The Geometrical Interpretation of Temporal Cone Norm in Almost Minkowski Manifold

Sorin Noaghi

Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

For almost Minkowski manifolds we prove that the norm determined by a unitary vector field which belongs to the timelike cone is the sum of two fundamental forms induced by the Lorentzian metrics on two submanifolds.

Mathematics Subject Classification: 53C50, 53B30, 53C80.
Key words: almost Minkowski manifold, topological cone norm.

1 Introduction

A Lorentz manifold is a pair (M, g) where M is an $n+1$ dimensional smooth paracompact manifold and g is a global smooth two-times covariant symmetric tensor field which is nondegenerate and has $n-1$ signature.

A time-normalized space-time $[M, g, Z]$ is a Lorentz manifold (M, g) for which a global unitary (i.e., $g(Z, Z) = -1$) tangent vector field Z of timelike vectors is fixed; this will be denoted by $[M, g, Z]$.

Definition 1 An almost Minkowski manifold is a time-normalized space-time $[M, g, Z]$ provided that the distribution

$$\Delta : x \in M \to \Delta_x \overset{def}{=} \{Y \in T_x M \mid g(Y, Z) = 0\}$$

is totally integrable.

Proposition 2 *The necessary and sufficient condition that a time-normalized space-time manifold $[M, g, Z]$ be an almost Minkowski manifold is the existence of a preferential atlas*

A$= \left\{(U_\alpha, \chi_\alpha) \mid \alpha \in \Gamma,\ \chi_\alpha(x) = \left(x^i\right),\ i = \overline{1, n+1}\ \partial_{n+1} = Z|_{U_\alpha}\right\}$,

where

$$\frac{\partial g_{in+1}}{\partial x^j} = \frac{\partial g_{jn+1}}{\partial x^i}\ \forall i, j \in \overline{1, n+1}.$$

Balkan Journal of Geometry and Its Applications, Vol.8, No.2, 2003, pp. 37-42.

Proof. For X,Y belonging to Δ we have

$$g\left(X,Z\right)=g\left(Y,Z\right)=0,$$

which infers

$$\begin{aligned} g\left(\nabla_Y X,Z\right)+g\left(X,\nabla_Y Z\right) &= 0 \\ g\left(\nabla_X Y,Z\right)+g\left(Y,\nabla_X Z\right) &= 0, \end{aligned}$$

where ∇ is the Levi-Civita connection of Lorentz manifold (M,g). In the local charts of atlas A where $Z=\partial_{n+1},\ X=X^i\partial_i$ and $Y=Y^j\partial_j$ we have

$$\begin{aligned} g\left(\left[X,Y\right],Z\right) &= g\left(\nabla_X Y,Z\right)-g\left(\nabla_Y X,Z\right)= \\ &= g\left(X,\nabla_Y Z\right)-g\left(Y,\nabla_X Z\right)= \\ &= X^iY^j\left[\Gamma^k_{jn+1}g_{in+1}-\Gamma^k_{in+1}g_{jn+1}\right]= \\ &= X^iY^j\left(\frac{\partial g_{in+1}}{\partial x^j}-\frac{\partial g_{jn+1}}{\partial x^i}\right). \end{aligned}$$

Therefore $[X,Y]$ belongs to Δ if and only if $\dfrac{\partial g_{in+1}}{\partial x^j}=\dfrac{\partial g_{jn+1}}{\partial x^i}$

□

Remark 3 The existence of almost Minkowski manifolds is obvious, since it is possible to choose Z so that $g\left(\partial_i,Z\right)\partial_i$ be irrotational. If (M,g) is stable causal then there exists a real global function f with the gradient ∇f of timelike type, (see e.g. [1], [6]), and the corresponding 1-form of $Z=\dfrac{1}{\sqrt{-g\left(\nabla f,\nabla f\right)}}\nabla f$ closed, and therefore it respects the previous conditions.

Remark 4 If the corank one distribution is not integrable, then any two points can be connected by a curve $\gamma:[0,1]\to M$ where $g\left(\gamma'\left(t\right),Z_{\gamma(t)}\right)=0$, according to the Carathéodory theorem, ([3, p.10]).

