Pseudo-Riemannian structures with the same connection

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Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. We study the PDEs systems determined by the equalities between two connections, one produced by a pseudo-Riemannian metric g and other produced by pseudo-Riemannian Hessian metrics $h = \nabla_g^2 f$ respectively $k = \nabla_h^2 f$, where f is the unknown function. In this context we introduce the notion of Hessian-harmonic function. Some solutions of our geometrical PDEs systems are important for Mathematical Optimization and for string theory (WDVV equations) on pseudo-Riemannian manifolds.

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1 Introduction

Studying optimization on the Riemannian manifold we got acquainted with the notion of the Riemannian Hessian metric.

Trying to solve some open problems formulated by C. Udrişte in [9], we replaced the initial Euclidean space $\mathbf{R}^n$ with an arbitrary pseudo-Riemannian manifold. For the beginning, we studied [11] the properties of the pseudo-Riemannian manifold $(M, h = \nabla_g^2 f)$, where (M, g) is an initial pseudo-Riemannian manifold and $f\colon M \to \mathbf{R}$ is a function whose Hessian $\nabla_g^2 f$ with respect to g is non-degenerate and with constant signature. Now, we develope further this theory using an ideea from the paper [7], where is studied the geometry of a general diagonal metric defined on the positive orthant $\mathbf{R}_+^n$ and on the hypercube $C_0^n = (0,1)^n$.

The purpose of this paper is to analyse the geometrical PDEs determined by the equalities between two connections, one produced by a diagonal metric and other produced by pseudo-Riemannian Hessian metrics $h = \nabla_g^2 f$ and, respectively, $k = \nabla_h^2 f$. We are motivated by many important examples and applications of pseudo-Riemannian Hessian structures. For example, see [1], [2] and [10].

The paper is organized as follows:

Section 2 recalls some basic facts of pseudo-Riemannian Hessian geometry and general diagonal geometry. In Section 3 we start with the study of PDEs system determined by the conditions $\overline{\Gamma}_{ij}^k = \Gamma_{ij}^k$ for all $i, j, k = \overline{1, n}$, where Γ_{ij}^k are the Christoffel

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symbols of the general diagonal metric g and $\overline{\Gamma}_{ij}^{k}$ are the Christoffel symbols of the pseudo-Riemannian Hessian metric $h = \nabla_g^2 f$. We determine some particular solutions when f is a separable function. Then we continue with the study of PDEs system $\overline{\overline{\Gamma}}_{ij}^{k} = \Gamma_{ij}^{k}$ for all $i, j, k = \overline{1, n}$, where $\overline{\overline{\Gamma}}_{ij}^{k}$ are the Christoffel symbols of the pseudo-Riemannian Hessian metric $k = \nabla_h^2 f$ of an unknown separable function f. In Section 4 we determine a class of Hessian-selfharmonic functions. In section 5 we characterize the associativity of the algebra $\mathcal{U}(\mathcal{M}, \nabla, \bar{\nabla})$ by a system of PDEs that reduces to zero curvature when the initial metric is Euclidean.

2 Preliminaries on pseudo-Riemannian Hessian Geometry

Let (M, g) be a pseudo-Riemannian manifold and $f\colon M \to \mathbf{R}$ a smooth function. If the Hessian $\nabla_g^2 f$ is non-degenerate and with constant signature, then $h = \nabla_g^2 f$ is a pseudo-Riemannian Hessian metric.

Theorem 2.1 *[11] Let Γ_{ij}^{p} be the Christoffel symbols and R_{ijk}^{m} be the components of the curvature tensor field produced by the pseudo-Riemannian metric g_{ij}. If $f,^{pk}$ are the contravariant components of the pseudo-Riemannian metric $h_{pk} = f_{,pk}$, then the components of Levi-Civita connection ∇_h are given by the following formula*

$$\overline{\Gamma}_{ij}^{p} = \Gamma_{ij}^{p} + \frac{1}{2} f,^{pk} \left[f_{,ijk} + \left(R_{ikj}^{m} + R_{jki}^{m} \right) f_{,m} \right].$$

