space $T_o(M)$. Consequently, the scalar product g_o on $T_o(M)$ induces a scalar product $<,>$ on $\mathbf{m}$ which is $\mathrm{Ad}(H)$-invariant. A non-zero vector $Z \in \underline{\mathbf{g}}$ is called a *geodesic vector* if the curve $\exp tX(o)$ is a geodesic. We recall the following characterization of geodesic vectors:

Lemma 2.1 [8] *A non-zero vector $X \in \underline{\mathbf{g}}$ is a geodesic vector if and only if*

$$< [X, Y]_{\mathbf{m}}, X_{\mathbf{m}} >= 0 \tag{2.2}$$

for all $Y \in \mathbf{m}$ (the subscript $\mathbf{m}$ denotes the projection into $\mathbf{m}$).

Therefore, looking for all homogeneous geodesics of a homogeneous Riemannian manifold $(M = G/H, g)$, we first calculate the connected component G of the full isometry group I(M), or at least the corresponding Lie algebra $\underline{\mathbf{g}}$. Then, we find a decomposition of the form (2.1) and look for the geodesic vectors in the form

$$Z = \sum_{i=1}^{r} x_i e_i + \sum_{j=1}^{s} a_j A_j \tag{2.3}$$

where $\{e_i\}_{i=1,2,...,r}$ is a convenient basis of $\mathbf{m}$ and $\{A_j\}_{j=1,2,...,s}$ is a basis of $\underline{\mathbf{h}}$. When we take $Y = e_i$, $i = 1, 2, ..., r$, the condition (2.2) produces a system of r quadratic equations for the variables x_i and a_j . Then, we see for which values of $x_1, x_2, ..., x_r$ and $a_1, a_2, ..., a_s$ this system is satisfied. To such solutions, for which $x_1, x_2, ..., x_r$ are not all equal to zero, correspond geodesic vectors (see also [6]).

A finite family $\{\gamma_1, \gamma_2, ..., \gamma_k\}$ of homogeneous geodesics through $o \in M$ is said to be *orthogonal* (respectively, *linearly independent*) if the corresponding initial tangent vectors at o are orthogonal (resp., linearly independent). The following result holds:

Proposition 2.2 [6] *A finite family $\{\gamma_1, \gamma_2, ..., \gamma_k\}$ of homogeneous geodesics through $o \in M$ is orthogonal (respectively, linearly independent) if the* $\mathbf{m}$*-components of the corresponding geodesic vectors are orthogonal (respectively, linearly independent).*

We now recall some basic facts about generalized symmetric spaces. A *generalized symmetric space* is a connected Riemannian manifold (M, g) admitting a *regular s-structure*, that is, a family $\{s_x : x \in M\}$ of symmetries on M, such that

$$s_x \circ s_y = s_z \circ s_x, \quad z = s_x(y),$$

for every points $x, y, \in M$ [5]. As it is well-known, every generalized symmetric space is a homogeneous Riemannian space G/H [4]. An s-structure $\{s_x : x \in M\}$ is said to be *of order* $k \geq 2$ if $(s_x)^k = id$ for all $x \in M$ and $(s_x)^i \neq id$ for $i < k$. A Riemannian manifold (M, g) is said to be *k-symmetric* if it admits a regular s-structure of order k. Each generalized symmetric space is k-symmetric for some k [4]. The *order* of a generalized symmetric space is the least integer k such that (M, g) is k-symmetric.

Low-dimensional generalized symmetric spaces have been classified [4], [5]. Comparing this classification with the classification of low-dimensional g.o. spaces given in [8], we see that the generalized symmetric spaces which are not g.o. spaces are the ones of type 2, 3, 4, 7, 8 (all of order 4) and 9 (of order 6) in the classification given in [5].

For all these spaces, in [4] it is given a basis of $\underline{\mathbf{g}}$, containing a basis $\{X_1, Y_1, X_2, Y_2, W\}$ of $\mathbf{m}$, with respect to which the Lie bracket $[,]$ of $\underline{\mathbf{g}}$ and the scalar product $<, >$ of $\mathbf{m}$ are explicitly described.

According to Lemma 2.1, a non-zero vector $X \in \underline{\mathbf{g}}$ is geodesic if and only if

$$< [X, Y]_{\mathbf{m}}, X_{\mathbf{m}} >= 0$$

for all $Y \in \mathbf{m}$ or, equivalently, if and only if

$$\left\{\begin{array}{l} < [X, X_1]_{\mathbf{m}}, X_{\mathbf{m}} >= 0, \\ < [X, X_2]_{\mathbf{m}}, X_{\mathbf{m}} >= 0, \\ < [X, Y_1]_{\mathbf{m}}, X_{\mathbf{m}} >= 0, \\ < [X, Y_2]_{\mathbf{m}}, X_{\mathbf{m}} >= 0, \\ < [X, W]_{\mathbf{m}}, X_{\mathbf{m}} >= 0. \end{array}\right. \tag{2.4}$$

3 Homogeneous geodesics of generalized symmetric spaces of type 2

A five-dimensional generalized symmetric space M of type 2 is $\mathbb{R}^5(x,y,z,w,t)$, equipped with the Riemannian metric

$$\begin{aligned} g \;=\;& e^{-2\lambda_1 t}dx^2 + e^{2\lambda_1 t}dy^2 + e^{-2\lambda_2 t}dz^2 + e^{2\lambda_2 t}dw^2 + dt^2 + \\ & 2\alpha[e^{-(\lambda_1+\lambda_2)t}dxdz + e^{(\lambda_1+\lambda_2)t}dydw] + \\ & +2\beta[e^{(\lambda_1-\lambda_2)t}dydz - e^{(\lambda_2-\lambda_1)t}dxdw], \end{aligned}$$

where either $\lambda_1 > \lambda_2 > 0$, $\alpha^2+\beta^2 < 1$, or $\lambda_1 = \lambda_2 > 0$, $\alpha = 0$ and $0 \leq \beta < 1$, or $\lambda_1 < 0$, $\lambda_2 = 0$, $\alpha = 0$ and $0 < \beta < 1$. As homogeneous space, $M = G/H$, where G is the group of all matrices of the form

$$\begin{pmatrix} e^{\lambda_1 t} & 0 & 0 & 0 & x \\ 0 & e^{-\lambda_1 t} & 0 & 0 & y \\ 0 & 0 & e^{\lambda_2 t} & 0 & z \\ 0 & 0 & 0 & e^{-\lambda_2 t} & w \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The linear subspace $\mathbf{m}$ of $\underline{\mathbf{g}}$ admits a basis $\{X_1, Y_1, X_2, Y_2, W\}$ such that

$$[X_j, W] = -\lambda_j X_j, \quad [Y_j, W] = \lambda_j Y_j, \quad [\cdot,\cdot] = 0 \text{ otherwise}.$$

2a): $\lambda_1 > \lambda_2 > 0$ and $\alpha^2+\beta^2 < 1$.

