Higher order Osserman pseudo-Riemannian manifolds of neutral signature $(2,2)$

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Abstract

In this paper we construct a family of pseudo-Riemannian metrics of neutral signature $(2,2)$ which leads to k-Osserman manifolds for all k admissible. For these manifolds the generalized Jacobi operator is 2-nilpotent. Conditions for locally symmetry on the considered manifolds are established.

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Key words: generalized Jacobi operator, locally symmetric.

Let (M,g) be a pseudo-Riemannian manifold of signature (p,q) and dimension $n=p+q$. Let $R(\cdot,\cdot)$ be the Riemannian curvature operator. The Jacobi operator $J(X): Y \to R(Y,X)X$ is a self-adjoint operator and it plays an important role in the study of geodesic variations.

Let $S^{\pm}(M)$ be the pseudo-sphere bundles of unit spacelike $(+)$ and timelike $(-)$ vectors for the manifold (M,g). Then (M,g) is said to be *spacelike Osserman* (respectively *timelike Osserman*) if the eigenvalues of $J(\cdot)$ are constant on $S^+(M,g)$ (respectively on $S^-(M,g)$). The notions spacelike Osserman and timelike Osserman are equivalent and if (M,g) is either of them, then (M,g) is said to be Osserman.

In this paper we study the higher order Jacobi operator, which was first defined by Stanilov and Videv ([9]) in the Riemannian setting. This definition was extended to semi-Riemannian geometry in [6]. Let π be a nondegenerate k-plane in T_pM, with orthonormal basis $\{e_1,\dots,e_k\}$, where (M,g) is a pseudo-Riemannian manifold of signature (p,q). The generalized Jacobi operator is defined by

$$J_R(\pi) = \sum_{i=1}^{k} g(e_i,e_i)R(\cdot,e_i)e_i\,.$$

We say that a pair of integers (r,s) is an admissible pair for T_pM if $0 \le r \le p$, $0 \le s \le q$ and $1 \le s+r \le p+q-1$. This means that the Grassmannian $Gr_{(r,s)}(T_pM)$ of all non-degenerate planes in T_pM of signature (r,s) is non-empty and does not consist of a single point.

Let (r,s) be an admissible pair. We say that (M,g) is Ossermann of type (r,s) in $p \in M$ if the eigenvalues of the operator $J_R(\pi)$ do not depend on the choice of plane $\pi \in Gr_{(r,s)}(T_pM)$.

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P. Gilkey shows that if (M,g) is Osserman of type (r,s) then it is Osserman of type $(\tilde r,\tilde s)$ for all admissible pairs $(\tilde r,\tilde s)$ satisfying $r+s=\tilde r+\tilde s$ ([3], [4]). Thus, only the dimension $k=r+s$ of planes π is relevant and we simply talk about k-Osserman. A semi-Riemannian manifold (M,g) is said to be a k-Osserman manifold if for all points $p\in M$, (M,g) is k-Osserman in p with the eigenvalue structure of $J_{R_p}(\cdot)$ independent of the chosen point p.

Let $M=\mathbf{R}^4$ with coordinates $(x,y)=(x^1,x^2,y^1,y^2)$. Then $\mathcal{X}=Span\{\partial_1^x,\partial_2^x\}$ and $\mathcal{Y}=Span\{\partial_1^y,\partial_2^y\}$ define two distributions of TM. The splitting $TM=\mathcal{X}\bigoplus\mathcal{Y}$ is just the usual splitting $T\mathbf{R}^4=T\mathbf{R}^2\bigoplus T\mathbf{R}^2$. We define a semi-Riemannian metric of neutral signature $(2,2)$ by setting

$$
\begin{aligned}
g_{(f_1,f_2,h)} &= y^1f_1(x^1)dx^1\otimes dx^1+y^2f_2(x^2)dx^2\otimes dx^2+\\
&+ h(x^1,x^2)[dx^1\otimes dx^2+dx^2\otimes dx^1]+\\
&+ a[dx^1\otimes dy^1+dy^1\otimes dx^1+dx^2\otimes dy^2+dy^2\otimes dx^2],
\end{aligned}
\tag{0.1}
$$

where $a\in\mathbf{R}^*$ and f_1,f_2,h are smooth real valued functions. The coefficients of $g_{(f_1,f_2,h)}$ depend on x and y. Furthermore, the distribution $\mathcal{Y}$ is totally isotropic with respect to $g_{(f_1,f_2,h)}$.