Definition 5 We define the ordering relation for the elements of $T_xM,\ x\in M$:

$$X\leq Y\Leftrightarrow Y-X\in K_x,$$

where $K_x=\{X\in T_xM\mid g\left(X,X\right)<0,\ g\left(X,Z\right)<0\}$ is the interior of the timelike cone of the tangent vectors.

From ([4]) we have:

- (T_xM,K_x) is a Krein space, $\forall x\in M$
- The map $|\,|_Z:T_xM\to\mathbf{R},\ |\mathbf{X}|_\mathbf{Z}\overset{\mathbf{def}}{=}\min\left\{\lambda\geq\mathbf{0}\mid-\lambda\mathbf{Z}\leq\mathbf{X}\leq\lambda\mathbf{Z}\right\}$ is a topological norm of T_xM, named the Z-norm of the almost Minkowski manifold $\left[M,g,Z\right]$.
- An easy calculation implies

$$|X|_z = |g(X,Z)| + \sqrt{g(X,Z)^2 + g(X,X)}. \tag{1.1}$$

Proposition 6 *The Z-norm is invariant to a conformal change of Lorentzian metric.*

Proof. Let $g,\ \widehat{g}$ be two conformal metrics (i.e. $g = \widehat{g}\,\Omega^2$). The Z-norms expressions for the two almost Minkowski manifolds are, ([5])

$$\begin{aligned} |X|_Z^g &= \left|\frac{g(X,Z)}{g(Z,Z)}\right| + \sqrt{\left[\frac{g(X,Z)}{g(Z,Z)}\right]^2 + \frac{g(X,X)}{g(Z,Z)}} = \\ &= \left|\frac{\widehat{g}(X,Z)}{\widehat{g}(Z,Z)}\right| + \sqrt{\left[\frac{\widehat{g}(X,Z)}{\widehat{g}(Z,Z)}\right]^2 + \frac{\widehat{g}(X,X)}{\widehat{g}(Z,Z)}} = |X|_Z^{\widehat{g}} \end{aligned}$$

Remark 7 The Z-norm can be defined if the existent condition of the global timelike vector field Z is weakened and replaced with the existence of a line element field which is equivalent to the existence of Lorentzian metrics ([2]).

2 The Z-norm dependence on the first fundamental form of the hypersurface normal to Z

Consider the preferential atlas A of Proposition 2 (this exists, cf. [4]). We note by S the integral submanifold of distribution Δ with $p \in S$. Obviously S is a hypersurface imbedded in M with inclusion map $\theta : S \to M$. Let $n \in T_q^*M,\ q \in S$ be the 1-form $n(X) = g(X,Z),\ \forall X \in T_qM$. This implies $n(\theta_* X) = 0, \forall X \in T_qM$ and if we denote $H_q = \theta_*(T_qS)$, this is a hyperplane in T_qM. If Z is be tangent to $\theta(S)$, then there exist $X \in T_qS \setminus \{0\}$ such that $\theta_*(X) = Z$ and $-1 = g(Z,Z) = g(\theta_*(X), Z) = 0$, which is impossible. Therefore Z is not in the tangent space of $\theta(S)$. If $\{E_1, ..., E_n\}$ is a basis in T_qS, then $\{Z, \theta_*(E_1), ..., \theta_*(E_n)\}$ is linearly independent and hence is a basis for T_qM. The components of g with respect to this basis are

$$(g_{ab}) = \begin{pmatrix} g(Z,Z) & 0 & \cdots & 0 \\ 0 & g(\theta_*(E_1), \theta_*(E_1)) & \cdots & g(\theta_*(E_1), \theta_*(E_n)) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & g(\theta_*(E_n), \theta_*(E_1)) & \cdots & g(\theta_*(E_n), \theta_*(E_n)) \end{pmatrix} =$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & [g(\theta_*(E_i), \theta_*(E_j))] \end{pmatrix}$$

Because g has one negative eingenvalue, then $\theta^* g$ is positively definite. Let's consider $\theta^* : T_q^*M \to T_q^*S$, and $H_q^* = \left\{\omega \in T_q^*M \mid \omega(Z) = 0\right\}$.