In this case, $\underline{\mathbf{h}} = 0$ [4]. The Lie bracket $[,]$ and the Riemannian metric $<,>$ are respectively determined by

(3.1)

$[\,,\,]$	X_1	X_2	Y_1	Y_2	W
X_1	0	0	0	0	$-\lambda_1 X_1$
X_2	0	0	0	0	$-\lambda_2 X_2$
Y_1	0	0	0	0	$\lambda_1 Y_1$
Y_2	0	0	0	0	$\lambda_2 Y_2$
W	$\lambda_1 X_1$	$\lambda_2 X_2$	$-\lambda_1 Y_1$	$-\lambda_2 Y_2$	0

and

(3.2)

$<\,,\,>$	X_1	X_2	Y_1	Y_2	W
X_1	1	α	0	$-\beta$	0
X_2	α	1	β	0	0
Y_1	0	β	1	α	0
Y_2	$-\beta$	0	α	1	0
W	0	0	0	0	1

(see [4]). Using (3.1) and (3.2) to compute (2.4), we obtain that $X \in \underline{\mathbf{g}} = \mathbf{m}$ is geodesic if and only if its components (x_1, x_2, y_1, y_2, w) with respect to $\{X_1, X_2, Y_1, Y_2, W\}$ satisfy

$$(3.3) \qquad \begin{cases} w(x_1 + \alpha x_2 - \beta y_2) = 0, \\ w(\alpha x_1 + x_2 + \beta y_1) = 0, \\ w(\beta x_2 + y_1 + \alpha y_2) = 0, \\ w(-\beta x_1 + \alpha y_1 + y_2) = 0, \\ -\lambda_1 x_1(x_1 + \alpha x_2 - \beta y_2) - \lambda_2 x_2(\alpha x_1 + x_2 + \beta y_1) + \\ +\lambda_1 y_1(\beta x_2 + y_1 + \alpha y_2) + \lambda_2 y_2(-\beta x_1 + \alpha y_1 + y_2) = 0, \end{cases}$$

where we took into account that $\lambda_1 > \lambda_2 > 0$.

If $w \neq 0$, (3.4) gives easily $x_1 = x_2 = y_1 = y_2 = 0$. If $w = 0$, the solutions of (3.4) are given by $(x_1, x_2, y_1, y_2, 0)$, satisfying

$$(3.4) \qquad \begin{aligned} &-\lambda_1 x_1(x_1 + \alpha x_2 - \beta y_2) - \lambda_2 x_2(\alpha x_1 + x_2 + \beta y_1) + \\ &+\lambda_1 y_1(\beta x_2 + y_1 + \alpha y_2) + \lambda_2 y_2(-\beta x_1 + \alpha y_1 + y_2) = 0, \end{aligned}$$

Hence, we proved that X is a geodesic vector of a generalized symmetric space of type $2a$ if and only if

(i) $X = wW$, or
(ii) $X = x_1X_1 + x_2X_2 + y_1Y_1 + y_2Y_2$ and (3.4) holds.

Geometrically speaking, geodesic vectors of a generalized symmetric space of type $2a$ form *a straight line* (i) and *a hypercone* (of equation (3.4)) *in the orthogonal complement of such line* (ii).

Note that W is a geodesic vector of type (i), while X_1+Y_1, X_1-Y_1, $-(\alpha-\beta)X_1+X_2-(\alpha+\beta)Y_1+Y_2$ and $-(\alpha+\beta)X_1+X_2+(\alpha-\beta)Y_1-Y_2$ are geodesic vectors of type (ii) and all together they form an orthogonal basis of geodesic vectors.

2b): $\lambda_1 = \lambda_2 > 0$, $\alpha = 0$ and $\beta > 0$.

In this case, $\underline{\mathbf{h}} = so(2) = span(A)$, where A is determined by $AX_1 = X_2$, $AX_2 = -X_1$, $AY_1 = Y_2$, $AY_2 = -Y_1$, $AW = 0$ (see [4]). Since $[A, X] = AX$ for all $X \in \underline{\mathbf{g}}$, we get

(3.5)

$[\,,\,]$	X_1	X_2	Y_1	Y_2	W	A
X_1	0	0	0	0	$-\lambda_1 X_1$	$-X_2$
X_2	0	0	0	0	$-\lambda_2 X_2$	X_1
Y_1	0	0	0	0	$\lambda_1 Y_1$	$-Y_2$
Y_2	0	0	0	0	$\lambda_2 Y_2$	Y_1
W	$\lambda_1 X_1$	$\lambda_2 X_2$	$-\lambda_1 Y_1$	$-\lambda_2 Y_2$	0	0
A	X_2	$-X_1$	Y_2	$-Y_1$	0	0

and

(3.6)

$<\,,\,>$	X_1	X_2	Y_1	Y_2	W
X_1	1	0	0	$-\beta$	0
X_2	0	1	β	0	0
Y_1	0	β	1	0	0
Y_2	$-\beta$	0	0	1	0
W	0	0	0	0	1

We computed $<,>$ as in case $2a$), taking into account $\alpha = 0$. $X \in \underline{\mathbf{g}}$ is geodesic if and only if its components $(x_1, x_2, y_1, y_2, w, a)$ with respect to the basis $\{X_1, Y_1, X_2, Y_2, W, A\}$ of $\underline{\mathbf{g}}$ satisfy

$$(3.7) \qquad \begin{cases} \lambda_1 w(x_1 - \beta y_2) + a(x_2 + \beta y_1) = 0, \\ \lambda_2 w(x_2 + \beta y_1) - a(x_1 - \beta y_2) = 0, \\ -\lambda_1 w(\beta x_2 + y_1) + a(-\beta x_1 + y_2) = 0, \\ -\lambda_2 w(-\beta x_1 + y_2) - a(\beta x_2 + y_1) = 0, \\ -\lambda_1 x_1(x_1 - \beta y_2) - \lambda_2 x_2(x_2 + \beta y_1) + \\ +\lambda_1 y_1(\beta x_2 + y_1) + \lambda_2 y_2(-\beta x_1 + y_2) = 0. \end{cases}$$

If $a \neq 0$, then the solutions of (3.7) are given by $x_1 = x_2 = y_1 = y_2 = 0$. For $a = 0$, we put $a_1 = x_1 - \beta y_2$, $a_2 = x_2 + \beta y_1$, $a_3 = \beta x_2 + y_1$ and $a_4 = -\beta x_1 + y_2$. The system (3.7) reduces to

$$(3.8) \qquad \begin{cases} w a_i = 0, \quad i = 1, 2, 3, 4, \\ -\lambda_1 x_1 a_1 - \lambda_2 a_2 \lambda_1 y_1 a_3 + \lambda_2 y_2 a_4 = 0. \end{cases}$$

If $w \neq 0$, (3.8) gives $a_i = 0$, $i = 1, .., 4$, from which it follows easily $x_1 = x_2 = y_1 = y_2 = 0$. If $w = 0$, then (3.8) reduces to the last equation, which gives $x_1^2 + x_2^2 = y_1^2 + y_2^2$. Hence, X is a geodesic vector of a generalized symmetric space of type $2b$ if and only if its $\mathbf{m}$-component is

(i) $X_{\mathbf{m}} = wW$, or
(ii) $X_{\mathbf{m}} = x_1 X_1 + x_2 X_2 + y_1 Y_1 + y_2 Y_2$ and $x_1^2 + x_2^2 = y_1^2 + y_2^2$.

We can check easily that $\{W, X_1 + Y_2, X_1 - Y_2, X_2 + Y_1, X_2 - Y_1\}$ is an orthogonal basis of geodesic vectors of $\mathbf{m}$.

2c): $\lambda_1 > 0$, $\lambda_2 = 0$, $\alpha = 0$ and $0 < \beta < 1$.