In the paper [7] E. A. Papa Quiroz and P. Roberto Oliveira derived some geometric properties of the general diagonal Riemannian metric

$$g(x^1, \ldots, x^n) = \begin{pmatrix} \dfrac{1}{g_1^2(x^1)} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \dfrac{1}{g_n^2(x^n)} \end{pmatrix}$$

defined on the positive orthant $\mathbf{R}_+^n$, where $g_i\colon \mathbf{R}_+ \to \mathbf{R} \setminus \{0\}$ are differentiable functions. Due to the fact that the metric on the hypercube $C_0^n = (0, 1)^n$ is induced by the metric on $\mathbf{R}_+^n$, the properties hold on the hypercube.

Thus the Christoffel symbols of metric g are given by the formula

$$\Gamma_{ij}^{m} = -\frac{1}{g_i(x^i)} \frac{\partial g_i(x^i)}{\partial x^i} \delta_{im} \delta_{ij}$$

or equivalent $\Gamma_{ii}^{i} = -\dfrac{1}{g_i(x^i)} \dfrac{\partial g_i(x^i)}{\partial x^i}$ and 0 in rest. They also proved that the Riemannian manifold $\mathbf{R}_+^n$ endowed with the metric g has null curvature, i. e., $R_{ijk}^{\ell} = 0$ for all $i, j, k, \ell = \overline{1, n}$.

3 PDEs representing the equality of suitable connections

3.1 First step

Let us consider the pseudo-Riemannian manifold (M,g). Let us introduce a smooth function $f\colon M\to\mathbf{R}$ having a non-degenerate and of constant signature Hessian $h=\nabla_g^2 f$.

Then the pseudo-Riemannian Hessian metric $h=\nabla_g^2 f$ has the Christoffel symbols given by the formula

$$\overline{\Gamma}^p_{ij}=\Gamma^p_{ij}+\frac{1}{2}f,^{pk}\left[f_{,ijk}+\left(R^m_{ikj}+R^m_{jki}\right)f_{,m}\right].$$

The condition $\overline{\Gamma}=\Gamma$ is reduced to the PDEs system

$$f_{,ijk}+\left(R^m_{ikj}+R^m_{jki}\right)f_{,m}=0$$

with the unknown function f.

In the particular case when $M=\mathbf{R}^n_+$ and the Riemannian metric is of diagonal type, i.e.,

$$g(x^1,\dots,x^n)=\begin{pmatrix}\dfrac{1}{g_1^2(x^1)} & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \dfrac{1}{g_n^2(x^n)}\end{pmatrix}$$

it follows that $R^m_{ijk}=0$ for all $i,j,k,m=\overline{1,n}$. Thus the conditions $\overline{\Gamma}^p_{ij}=\Gamma^p_{ij}$ for all $i,j,p=\overline{1,n}$ are equivalent to $f_{,ijk}=0$ for all $i,j,k=\overline{1,n}$ or to the PDEs system

$$\frac{\partial f_{,ij}}{\partial x^k}-\Gamma^\ell_{ki}f_{,\ell j}-\Gamma^\ell_{kj}f_{,\ell i}=0,\quad \forall i,j,k=\overline{1,n}. \tag{3.1}$$

First case. $i=j=k$

We have

$$\frac{\partial f_{,ii}}{\partial x^i}-2\Gamma^i_{ii}f_{,ii}=0. \tag{3.2}$$

But $\Gamma^i_{ii}=-\dfrac{1}{g_i(x^i)}\dfrac{\partial g_i(x^i)}{\partial x^i}$ and $f_{,ii}=\dfrac{\partial^2 f}{\partial (x^i)^2}-\Gamma^m_{ii}f_{,m}$. Since $\Gamma^m_{ii}\neq 0$ only for $i=m$, we may write that

$$f_{,ii}=\frac{\partial^2 f}{\partial (x^i)^2}+\frac{1}{g_i(x^i)}\frac{\partial g_i(x^i)}{\partial x^i}\frac{\partial f}{\partial x^i}.$$