From $\theta^*|_{H_q^*} : H_q^* \to T_q^*S$, being obviously a bijection, we denote its inverse by $\widetilde{\theta}_* : T_q^*S \to H_q^*$. Therefore there exist two bijections θ_* and $\widetilde{\theta}_*$ between T_qS and H_q and respectively between T_q^*S and H_q^*. This map can be extended in a usual way to a map $\widetilde{\theta}$ of arbitrary tensors on S to $\theta(S)$ in M. Since n is normal to hypersurface $\theta(S)$, for a given tensor $T \in T_{s,q}^r S$ we obtain that $\widetilde{\theta}(T)$ has zero transvections with n in all indices:

$$\left(\widetilde{\theta}T\right)^{i_1..m..i_r}_{j_1...j_s} n_m = \left(\widetilde{\theta}T\right)^{i_1...i_r}_{j_1..m..j_s} g^{mp} n_p = 0.$$

Denote by h the metric on $\theta(S)$, defined by $h = \widetilde{\theta}(\theta^* g)$. In the preferential atlas A the components of h are

$$h_{ab} = g_{ab} + n_a n_b = g_{ab} + g_{an+1} g_{bn+1}\ , \forall a, b \in \overline{1, n+1}.$$

Proposition 8 *The (1.1) tensor associate to h having the components* h^b_a *is a projection operator,* $h^b_a = \delta^b_a + g_{an+1}\delta^b_{n+1}$ *and*

a) The projection of $X \in T_q M$ *onto the subspace* H_q *is*

$$h^b_a X^a \partial_b = X + g(X, Z)\, Z.$$

b) The projection of $\omega \in T^*_q M$ *onto the subspace* H^*_q *is*

$$h^b_a \omega_b dx^a = \omega + \omega(Z)\, n.$$

Proof.

$$\begin{aligned}
h^b_a &= h_{ac} g^{cb} = \left(g_{ac} + g_{an+1} g_{cn+1}\right) g^{cb} = \delta^b_a + g_{an+1}\delta^b_{n+1} \\
h^a_b h^b_c &= \left(\delta^a_b + g_{bn+1}\delta^a_{n+1}\right)\left(\delta^b_c + g_{cn+1}\delta^b_{n+1}\right) = h^a_c \\
X + g(X, Z)\, Z &= X^a \partial_a + X^a g_{an+1}\partial_{n+1} = X^a\left(\delta^b_a + g_{an+1}\delta^b_{n+1}\right)\partial_b = h^b_a X^a \partial_b \\
\omega + \omega(Z)\, n &= \omega_a dx^a + \omega_{n+1} g_{an+1} dx^a = \omega_a\left(\delta^a_b + g_{bn+1}\delta^a_{n+1}\right) dx^b = h^a_b \omega_a dx^b
\end{aligned}$$

Remark 9 Analogously we can project the tensor $T \in T\binom{r}{s}_q S$ to

$$H(T) \in \underbrace{H_q \otimes ... \otimes H_q}_{r\, times} \otimes \underbrace{H^*_q \otimes ... \otimes H^*_q}_{s\, factors} \overset{def}{=} H^r_s(q) \text{ via}$$

$$H(T) = T + (-1)^{r+s+1}\, T(Z^*, .., Z^*, Z, .., Z)\, \underbrace{Z \otimes ... \otimes Z}_{r\, factors} \otimes \underbrace{Z^* \otimes ... \otimes Z^*}_{s\, factors}$$

One can than verify the relation

$$H(H(T)) = H(T)\,,\ \forall T \in T\binom{r}{s}_q S.$$

Proposition 10 $(S, \theta^* g)$ *is a totally geodesic submanifold*

Proof. In the above notation, the coordinates of the second fundamental form of S are ([1, p. 46])

$$\chi_{ab} = h^c_a h^d_b n_{c\,;\,d} = h^c_a h^d_b g_{cn+1;\,d} = 0.$$

Remark 11 We will denote the covariant differentiation with respect to the Levi-Civita connection of $(S, \theta^* g)$ by double stroke.