In this case, $\underline{\mathbf{h}} = so(2) \bigoplus so(2) = span(A_1, A_2)$, where $A_1 = A$ of case $2b$), while A_2 is determined by $A_2 X_1 = X_2$, $A_2 X_2 = -X_1$, $A_2 Y_1 = -Y_2$, $A_2 Y_2 = Y_1$, $A_2 W = 0$ (see [4]). Similarly to case $2b$), we get

(3.9)

$[\,,\,]$	X_1	X_2	Y_1	Y_2	W	A_1	A_2
X_1	0	0	0	0	$-\lambda_1 X_1$	$-X_2$	$-X_2$
X_2	0	0	0	0	0	X_1	X_1
Y_1	0	0	0	0	$\lambda_1 Y_1$	$-Y_2$	Y_2
Y_2	0	0	0	0	0	Y_1	$-Y_1$
W	$\lambda_1 X_1$	0	$-\lambda_1 Y_1$	0	0	0	0
A_1	X_2	$-X_1$	Y_2	$-Y_1$	0	$-$	$-$
A_2	X_2	$-X_1$	$-Y_2$	Y_1	0	$-$	$-$

and

(3.10)

$<,>$	X_1	X_2	Y_1	Y_2	W
X_1	1	0	0	$-\beta$	0
X_2	0	1	β	0	0
Y_1	0	β	1	0	0
Y_2	$-\beta$	0	0	1	0
W	0	0	0	0	1

$X \in \underline{\mathbf{g}}$ is geodesic if and only if its components $(x_1, x_2, y_1, y_2, w, a, b)$ with respect to $\{X_1, Y_1, X_2, Y_2, W, A_1, A_2\}$ satisfy

$$\begin{cases} \lambda_1 w(x_1 - \beta y_2) + (a+b)(x_2 + \beta y_1) = 0, \\ -(a+b)(x_1 - \beta y_2) = 0, \\ -\lambda_1 w(\beta x_2 + y_1) + (a-b)(-\beta x_1 + y_2) = 0, \\ -(a-b)(\beta x_2 + y_1) = 0, \\ -\lambda_1 x_1(x_1 - \beta y_2) + \lambda_1 y_1(\beta x_2 + y_1) = 0. \end{cases} \tag{3.11}$$

If $a \neq \pm b$, then $x_1 = x_2 = y_1 = y_2 = 0$. If $a = b = 0$, then either $w = 0$ and $x_1(x_1 - \beta y_2) + y_1(\beta x_2 + y_1) = 0$, or $w \neq 0$ and $x_1 = \beta y_2$, $y_1 = -\beta x_2$. If $a = b \neq -b$, then $x_1 = \beta y_2$ and $x_2 = y_1 = 0$, while for $a = -b \neq b$, we obtain $x_1 = y_2 = 0$ and $y_1 = -\beta x_2$. Hence, if X is a geodesic vector of a generalized symmetric space of type $2c$, then its $\mathbf{m}$-component is

- (i) $X_{\mathbf{m}} = x_1 X_1 + x_2 X_2 + y_1 Y_1 + y_2 Y_2$ with $x_1(x_1 - \beta y_2) + y_1(\beta x_2 + y_1) = 0$, or
- (ii) $X_{\mathbf{m}} = \beta y_2 X_1 + x_2 X_2 - \beta x_2 Y_1 + y_2 Y_2 + wW$.

We can check easily that $X_1 + Y_1$, $X_1 - Y_1$ (type (i)), together with $X_2 - \beta Y_1 + W, X_2 - \beta Y_1 - (1 - \beta^2)W$ and $\beta X_1 + Y_2$ (type (ii)), form an orthogonal basis of $\mathbf{m}$.

From the study of cases $2a$), $2b$) and $2c$) and taking into account Proposition 2.2, we can conclude that *all five-dimensional generalized symmetric spaces of type* 2 *admit five mutually orthogonal homogeneous geodesics through the origin.*

4 Homogeneous geodesics of generalized symmetric spaces of type 3

A five-dimensional generalized symmetric space M of type 3 is the homogeneous space $M = SO(3, \mathbf{C})/SO(2)$, where $SO(3, \mathbf{C})$ is the special complex orthogonal group and the Riemannian metric of M is induced by a real invariant positive semi-definite form of $GL(3, C)$ (see [4]). The subalgebra is $\underline{\mathbf{h}} = so(2) = span(A)$, where $AX_1 = X_2$, $AX_2 = -X_1$, $AY_1 = Y_2$, $AY_2 = -Y_1$, $AW = 0$, with respect to a basis $\{X_1, X_2, Y_1, Y_2, W\}$ of $\mathbf{m}$. The Lie bracket $[,]$ and the Riemannian metric $<,>$ are respectively given by

(4.1)

[,]	X_1	X_2	Y_1	Y_2	W	A
X_1	0	0	0	$-W$	$-X_1$	$-X_2$
X_2	0	0	W	0	$-X_2$	X_1
Y_1	0	$-W$	0	0	Y_1	$-Y_2$
Y_2	W	0	0	0	Y_2	Y_1
W	X_1	X_2	$-Y_1$	$-Y_2$	0	0
A	X_2	$-X_1$	Y_2	$-Y_1$	0	0

and

(4.2)

< , >	X_1	X_2	Y_1	Y_2	W
X_1	a^2	0	0	$-\gamma$	0
X_2	0	a^2	γ	0	0
Y_1	0	γ	a^2	0	0
Y_2	$-\gamma$	0	0	a^2	0
W	0	0	0	0	b^2

where $a, b > 0$, γ are real numbers, $a^2 > |\gamma|$ (see [4]).

Using (4.1) and (4.2) to compute (2.4), we obtain that $X \in \underline{\mathbf{g}}$ is geodesic if and only if its components $(x_1, x_2, y_1, y_2, w, r)$ with respect to $\{X_1, X_2, Y_1, Y_2, W, A\}$ satisfy

$$
\left\{\begin{array}{l}
w(a^2x_1 + (b^2-\gamma)y_2) + r(a^2x_2 + \gamma y_1) = 0, \\
w(a^2x_2 - (b^2-\gamma)y_1) - r(a^2x_1 - \gamma y_2) = 0, \\
w(-a^2y_1 + (b^2-\gamma)x_2) + r(a^2y_2 - \gamma x_1) = 0, \\
w(-a^2y_2 - (b^2-\gamma)x_1) - r(a^2y_1 + \gamma x_2) = 0, \\
x_1^2 + x_2^2 = y_1^2 + y_2^2,
\end{array}\right. \tag{4.3}
$$

taking into account that $a^2 \neq 0$.

We can now find all the solutions of (4.3). If $r = w = 0$, then (4.3) reduces to $x_1^2 + x_2^2 = y_1^2 + y_2^2$. If $r = 0 \neq w$, the solutions of (4.3) are $x_1 = x_2 = y_1 = y_2 = 0$ if $a^2 - b^2 + \gamma \neq 0$, and $x_1 + y_2 = x_2 - y_1 = 0$ if $a^2 - b^2 + \gamma = 0$.