Then the relation (3.2) becomes

$$\frac{\partial^3 f}{\partial (x^i)^3}+\frac{1}{g_i(x^i)}\frac{\partial g_i(x^i)}{\partial x^i}\frac{\partial^2 f}{\partial (x^i)^2}+\left[-\frac{1}{g_i^2(x^i)}\left(\frac{\partial g_i(x^i)}{\partial x^i}\right)^2+\frac{1}{g_i(x^i)}\frac{\partial^2 g_i(x^i)}{\partial (x^i)^2}\right]\frac{\partial f}{\partial x^i}$$
$$+2\frac{1}{g_i(x^i)}\frac{\partial g_i(x^i)}{\partial x^i}\left[\frac{\partial^2 f}{\partial (x^i)^2}+\frac{1}{g_i(x^i)}\frac{\partial g_i(x^i)}{\partial x^i}\frac{\partial f}{\partial x^i}\right]=0$$

or equivalent
$$\frac{\partial}{\partial x^i}\left[g_i(x^i)\frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^i}g_i(x^i)\right)\right]=0,\quad \forall i=\overline{1,n}.$$

Second case. $i=j\neq k$

The system of PDEs (3.1) takes the form $\dfrac{\partial f_{,ii}}{\partial x^k}=0$ or equivalent
$$\frac{\partial}{\partial x^i}\left[\frac{\partial^2 f}{\partial x^k\partial x^i}g_i(x^i)\right]=0,\quad \forall i\neq k.$$

The third case. $i\neq j$

1) $i\neq k$ and $j\neq k$. The system of PDEs (3.1) has the form $\dfrac{\partial f_{,ij}}{\partial x^k}=0$ or equivalent $\dfrac{\partial^3 f}{\partial x^i\partial x^j\partial x^k}=0$ for all $i\neq j\neq k$.

2) $i\neq j$ and $i=k$. The system of PEDs (3.1) takes the form $\dfrac{\partial f_{,ij}}{\partial x^i}-\Gamma^i_{ii}f_{,ij}=0$ or equivalent $\dfrac{\frac{\partial}{\partial x^i}\left[g_i(x^i)f_{,ij}\right]}{g_i(x^i)}=0$. Since $i\neq j$, it follows that $f_{,ij}=\dfrac{\partial^2 f}{\partial x^i\partial x^j}$.

Hence we obtain $\dfrac{\partial}{\partial x^i}\left[g_i(x^i)\dfrac{\partial^2 f}{\partial x^i\partial x^j}\right]=0$.

3) $j=k$ and $i\neq k$. We find $\dfrac{\partial}{\partial x^j}\left[g_j(x^j)\dfrac{\partial^2 f}{\partial x^i\partial x^j}\right]=0$. Thus the system of PDEs (3.1) is

$$\text{(3.3)}\qquad\begin{cases}\dfrac{\partial}{\partial x^i}\left[g_i(x^i)\dfrac{\partial}{\partial x^i}\left(\dfrac{\partial f}{\partial x^i}g_i(x^i)\right)\right]=0, & \forall i=\overline{1,n}\\ \dfrac{\partial}{\partial x^i}\left[\dfrac{\partial^2 f}{\partial x^k\partial x^i}g_i(x^i)\right]=0, & \forall i\neq k\\ \dfrac{\partial^3 f}{\partial x^i\partial x^j\partial x^k}=0, & \forall i\neq j\neq k\\ \dfrac{\partial}{\partial x^i}\left[g_i(x^i)\dfrac{\partial^2 f}{\partial x^i\partial x^j}\right]=0, & \forall i\neq j\\ \dfrac{\partial}{\partial x^j}\left[g_j(x^j)\dfrac{\partial^2 f}{\partial x^i\partial x^j}\right]=0, & \forall i\neq j.\end{cases}$$

Remark 3.1 *We also have to impose the condition that $h=\nabla^2_g f$ is non-degenerate, which is equivalent to* $\det(\mathrm{h_{ij}})\neq 0$*, where* $h_{ij}=f_{,ij}=\dfrac{\partial^2 f}{\partial x^i\partial x^j}$*, for all $i\neq j$ and* $h_{ii}=f_{,ii}=\dfrac{\partial^2 f}{\partial (x^i)^2}+\dfrac{1}{g_i(x^i)}\dfrac{\partial g_i(x^i)}{\partial x^i}\dfrac{\partial f}{\partial x^i}$.