Lemma 1 *The only nonvanishing covariant derivatives are given by*

$$
\begin{aligned}
\nabla_{\partial_1^x}\partial_1^x &= -\tfrac{1}{2a}f_1(x^1)\partial_1^x+\left[\tfrac{1}{2a}y^1f_1'(x^1)+\tfrac{y^1}{2a^2}f_1^2(x^1)\right]\partial_1^y+\\
&+ \left[\tfrac{1}{a}\tfrac{\partial h}{\partial x^1}(x^1,x^2)+\tfrac{1}{2a^2}f_1(x^1)h(x^1,x^2)\right]\partial_2^y\,,\\
\nabla_{\partial_2^x}\partial_2^x &= -\tfrac{1}{2a}f_2(x^2)\partial_2^x+\left[\tfrac{1}{2a^2}f_2(x^2)h(x^1,x^2)+\tfrac{1}{a}\tfrac{\partial h}{\partial x^2}(x^1,x^2)\right]\partial_1^y+\\
&+ \left[\tfrac{1}{2a}y^2f_2'(x^2)+\tfrac{y^2}{2a^2}f_2^2(x^2)\right]\partial_2^y\,,\\
\nabla_{\partial_1^x}\partial_1^y &= \tfrac{1}{2a}f_1\partial_1^y\,,\\
\nabla_{\partial_2^x}\partial_2^y &= \tfrac{1}{2a}f_2\partial_2^y\,.
\end{aligned}
\tag{0.2}
$$

From (0.1) we have the following:

Proposition 1 *The only nonvanishing components of the curvature tensor of* $(\mathbf{R}^4,g_{(f_1,f_2,h)})$ *are given by*

$$
\begin{aligned}
R(\partial_1^x,\partial_2^x)\partial_1^x &= -\tfrac{1}{a}\left[\tfrac{\partial^2h}{\partial x^1\partial x^2}+\tfrac{1}{2a}f_2\tfrac{\partial h}{\partial x^1}+\tfrac{1}{2a}f_1\tfrac{\partial h}{\partial x^2}+\tfrac{1}{4a^2}f_1f_2h\right]\partial_2^y\,,\\
R(\partial_1^x,\partial_2^x)\partial_2^x &= \tfrac{1}{a}\left[\tfrac{\partial^2h}{\partial x^1\partial x^2}+\tfrac{1}{2a}f_2\tfrac{\partial h}{\partial x^1}+\tfrac{1}{2a}f_1\tfrac{\partial h}{\partial x^2}+\tfrac{1}{4a^2}f_1f_2h\right]\partial_2^y\,.
\end{aligned}
\tag{0.3}
$$

Theorem 1 *Let $p\geq 2$. Then $(M,g_{(f_1,f_2,h)})$ is k-Osserman for every admissible k.*

Proof. Let be X_1, X_2, X_3 coordinate vector fields. By proposition 1, $J(X_1)X_3=R(X_3,X_1)X_1=0$ if $X_1\in\mathcal{Y}$. Thus $\mathcal{Y}\subset Ker(J(X_1))$. Furthermore, $range\,(J(X_2))\subset span\,\{R(\partial_i^x,\partial_j^x)\partial_k^x\}\subset\mathcal{Y}$. Thus $J(X_1)J(X_2)=0$.
If $\{X_1,X_2,\dots,X_k\}$ is an orthonormal basis for $\pi\in Gr_{(r,s)}(M,g_{(f_1,f_2,h)})$, then we have

$$
J(\pi)^2=\sum_{i,j=1}^{k}g_{(f_1,f_2,h)}(X_i,X_i)g_{(f_1,f_2,h)}(X_j,X_j)J(X_i)J(X_j)=0\,.
$$