The case $r \neq 0$ is much more complicated. After some standard but quite long calculations, we eventually find that, when $r \neq 0$, the solutions of (4.3) are given by

$$
\left\{\begin{array}{l}
x_1 = -\dfrac{b^2w(ry_1 + wy_2)}{a^2(r^2+w^2)} + \dfrac{\gamma}{a^2}y_2, \\
x_2 = \dfrac{b^2w(wy_1 - ry_2)}{a^2(r^2+w^2)} - \dfrac{\gamma}{a^2}y_1, \quad \text{with } r^2 = \dfrac{b^2w^2(b^2-2\gamma)}{a^4-\gamma^2} - w^2.
\end{array}\right.
$$

Thus, if X is a geodesic vector of a generalized symmetric space of type 3, then its $\mathbf{m}$-component is

(i) $X_{\mathbf{m}} = x_1X_1 + x_2X_2 + y_1Y_1 + y_2Y_2$ and $x_1^2 + x_2^2 = y_1^2 + y_2^2$, or

(ii) $X_{\mathbf{m}} = wW$ if $a^2 - b^2 + \gamma \neq 0$, while
$X_{\mathbf{m}} = x_2X_2 + x_2Y_1 + wW$ if $\mathrm{a}^2 - \mathrm{b}^2 + \gamma = 0$, or

(iii) $X_{\mathbf{m}} = (-\dfrac{b^2w(ry_1 + wy_2)}{a^2(r^2+1)} + \dfrac{\gamma}{a^2}y_2)X_1 + (\dfrac{b^2w(wy_1 - ry_2)}{a^2(r^2+1)} - \dfrac{\gamma}{a^2}y_1)X_2 + y_1Y_1 +$
$+ y_2Y_2 + wW$, where $r^2 = \frac{b^2w^2(b^2-2\gamma)}{a^4-\gamma^2} - w^2 \neq 0$.

One can check that W is a vector of type (ii), while $V_1 = X_1 + Y_1$, $V_2 = X_1 - Y_1$, $V_3 = \gamma X_1 + a^2 X_2 - \gamma Y_1 + a^2 Y_2$ and $V_4 = \gamma X_1 - a^2 X_2 + \gamma Y_1 + a^2 Y_2$ are of type (i), and $\{V_1, V_2, V_3, V_4, W\}$ is an orthogonal basis of $\mathbf{m}$. Therefore, Proposition 2.2 implies that *five-dimensional generalized symmetric spaces of type* 3 *admit five mutually orthogonal homogeneous geodesics through the origin* o.

5 Homogeneous geodesics of generalized symmetric spaces of type 4

Five-dimensional generalized symmetric spaces M of type 4 are complex matrix groups

$$\begin{pmatrix} e^{\lambda t} & 0 & z \\ 0 & e^{-\lambda t} & w \\ 0 & 0 & 1 \end{pmatrix}$$

where $z, w \in \mathbf{C}$ and $t \in I\!R$. M is also the space $\mathbf{C}^2(z,w) \times I\!R(t)$, equipped with a Riemannian metric

$$\begin{aligned} g \quad = \quad & e^{-(\lambda+\bar\lambda)t} dz d\bar z + e^{(\lambda+\bar\lambda)t} dw d\bar w + dt^2 + 2c[e^{(\bar\lambda-\lambda)t} dz d\bar w + e^{(\lambda-\bar\lambda)t} d\bar z dw] + \\ & + \gamma e^{-2\lambda t} dz^2 + \bar\gamma e^{-2\bar\lambda t} d\bar z^2 - \gamma e^{2\lambda t} dw^2 - \bar\gamma e^{2\bar\lambda t} d\bar w^2, \end{aligned}$$

with $\lambda, \gamma \in \mathbf{C}$, $c \in I\!R$, $\gamma\bar\gamma + c^2 < 1/4$ [4]. Put $\nu = (1+b^2)\gamma$, where $c = \dfrac{1-b^2}{2(1+b^2)}$. Then, $\gamma\bar\gamma + c^2 < 1/4$ is equivalent to $\nu\bar\nu < b^2$.

2a): $\lambda + \bar\lambda \neq 0$ and $\nu \neq 0$.

In this case, $\underline{\mathbf{h}} = 0$ [4]. With respect to a suitable basis $\{X_1, X_2, Y_1, Y_2, W\}$ of $\underline{\mathbf{g}}$, the Lie bracket $[,]$ and the Riemannian metric $<,>$ are given by (see [4])

(5.1)

$[\,,\,]$	X_1	X_2	Y_1	Y_2	W
X_1	0	0	0	0	$-\eta X_2 - \mu Y_2$
X_2	0	0	0	0	$\eta X_1 - \mu Y_1$
Y_1	0	0	0	0	$-\mu X_2 + \eta Y_2$
Y_2	0	0	0	0	$-\mu X_1 - \eta Y_1$
W	$\eta X_2 + \mu Y_2$	$-\eta X_1 + \mu Y_1$	$\mu X_2 - \eta Y_2$	$\mu X_1 + \eta Y_1$	0

and

(5.2)

$<\,,\,>$	X_1	X_2	Y_1	Y_2	W
X_1	1	α	0	$-\beta$	0
X_2	α	b^2	β	0	0
Y_1	0	β	1	α	0
Y_2	$-\beta$	0	α	b^2	0
W	0	0	0	0	1

where $\lambda = \eta + i\mu$ and $\nu = \alpha + i\beta$. Using (5.1) and (5.2) in (2.4), we obtain that $X \in \underline{\mathbf{g}}$ is geodesic if and only if its components satisfy

$$(5.3)\quad \begin{cases} w\eta(\alpha x_1 + b^2 x_2 + \beta y_1) + w\mu(-\beta x_1 + \alpha y_1 + b^2 y_2) = 0, \\ -w\eta(x_1 + \alpha x_2 - \beta y_2) + w\mu(\beta x_2 + y_1 + \alpha y_2) = 0, \\ w\mu(\alpha x_1 + b^2 x_2 + \beta y_1) - w\eta(-\beta x_1 + \alpha y_1 + b^2 y_2) = 0, \\ w\mu(x_1 + \alpha x_2 - \beta y_2) + w\eta(\beta x_2 + y_1 + \alpha y_2) = 0, \\ (\eta x_1 + \mu y_1)(\alpha x_1 + b^2 x_2 + \beta y_1) + (\mu x_1 - \eta y_1)(-\beta x_1 + \alpha y_1 + b^2 y_2) + \\ +(-\eta x_2 + \mu y_2)(x_1 + \alpha x_2 - \beta y_2) + (\mu x_2 + \eta y_2)(\beta x_2 + y_1 + \alpha y_2) = 0 \end{cases}$$

It is not difficult to show that if $w \neq 0$, the solutions of (5.3) are given by $(0,0,0,0,w)$, while if $w = 0$, (5.3) reduces to the fifth equation. So, in this case the solutions are $(x_1, x_2, y_1, y_2, 0)$ such that

$$(5.4)\quad \begin{aligned} &(\eta x_1 + \mu y_1)(\alpha x_1 + b^2 x_2 + \beta y_1) + (\mu x_1 - \eta y_1)(-\beta x_1 + \alpha y_1 + b^2 y_2) + \\ &+(-\eta x_2 + \mu y_2)(x_1 + \alpha x_2 - \beta y_2) + (\mu x_2 + \eta y_2)(\beta x_2 + y_1 + \alpha y_2) = 0 \end{aligned}$$

We then proved that X is a geodesic vector of a generalized symmetric space of type $4a$ if and only if

(i) $X = wW$, or
(ii) $X = x_1 X_1 + x_2 X_2 + y_1 Y_1 + y_2 Y_2$ and (5.4) holds.