Particular case

We suppose that the unknown function f is a separable function $f\colon \mathbf{R}^n_+\to\mathbf{R}$, $f\left(x^1,\dots,x^n\right)=f_1(x^1)+f_2(x^2)+\cdots+f_n(x^n)$, where $f_i\colon\mathbf{R}_+\to\mathbf{R}$ are differentiable functions. It follows that

$$\frac{\partial f}{\partial x^i} = \frac{\partial f_i}{\partial x^i}, \quad \frac{\partial^2 f}{\partial (x^i)^2} = \frac{\partial^2 f_i}{\partial (x^i)^2} \quad \text{and} \quad \frac{\partial^2 f}{\partial x^i \partial x^j} = 0, \quad \forall i \neq j.$$

All equations of system of PDEs (3.3), excepting the first, are satisfied identically.

The first equation takes the form

$$\frac{\partial}{\partial x^i}\left[g_i(x^i)\frac{\partial}{\partial x^i}\left(\frac{\partial f_i}{\partial x^i} g_i(x^i)\right)\right] = 0.$$

The condition $\det(\mathrm{h_{ij}}) \neq 0$ is equivalent to $h_{11}h_{22}\cdots h_{nn} \neq 0$ or $h_{ii} \neq 0$ for all $i = \overline{1,n}$. Hence we have to impose the conditions

$$\frac{\partial^2 f_i}{\partial (x^i)^2} + \frac{1}{g_i(x^i)}\frac{\partial g_i(x^i)}{\partial x^i}\frac{\partial f_i}{\partial x^i} \neq 0, \quad \forall i = \overline{1,n}$$

or equivalent $\frac{\partial}{\partial x^i}\left[g_i(x^i)\frac{\partial f_i}{\partial x^i}\right] \neq 0$ for all $i = \overline{1,n}$.

Therefore the system of PDEs (3.3) takes the form

$$\begin{cases} \frac{\partial}{\partial x^i}\left[g_i(x^i)\frac{\partial}{\partial x^i}\left(\frac{\partial f_i}{\partial x^i} g_i(x^i)\right)\right] = 0 \\ \frac{\partial}{\partial x^i}\left[\frac{\partial f_i}{\partial x^i} g_i(x^i)\right] \neq 0, \quad i = \overline{1,n}. \end{cases}$$

Since in each equation it appears only the variable x^i, we replace x^i by x, g_i by G and f_i by F.

Hence we may write $\begin{cases} \left[G(GF')'\right]' = 0 \\ (GF')' \neq 0 \end{cases}$ or equivalent $\begin{cases} G(GF')' = c \\ GF' \neq k, \end{cases}$ where c and k are real constants.

In the following we shall derive some solutions of this system.

1) We seek F such that $F'G = \ln G$. Then the equation $G(F'G)' = c$ produces a constraint for G, i. e., $G\frac{G'}{G} = c$. Consequently G must have the form $G(x) = cx + b$, where b and c are real positive constants.

From the relation $F'G = \ln G$ it follows that $F' = \frac{\ln G}{G}$, hence

$$F(x) = \frac{\ln^2(cx+b)}{2c}.$$

Therefore on $M = \mathbf{R}^n_+$, the initial metric is

$$g(x^1,\ldots,x^n) = \begin{pmatrix} \frac{1}{(cx^1+b)^2} & 0 & \cdots & 0 \\ \cdots & & & \\ 0 & 0 & \cdots & \frac{1}{(cx^n+b)^2} \end{pmatrix}$$

and the function f is

$$f(x^1,\dots,x^n)=\frac{1}{2c}\ln^2(cx^1+b)+\dots+\frac{1}{2c}\ln^2(cx^n+b),$$

where $b,c>0$ and $(x^1,\dots,x^n)\in\mathbf{R}^n_+$. Moreover $F'G=\ln G\neq k$.