Then for any tensor $T \in T\binom{r}{s}_q S$ we have:

$$H(T)^{i_1...i_r}_{j_1...j_s||m} = \overline{T}^{k_1...k_r}_{l_1...l_s\,;\,p} h^{i_1}_{k_1} ... h^{i_r}_{k_r} h^{l_1}_{j_1} ... h^{l_s}_{j_s} h^p_m\ ,$$

where $\overline{T}$ is an extension of $H(T)$ to a neighborhood of $\theta(S)$. This formula is correct because the double stroke of the induced metric is zero and the torsion vanishes,

$$\begin{aligned}
h_{ab||c} &= \left(g_{ef} + g_{en+1} g_{fn+1}\right)_{;i} h^e_a h^f_b h^i_c = 0 \\
f_{||ab} &= h^c_a h^d_b f_{;cd} = h^c_a h^d_b f_{;dc} = f_{||ba}.
\end{aligned}$$

For $p \in M$, we denote by s the 1-dimensional submanifold which passes through p and $T_q s = \langle Z_q \rangle$, $\forall q \in s$. The local imbedding map $i : s \hookrightarrow M$ is the inclusion which determines the applications $i_{*,q} : T_q s \to T_q M$ and $i_q^* : T_q^* M \to T_q^* s$. Denote $N_q = i_{*,q}(T_q s)$, $N_q^* = \left\{\omega \in T_q^* M \mid \omega(X) = g(X,Z),\ \lambda \in \mathbf{R}\right\}$; it is obvious that $i_{*,q} : T_q s \to N_q$ and $i_q^*|_{N_q^*} : N_q^* \to T_q^* s$ are bijections.

For $l = \left[i_q^*|_{N_q^*}\right]^{-1} \circ \left(i_q^* g\right)$, we have $l(X,Y) = -\overline{XY}$, where

$$X = \overline{X} Z_q,\ Y = \overline{Y} Z_q,\ X, Y \in T_q s,$$

l must be a two symmetric 2-form negatively definite on $i(s)$ and for $\forall q \in \theta(S) \cap i(s)$

$$T_q M = H_q \oplus N_q,\ T_q^* M = H_q^* \oplus N_q^*.$$

Proposition 12 *The Z-norm on the almost Minkowski manifold $[M, g, Z]$ is the sum of the Riemannian norms associated to the projections onto the submanifolds $(\theta(h),\ h)$ and $(i(s),\ -l)$.*

Proof. In $(\theta(S), h)$ the Riemannian norm is

$$|X|_h = \sqrt{h(X,X)} = \sqrt{g(X,X) + g(X,Z)^2},\ \forall X \in H_q.$$

In $(i(s), -l)$ the Riemannian norm is

$$|X|_l = \sqrt{-l(X,X)} = \left|\overline{X}\right| = |g(X,Z)|,\ \forall X \in N_q.$$

If $X \in T_q M = H_q \oplus N_q$, by Proposition 8 a) we have $X = X_1 + X_2$, where $X_1 = X + g(X,Z)Z$ and $X_2 = -g(X,Z)Z$;

$$\begin{aligned} h(X_1, X_1) &= g(X_1, X_1) + g(X_1, Z)^2 = g(X,X) + g(X,Z)^2 \\ l(X_2 X_2) &= -g(X,Z)^2 \end{aligned}$$

Then

$$\begin{aligned} |X_1|_h + |X_2|_l &= \sqrt{-l(X_2, X_2)} + \sqrt{h(X_1, X_1)} = \\ &= |g(X,Z)| + \sqrt{g(X,X) + g(X.Z)^2} \overset{1.1}{=} |X|_Z . \end{aligned}$$

□

References

[1] S.W. Hawking, *The existence of cosmic time functions*, Proc. Roy. Soc. Lond. A308, 1968, 433-435.

[2] S.W. Hawking, G.F.R. Ellis, *The large scalar structure of space-time*, Cambridge University Press, 1973.

[3] R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Math. Surveys and Monographs, 1991.

[4] D.I.Papuc, *The geometry of vector bundles endowen with a cone field* , Gen. Math. 5, (1997), 313-323.

[5] D.I.Papuc, *A new geometry of Lorentzian manifold*, Publ. Math. Debrecen Tomus 52, 1998, Fasc 1-2 , 145-158.

[6] H.J. Seifert, *Smoothing and extending cosmic time functions*, Gen. Rel. Grav. 8, 1977, 815-831.

University of Petroşani
University Street, No:20
2675 Petroşani, Romania
email:snoaghi@mail.recep.ro