Then, for generalized symmetric spaces of type $4a$, geodesic vectors form *a straight line* (i) and *a hypercone* (of equation (5.4)) *in the orthogonal complement of such line* (ii).

About the existence of an orthogonal basis of geodesic vectors, we found that such basis always exists, but it strongly varies with the different values of α, β, η and μ. Taking into account that $\eta \neq 0$ and $\alpha + i\beta \neq 0$, an orthogonal basis of geodesic vectors is given by:

a) $\{X_1, Y_1, X_2 - \beta Y_1, \beta X_1 + Y_2, W\}$ when $\alpha = \mu = 0$;

b) $\{X_1 + \frac{\alpha\mu + \beta\eta + \sqrt{\Delta}}{\alpha\eta - \beta\mu} Y_1, X_1 + \frac{\alpha\mu + \beta\eta - \sqrt{\Delta}}{\alpha\eta - \beta\mu} Y_1, (\beta k - \alpha) X_1 + X_2 - (\alpha k + \beta) Y_1 + k Y_2, (\beta + \alpha k) X_1 - k X_2 + (\beta k - \alpha) Y_1 + Y_2, W\}$, with $\Delta = (\alpha^2 + \beta^2)(\eta^2 + \mu^2)$ and $k = \frac{\alpha\mu - \beta\eta + \sqrt{\Delta}}{\alpha\eta + \beta\mu}$, when $\alpha \neq \pm\frac{\mu}{\eta}\beta$;

c) $\{X_1, Y_1, X_2 - (1 + \frac{\mu^2}{\eta^2})\beta Y_1 + \frac{\mu}{\eta} Y_2, -\frac{\eta^2 + \mu^2}{\eta\mu} X_1 + X_2 - \frac{\eta}{\mu} Y_2, W\}$ when $\alpha = \frac{\mu}{\eta}\beta \neq 0$;

d) $\{X_1 + \frac{\mu}{\eta} Y_1, X_1 - \frac{\eta}{\mu} Y_1, \frac{\mu}{\eta}\beta X_1 + X_2 - \beta Y_1, \beta X_1 + \frac{\mu}{\eta}\beta Y_1 + Y_2, W\}$ if $\alpha = -\frac{\mu}{\eta}\beta \neq 0$.

Therefore, *five-dimensional generalized symmetric spaces of type* $4a$ *admit five mutually orthogonal homogeneous geodesics through the origin.*

2b): $\lambda + \bar{\lambda} = 0$, $\nu = 0$ and $b^2 \neq 1$.

Since $\lambda+\bar\lambda=0$, we have $\lambda=i\mu$, $\mu\in I\!R$. In this case, $\underline{\mathbf{h}}=so(2)=span(A)$, where A is determined by $AX_1=-Y_1$, $AX_2=Y_2$, $AY_1=X_1$, $AY_2=-X_2$, $AW=0$ (see [4]). Computing $[,]$ and $<,>$, taking into account that now $\nu=0$ and $\lambda=i\mu$, we get

(5.5)

[,]	X_1	X_2	Y_1	Y_2	W	A
X_1	0	0	0	0	$-\mu Y_2$	Y_1
X_2	0	0	0	0	$-\mu Y_1$	$-Y_2$
Y_1	0	0	0	0	$-\mu X_2$	$-X_1$
Y_2	0	0	0	0	$-\mu X_1$	X_2
W	μY_2	μY_1	μX_2	μX_1	0	0
A	$-Y_1$	Y_2	X_1	$-X_2$	0	0

and $\{X_1,X_2,Y_1,Y_2,W\}$ is an orthogonal basis, with $<X_1,X_1>=<Y_1,Y_1>=<W,W>=1$ and $<X_2,X_2>=<Y_2,Y_2>=b^2\neq 1$.

Therefore, $X\in\underline{\mathbf{g}}$ is geodesic if and only if its components (x_1,x_2,y_1,y_2,w,a) with respect to the basis $\{X_1,Y_1,X_2,Y_2,W,A\}$ satisfy

$$\left\{\begin{array}{l}\mu wb^2y_2-ay_1=0,\\ \mu wy_1+ab^2y_2=0,\\ \mu wb^2x_2+ax_1=0,\\ \mu wx_1-ab^2x_2=0,\\ 2\mu(x_1y_2+x_2y_1)=0.\end{array}\right. \tag{5.6}$$

It is easy to show that if $\mu=0$, then any vector of $\mathbf{m}$ is the component of a geodesic vector. Hence, we now focus on the case $\mu\neq 0$. When $w\neq 0$, from (5.6) we get $x_1=x_2=y_1=y_2=0$, while if $w=0$, then (5.6) reduces to $x_1y_2+x_2y_1=0$. Therefore, if $\mu\neq 0$, X is a geodesic vector of a generalized symmetric space of type $4b$ if and only if

(i) $X_{\mathbf{m}}=wW$, or
(ii) $X_{\mathbf{m}}=x_1X_1+x_2X_2+y_1Y_1+y_2Y_2$ and $x_1y_2+x_2y_1=0$.

We can check easily that X_1, X_2, Y_1, Y_2 and W are mutually orthogonal geodesic vectors. Thus, five-dimensional generalized symmetric spaces of type $4b$ admit five linearly independent homogeneous geodesics through the origin o.

2c): $\lambda+\bar\lambda\neq 0$, $\nu=0$ and $b^2=1$.

In this case, $\underline{\mathbf{h}}=so(2)\bigoplus so(2)=span(A_1,A_2)$, where $A_1=A$ of type $4b$, while A_2 is determined by $A_2X_1=X_2$, $A_2X_2=-X_1$, $A_2Y_1=-Y_2$, $AY_2=Y_1$, $AW=0$ (see [4]). Hence, computing the Lie bracket, we get

[,]	X_1	X_2	Y_1	Y_2	W	A_1	A_2
X_1	0	0	0	0	$-\eta X_2-\mu Y_2$	Y_1	$-X_2$
X_2	0	0	0	0	$\eta X_1-\mu Y_1$	$-Y_2$	X_1
Y_1	0	0	0	0	$-\mu X_2+\eta Y_2$	$-X_1$	Y_2
Y_2	0	0	0	0	$-\mu X_1-\eta Y_1$	X_2	$-Y_1$
W	$\eta X_2+\mu Y_2$	$-\eta X_1+\mu Y_1$	$\mu X_2-\eta Y_2$	$\mu X_1+\eta Y_1$	0	0	0
A_1	$-Y_1$	Y_2	X_1	$-X_2$	0	–	–
A_2	X_2	$-X_1$	$-Y_2$	Y_1	0	–	–

Note that in this case $\{X_1, X_2, Y_1, Y_2, W\}$ is an orthonormal basis of $\mathbf{m}$.