Theorem 2 *Let $p \geq 2$. The manifold $(\mathbf{R}^4, g_{(f_1,f_2,h)})$ is a locally symmetric space if and only if the functions f_1, f_2, h are solutions of the following partial differential equations in $\mathbf{R}^2$:*

$$\frac{\partial \Phi}{\partial x^k} + \frac{f_k}{2a}\Phi = 0\,,\ k = 1, 2\,, \tag{0.4}$$

where we note

$$\Phi(x^1, x^2) = \frac{1}{a}\left[\frac{\partial^2 h}{\partial x^1 \partial x^2} + \frac{1}{2a} f_2 \frac{\partial h}{\partial x^1} + \frac{1}{2a} f_1 \frac{\partial h}{\partial x^2} + \frac{1}{4a^2} f_1 f_2 h\right].$$

Proof. If we take in account this notation, we obtain by (0.3)

$$R(\partial_1^x, \partial_2^x)\partial_k^x = (-1)^k \Phi(x^1, x^2)\partial_{3-k}^y\,,\ k = 1, 2\,.$$

Let $X_k = \alpha_i^k \partial_i^x$, $k = \overline{1,4}$, $i = \overline{1,4}$. The condition $\nabla_{X_1} R(X_2, X_3)X_4 = 0$ leads to

$$\nabla_{\alpha_i^1 \partial_i^x} R(\alpha_j^2 \partial_j^x, \alpha_l^3 \partial_l^x)\alpha_s^4 \partial_s^x = 0\,,\ i, j, k, s = \overline{1,4}\,.$$

Equivalently,

$$\alpha_2^1 \alpha_1^2 \alpha_2^3 \alpha_1^4 \nabla_{\partial_2^x} R(\partial_1^x, \partial_2^x)\partial_1^x + \alpha_1^1 \alpha_1^2 \alpha_2^3 \alpha_2^4 \nabla_{\partial_1^x} R(\partial_1^x, \partial_2^x)\partial_2^x +$$

$$+\alpha_2^1 \alpha_2^2 \alpha_1^3 \alpha_1^4 \nabla_{\partial_2^x} R(\partial_2^x, \partial_1^x)\partial_1^x + \alpha_1^1 \alpha_2^2 \alpha_1^3 \alpha_2^4 \nabla_{\partial_1^x} R(\partial_2^x, \partial_1^x)\partial_2^x = 0\,.$$

But

$$\nabla_{\partial_1^x} R(\partial_1^x, \partial_2^x)\partial_2^x = -\nabla_{\partial_1^x} R(\partial_2^x, \partial_1^x)\partial_2^x = \nabla_{\partial_1^x} \Phi \partial_1^y = \left(\frac{\partial \Phi}{\partial x^1} + \frac{f_1}{2a}\Phi\right)\partial_1^y$$

$$\nabla_{\partial_2^x} R(\partial_1^x, \partial_2^x)\partial_1^x = -\nabla_{\partial_2^x} R(\partial_2^x, \partial_1^x)\partial_1^x = -\left(\frac{\partial \Phi}{\partial x^2} + \frac{f_2}{2a}\Phi\right)\partial_2^y\,.$$

The proof is complete.

Corollary 1 *If $h(x^1, x^2) \equiv C$ (h is a constant function), the conditions (0.4) for locally symmetry becames*

$$\begin{cases} f_1'(x^1) f_2(x^2) + \frac{1}{2a} f_1^2(x^1) f_2(x^2) = 0\,, \\ f_2'(x^2) f_1(x^1) + \frac{1}{2a} f_2^2(x^2) f_1(x^1) = 0\,. \end{cases} \tag{0.5}$$

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