$X \in \underline{\mathbf{g}}$ is geodesic if and only if its components $(x_1, x_2, y_1, y_2, w, a, b)$ with respect to $\{X_1, Y_1, X_2, Y_2, W, A_1, A_2\}$ satisfy

$$\text{(5.7)} \qquad \begin{cases} w(\eta x_2 + \mu y_2) - a y_1 + b x_2 = 0, \\ w(\eta x_1 - \mu y_1) - a y_2 + b x_1 = 0, \\ w(\mu x_2 - \eta y_2) + a x_1 - b y_2 = 0, \\ w(\mu x_1 + \eta y_1) - a x_2 + b y_1 = 0, \\ 2\mu(x_1 y_2 + y_1 x_2) = 0. \end{cases}$$

When $w \neq 0$, (5.7) gives $x_1 = x_2 = y_1 = y_2 = 0$. For $w = 0$, (5.7) reduces to its last equation $x_1 y_2 + y_1 x_2 = 0$. Thus, if X is a geodesic vector of a generalized symmetric space of type $4c$, then its $\mathbf{m}$-component is

(i) $\quad X_{\mathbf{m}} = wW$, or
(ii) $\quad X_{\mathbf{m}} = x_1 X_1 + x_2 X_2 + y_1 Y_1 + y_2 Y_2$ and $x_1 y_2 + x_2 y_1 = 0$.

Note that $\{X_1, X_2, Y_1, Y_2, W\}$ is an orthonormal basis of $\mathbf{m}$, where X_1, X_2, Y_1, Y_2 are of type (i) while W is of type (ii). Therefore, from Proposition 2.2 it follows that *five-dimensional generalized symmetric spaces of type* $4c$ *admit five mutually orthogonal homogeneous geodesics through the origin.*

6 Homogeneous geodesics of generalized symmetric spaces of type 7

As homogeneous spaces, five-dimensional generalized symmetric spaces M of type 7 are real matrix groups

$$\begin{pmatrix} e^{\lambda t} & 0 & 0 & 0 & x \\ 0 & e^{-\lambda t} & 0 & 0 & y \\ te^{\lambda t} & 0 & e^{\lambda t} & 0 & u \\ 0 & -te^{-\lambda t} & 0 & e^{-\lambda t} & v \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

M is also $\mathbb{R}^5(x, y, u, v, t)$, equipped with a Riemannian metric

$$\begin{aligned} g \quad = \quad & dt^2 + e^{-2\lambda t}(tdx - du)^2 + e^{2\lambda t}(tdy + dv)^2 + a^2(e^{-2\lambda t}dx^2 + e^{2\lambda t}dy^2) + \\ & +2\gamma(dydu - dxdv), \end{aligned}$$

where $\lambda, a, \gamma \in \mathbb{R}$, $\lambda \geq 0$, $a > 0$ and $\gamma^2 < a^2$.

2a): $\lambda \neq 0$.

In this case, $\underline{\mathbf{h}} = 0$ [4]. Following [4], there exists a basis $\{X_1, X_2, Y_1, Y_2, W\}$ of $\underline{\mathbf{g}}$ such that

$$
\begin{array}{c|ccccc}
[\,,\,] & X_1 & X_2 & Y_1 & Y_2 & W \\
\hline
X_1 & 0 & 0 & 0 & 0 & -\lambda X_1 - X_2 \\
X_2 & 0 & 0 & 0 & 0 & -\lambda X_2 \\
Y_1 & 0 & 0 & 0 & 0 & \lambda Y_1 + Y_2 \\
Y_2 & 0 & 0 & 0 & 0 & \lambda Y_2 \\
W & \lambda X_1 + X_2 & \lambda X_2 & -\lambda Y_1 - Y_2 & -\lambda Y_2 & 0
\end{array} \tag{6.1}
$$

and

$$
\begin{array}{c|ccccc}
<\,,\,> & X_1 & X_2 & Y_1 & Y_2 & W \\
\hline
X_1 & a^2 & 0 & 0 & -\gamma & 0 \\
X_2 & 0 & 1 & \gamma & 0 & 0 \\
Y_1 & 0 & \gamma & a^2 & 0 & 0 \\
Y_2 & -\gamma & 0 & 0 & 1 & 0 \\
W & 0 & 0 & 0 & 0 & 1
\end{array} \tag{6.2}
$$

We now use (6.1) and (6.2) to compute (2.4). Taking into account that $\lambda \neq 0$, we get

$$
\begin{cases}
a^2 x_1 - \gamma y_2 = 0, \\
x_2 + \gamma y_1 = 0, \\
\gamma x_2 + a^2 y_1 = 0, \\
y_2 - \gamma x_1 = 0,
\end{cases} \tag{6.3}
$$

which only admits the solutions $(0,0,0,0,w)$. So, *X is a geodesic vector of a generalized symmetric space of type 7a if and only if X is parallel to W*. In other words, *geodesic vectors of a five-dimensional generalized symmetric space of type 7a form a straight line.* As a consequence, we clearly have the following

Theorem 6.1 *Five-dimensional generalized symmetric spaces of type 7a only admit one homogeneous geodesic through the origin.*

2b): $\lambda = 0$.

In this case, $\underline{\mathbf{h}} = so(2) = span(A)$, where A is determined by $AX_1 = -Y_1$, $AX_2 = Y_2$, $AY_1 = X_1$, $AY_2 = -X_2$, $AW = 0$ (see [4]). Computing $[,]$ and $<,>$, taking into account that now $\lambda = 0$ and $\gamma = 0$ [4], we get

$$
\begin{array}{c|cccccc}
[\,,\,] & X_1 & X_2 & Y_1 & Y_2 & W & A \\
\hline
X_1 & 0 & 0 & 0 & 0 & -X_2 & Y_1 \\
X_2 & 0 & 0 & 0 & 0 & 0 & -Y_2 \\
Y_1 & 0 & 0 & 0 & 0 & Y_2 & -X_1 \\
Y_2 & 0 & 0 & 0 & 0 & 0 & X_2 \\
W & X_2 & 0 & -Y_2 & 0 & 0 & 0 \\
A & -Y_1 & Y_2 & X_1 & -X_2 & 0 & 0
\end{array} \tag{6.4}
$$

and $\{X_1, X_2, Y_1, Y_2, W\}$ is an orthogonal basis of $\underline{\mathbf{g}}$, with $< X_1, X_1 >=< Y_1, Y_1 >= a^2$ and $< X_2, X_2 >=< Y_2, Y_2 >=< W, W >= 1$.

Using (6.4) to compute (2.4), it is easy to show that $X \in \underline{\mathbf{g}}$ is geodesic if and only if its components $(x_1, x_2, y_1, y_2, w, a)$ with respect to the basis $\{X_1, Y_1, X_2, Y_2, W, A\}$ satisfy

$$\text{(6.5)} \qquad \begin{cases} ay_1 - wx_2 = 0, \\ ay_2 = 0, \\ ax_1 - wy_2 = 0, \\ ax_2 = 0, \\ x_1x_2 - y_1y_2 = 0. \end{cases}$$

For $a \neq 0$, (6.5) gives $x_1 = x_2 = y_1 = y_2 = 0$. For $a = 0$, we have either $w = 0$ and $x_1x_2 - y_1y_2 = 0$, or $w \neq 0$ and $x_2 = y_2 = 0$. Therefore, if X is a geodesic vector of a generalized symmetric space of type $7b$, then its $\mathbf{m}$-component is given by

(i) $\quad X_{\mathbf{m}} = x_1X_1 + x_2X_2 + y_1Y_1 + y_2Y_2$ and $x_1x_2 - y_1y_2 = 0$, or
(ii) $\quad X_{\mathbf{m}} = x_1X_1 + y_1Y_1 + wW$.

Finally, since all the vectors of the orthogonal basis $\{X_1, X_2, Y_1, Y_2, W\}$ are of type (i) or (ii), we can conclude that *five-dimensional generalized symmetric spaces of type $7b$ admit five mutually orthogonal homogeneous geodesics through the origin.*

7 Homogeneous geodesics of generalized symmetric spaces of type 8

As homogeneous spaces, five-dimensional generalized symmetric spaces M of type 8 are $I^e(\mathbb{R}^3)/SO(2)$ or $I^h(\mathbb{R}^3)/SO(2)$, where I^e (respectively, I^h) denotes the group of all positive affine transformations of $\mathbb{R}^3$ that preserve $dx^2 + dy^2 + dz^2$ (respectively, $dx^2 + dy^2 - dz^2$). M is also described as submanifold of $\mathbb{R}^6(x, y, z, \alpha, \beta, \gamma)$, such that $\alpha^2 + \beta^2 \pm \gamma^2 = \pm 1$. The Riemannian metric of M is induced by the regular invariant quadratic form

$$\bar{g} \;=\; dx^2 + dy^2 \pm dz^2 + \lambda^2(d\alpha^2 + d\beta^2 \pm d\gamma^2) + [\mu \pm (-1)](\alpha dx + \beta dy \pm \gamma dz)^2,$$

where $\lambda, \mu > 0$ [4]. The five-dimensional generalized symmetric spaces of type $8a$ (respectively, $8b$) are obtained when we have the sign "$+$" (respectively, "$-$") in the previous formulas. Here we analyze the case $8a$, the case $8b$ can be treated similarly and it leads to the same conclusions. In both cases, $\underline{\mathbf{h}} = so(2) = span(A)$, where A is determined by $AX_1 = -Y_1$, $AX_2 = Y_2$, $AY_1 = X_1$, $AY_2 = -X_2$, $AW = 0$, with $\{X_1, X_2, Y_1, Y_2, W\}$ a basis of $\mathbf{m}$. The Lie bracket on M is determined by

(7.1)

$[\,,\,]$	X_1	X_2	Y_1	Y_2	W	A
X_1	0	W	0	0	$-X_2$	Y_1
X_2	$-W$	0	0	0	0	$-Y_2$
Y_1	0	0	0	$-W$	Y_2	$-X_1$
Y_2	0	0	W	0	0	X_2
W	X_2	0	$-Y_2$	0	0	0
A	$-Y_1$	Y_2	X_1	$-X_2$	0	0

Moreover, $\{X_1, X_2, Y_1, Y_2, W\}$ is an orthogonal basis of $\mathbf{m}$, with $< X_1, X_1 >=< Y_1, Y_1 >= b^2$, $< X_2, X_2 >=< Y_2, Y_2 >= 1$ and $< W, W >= c^2$, where $b, c > 0$

[4]. Using (7.1) to compute (2.4), we obtain that $X \in \underline{\mathbf{g}}$ is geodesic if and only if its components $(x_1, x_2, y_1, y_2, w, a)$ with respect to the basis $\{X_1, Y_1, X_2, Y_2, W, A\}$ satisfy

$$\begin{cases} x_2 w(1-c^2) - y_1 ab^2 = 0, \\ x_1 wc^2 + ay_2 = 0, \\ -y_2 w(1-c^2) + x_1 ab^2 = 0, \\ y_1 wc^2 + ax_2 = 0, \\ x_1 x_2 - b^2 y_1 y_2 = 0. \end{cases} \tag{7.2}$$

For $a = w = 0$, (7.2) reduces to $x_1x_2 - b^2y_1y_2 = 0$. If $a = 0 \neq w$, we get $x_1 = x_2 = y_1 = y_2 = 0$ when $c^2 \neq 1$, while if $c^2 = 1$ we only have $x_1 = y_1 = 0$. If $a \neq 0$, we must distinguish different cases. We eventually obtain:

1) If $c^2 - 1 < 0$, then $x_1 = x_2 = y_1 = y_2 = 0$.

2) If $c^2 - 1 > 0$ and $b^2 = 1$, then, in addition to $x_1 = x_2 = y_1 = y_2 = 0$, we also have the solutions $x_2 = -\frac{wc^2}{a}y_1$, $y_2 = -\frac{wc^2}{a}x_1$ and $w = \pm\frac{ab}{c\sqrt{c^2-1}}$.

3) If $c^2 - 1 > 0$ and $b^2 \neq 1$, the solutions are either $x_1 = x_2 = y_1 = y_2 = 0$, or $x_2 = y_1 = 0$, $y_2 = -\frac{wc^2}{a}x_1$ and $w = \pm\frac{ab}{c\sqrt{c^2-1}}$, or $x_1 = y_2 = 0$, $x_2 = -\frac{wc^2}{a}y_1$ and $w = \pm\frac{ab}{c\sqrt{c^2-1}}$.

In this way, we proved that the if X is a geodesic vectors of a five-dimensional generalized symmetric space of type $8a$, then its $\mathbf{m}$-component is:

(*i*) $X_{\mathbf{m}} = wW$, or

(*ii*) $X_{\mathbf{m}} = x_1X_1 + x_2X_2 + y_1Y_1 + y_2Y_2$ and $x_1x_2 - b^2y_1y_2 = 0$, or

(*iii*) $X_{\mathbf{m}} = x_2X_2 + y_2Y_2 + wW$ (only when $c^2 = 1$), or

(*iv*) $X_{\mathbf{m}} = x_1X_1 - \frac{wc^2}{a}y_1X_2 + y_1Y_1 - \frac{wc^2}{a}x_1Y_2 \pm \frac{ab}{c\sqrt{c^2-1}}W$ (only when $c^2 > 1$ and $b^2 = 1$), or

(*v*) $X_{\mathbf{m}} = x_1X_1 - \frac{wc^2}{a}x_1Y_2 \pm \frac{ab}{c\sqrt{c^2-1}}W$ and
$X_{\mathbf{m}} = -\frac{wc^2}{a}y_1X_2 + y_1Y_1 \pm \frac{ab}{c\sqrt{c^2-1}}W$(only when $c^2 > 1$ and $b^2 \neq 1$).

The calculations for spaces of type $8b$ are similar. Since all the vectors of the orthogonal basis $\{X_1, X_2, Y_1, Y_2, W\}$ are geodesic vectors of type (*i*) or (*ii*), which exist for all values of b and c, we can conclude that *five-dimensional generalized symmetric spaces of type* 8 *admit five mutually orthogonal homogeneous geodesics through the origin.*

8 Homogeneous geodesics of generalized symmetric spaces of order 6 (type 9)

Five-dimensional generalized symmetric spaces of order 6 can be described in the following way. The underlying manifold is $I\!R^5(x,y,z,u,v)$, equipped with the Riemannian metric

$$(8.1)\qquad g = \frac{2}{3}a^2(du^2+dudv+dv^2)+(2b^2+1)(e^{2(u+v)}dx^2+e^{-2u}dy^2+ \\ +e^{-2v}dz^2)+2(b^2-1)(e^v dxdy+e^u dxdz-e^{-(u+v)}dydz),$$

where $a>0$ and $b>0$ are real numbers. The space $(I\!R^5,g)$ can be identified with the homogeneous space G/H, where G is the group of all matrices of the form

$$\begin{pmatrix} e^{-(u+v)} & 0 & 0 & x \\ 0 & e^u & 0 & y \\ 0 & 0 & e^v & y \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Generalized symmetric spaces of type 9 have a special interest because they are *of solvable type*, that is, G is a solvable Lie group. In a forthcoming paper [1], the authors and O. Kowalski will study homogeneous geodesics in some examples of generalized symmetric spaces of solvable type of arbitrary odd dimension.

Following [4], the Lie algebra $\underline{\mathbf{g}}$ of G admits a basis $\{X_1,X_2,Y_1,Y_2,W\}$ such that

(8.2)

$[\,,\,]$	X_1	X_2	Y_1	Y_2	W
X_1	0	$-X_2$	0	0	W
X_2	X_2	0	X_2	0	0
Y_1	0	$-X_2$	0	Y_2	0
Y_2	0	0	$-Y_2$	0	0
W	$-W$	0	0	0	0

and

(8.3)

$<\,,\,>$	X_1	X_2	Y_1	Y_2	W
X_1	$\frac{2}{3}a^2$	0	$\frac{1}{3}a^2$	0	0
X_2	0	$2b^2+1$	0	b^2-1	b^2-1
Y_1	$\frac{1}{3}a^2$	0	$\frac{2}{3}a^2$	0	0
Y_2	0	b^2-1	0	$2b^2+1$	$-(b^2-1)$
W	0	b^2-1	0	$-(b^2-1)$	$2b^2+1$

Since $\underline{\mathbf{h}}=0$ [4], each geodesic vector must be an element of $\underline{\mathbf{g}}$. We use (8.2) and (8.3) to compute (2.4). Putting

$$h=\frac{b^2-1}{2b^2+1},$$

we get easily that $X \in \underline{\mathbf{g}}$ is geodesic if and only if its components satisfy

$$
\left\{
\begin{array}{l}
(x_2+w)(x_2+hy_2-w)=0,\\
(x_1+y_1)(x_2+hy_2+hw)=0,\\
(x_2+y_2)(x_2-y_2+hw)=0,\\
y_1(hx_2+y_2-hw)=0,\\
x_1(hx_2-hy_2+w)=0.
\end{array}
\right.
\tag{8.4}
$$

By the definition of h, it follows easily that $h-1 \neq 0$, $h+1 \neq 0$ and $2h-1 \neq 0$, for all $b > 0$. We now find all the solutions of (8.4).

a) If $x_1 = 0$, (8.4) reduces to

$$
\left\{
\begin{array}{l}
(x_2+w)(x_2+hy_2-w)=0,\\
y_1(x_2+hy_2+hw)=0,\\
(x_2+y_2)(x_2-y_2+hw)=0,\\
y_1(hx_2+y_2-hw)=0.
\end{array}
\right.
\tag{8.5}
$$

Adding the second and the fourth equation of (8.5) and taking into account that $h+1 \neq 0$, we get that either $y_1 = 0$ or $x_2 + y_2 = 0$.

If also $y_1 = 0$, then (8.5) reduces to

$$
\left\{
\begin{array}{l}
(x_2+w)(x_2+hy_2-w)=0,\\
(x_2+y_2)(x_2-y_2+hw)=0,
\end{array}
\right.
$$

whose solutions are $(0,-w,0,w,w)$, $(0,-w,0,(h-1)w,w)$, $(0,-y_2,0,y_2,(h-1)y_2)$, $(0,(1-h)w,0,w,w)$.

When $y_1 \neq 0$, the only solutions are $(0,0,y_1,0,0)$.

b) If $x_1 \neq 0$, from the last equation of (8.4) we get $hx_2 - hy_2 + w = 0$.

If $y_1 = 0$, the case is similar to the case $x_1 = 0, y_1 \neq 0$. Proceeding in the same way, we get the solutions $(x_1,0,0,0,0)$.

If $y_1 \neq 0$, (8.4) gives

$$
\left\{
\begin{array}{l}
hx_2-hy_2+w=0,\\
hx_2+y_2-hw)=0,\\
(x_2+w)(x_2+hy_2-w)=0,\\
(x_1+y_1)(x_2+hy_2+hw)=0,\\
(x_2+y_2)(x_2-y_2+hw)=0.
\end{array}
\right.
\tag{8.6}
$$

whose solutions are $(x_1,0,y_1,0,0)$.

So, we proved the following

Proposition 8.1 *X is a geodesic vector of a five-dimensional generalized symmetric space of type 9 if and only if*

$$
\begin{array}{ll}
(i) & X=-wX_2+wY_2+wW, \text{ or}\\
(ii) & X=-wX_2+(h-1)wY_2+wW, \text{ or}\\
(iii) & X=-y_2X_2+y_2Y_2+(h-1)y_2W, \text{ or}\\
(iv) & X=(1-h)wX_2+wY_2+wW, \text{ or}\\
(v) & X=x_1X_1+y_1Y_1.
\end{array}
$$

Note that two geodesic vectors of two distinct types among (i), (ii), (iii) and (iv) are always distinct, since $h-1 \neq 1$.

If we add to $\{X_1, Y_1\}$ a triplet of vectors chosen in $\{V_1 = -X_2 + Y_2 + W, V_2 = -X_2 + (h-1)Y_2 + W, V_3 = -X_2 + Y_2 + (h-1)W, V_4 = -(h-1)X_2 + Y_2 + W\}$, we then always get five linearly independent geodesic vectors, taking into account that $h-2 \neq 0$. Hence, there exist five linearly independent geodesic vectors in M.

About the orthogonality, it is easy to prove that a geodesic vector of type (v) is orthogonal to all geodesic vectors of type (i), (ii), (iii) or (iv). Moreover, for example X_1 and $V = X_1 - 2Y_1$ are two orthogonal geodesic vectors of type (v). Finally, two geodesic vectors chosen in two different types among (i), (ii), (iii) and (iv) are never mutually orthogonal. In fact, $< V_1, V_2 >=< V_1, V_3 >= < V_1, V_4 >= 3(h+1) \neq 0$, while $< V_2, V_3 >=< V_2, V_4 >=< V_3, V_4 >= (2b^2+1)(2h-1)-(b^2-1)(h^2+2) \neq 0$. So, we can conclude that there are at most three geodesic vectors mutually orthogonal, for example taking X_1, $V = X_1 - 2Y_1$ and V_1. So, from Proposition 8.1 it follows

Corollary 8.2 *Geodesic vectors of five-dimensional generalized symmetric spaces of type* 9 *form*

*(a) a plane (*2*-dimensional vector subspace)* $\mathcal{P}$ *of* $\underline{\mathbf{g}}$ *(type* (v)*), and*

(b) four straight lines, respectively generated by the geodesic vectors $-X_2+Y_2+W$, $-X_2+(h-1)Y_2+W$, $-X_2+Y_2+(h-1)W$ *and* $(1-h)X_2+Y_2+W$ *(types* (i), (ii), (iii) *and* (iv)*, respectively), in the orthogonal complement of* $\mathcal{P}$*.*

Finally, we can conclude with the following result.

Theorem 8.3 *Five-dimensional generalized symmetric space of type* 9 *admit five linearly independent homogeneous geodesics through the origin* o*, but never five orthogonal ones. There are at most three mutually orthogonal homogeneous geodesics through* o*.*

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