2) We seek F such that $F'G=\dfrac{1}{G}$. Then the constraint for G becomes $G'=cG$, hence $G(x)=e^{-cx+b}$. We consider the solution $G(x)=e^{-cx}$.

From the relation $F'G=\dfrac{1}{G}$, it follows that $F'=\dfrac{1}{G^2}=e^{2cx}$, hence

$$F(x)=\frac{e^{2cx}}{2c},\quad c\neq 0.$$

Therefore on $M=\mathbf{R}^n_+$, the initial metric is

$$g(x^1,\dots,x^n)=\begin{pmatrix} e^{2cx^1} & 0 & \cdots & 0\\ \cdots & & & \\ 0 & 0 & \cdots & e^{2cx^n}\end{pmatrix}$$

and $f(x^1,\dots,x^n)=\dfrac{1}{2c}e^{2cx^1}+\dots+\dfrac{1}{2c}e^{2cx^n}$. Moreover $F'G=\dfrac{1}{G}\neq k$.

Remark 3.2 *a) The Riemannian metric g from the previous example was related to an n-dimensional ecological Volterra-Hamiltonian system of ordinary differential equations by Antonelli [2].*

b) In the paper [8] T. Rapcsák and T. Csendes use the metric g in order to discuss nonlinear coordinate transformations.

3) If the unknown function F satisfy the relation $F'G=G^\alpha$, $\alpha>0$, then the constraint for G is $\alpha G^\alpha G'=c$. In other words, $G(x)=\left[\dfrac{\alpha+1}{\alpha}(cx+\alpha b)\right]^{\frac{1}{\alpha+1}}$ and

$$F(x)=\int G^{\alpha-1}(x)dx=\frac{1}{2c}\left(\frac{\alpha+1}{\alpha}\right)^\alpha(cx+\alpha b)^{\frac{2\alpha}{\alpha+1}},\quad b,c>0.$$

Therefore on $M=\mathbf{R}^n_+$, the initial metric is

$$g(x^1,\dots,x^n)=\begin{pmatrix} \dfrac{1}{\left[\dfrac{\alpha+1}{\alpha}(cx^1+\alpha b)\right]^{\frac{2}{\alpha+1}}} & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \dfrac{1}{\left[\dfrac{\alpha+1}{\alpha}(cx^n+\alpha b)\right]^{\frac{2}{\alpha+1}}}\end{pmatrix}$$

and

$$f(x^1,\dots,x^n)=\frac{1}{2c}\left(\frac{\alpha+1}{\alpha}\right)^\alpha(cx^1+\alpha b)^{\frac{2\alpha}{\alpha+1}}+\dots+\frac{1}{2c}\left(\frac{\alpha+1}{\alpha}\right)^\alpha(cx^n+\alpha b)^{\frac{2\alpha}{\alpha+1}},\ b,c>0.$$

4) We seek F such that $F'G = F'$. Then the equation $G(F'G)' = c$ produces a constraint for G, i. e. $G(x) = 1$ and the equation $G(F'G)' = c$ becomes $F'' = c$, hence $F(x) = \dfrac{cx^2}{2} + bx + d$.

Thus on $M = \mathbf{R}^n_+$, the initial metric $g(x^1, \ldots, x^n) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ is Euclidean and $f(x^1, \ldots, x^n) = \left(\dfrac{c(x^1)^2}{2} + bx^1 + d\right) + \cdots + \left(\dfrac{c(x^n)^2}{2} + bx^n + d\right)$.

5) We seek F such that $F'G = \dfrac{1}{F'}$. Then $G = \dfrac{1}{(F')^2}$. Introducing G into the equation $G(F'G)' = c$ we have $F''(F')^{-4} = -c$, hence $F'^{-3} = 3cx - 3b$. Then $F(x) = \dfrac{1}{2c}(3cx - 3b)^{\frac{2}{3}}$ and $G(x) = \dfrac{1}{(F'(x))^2} = (3cx - 3b)^{\frac{2}{3}}$. Since $x > 0$, we choose $b < 0$ and $c > 0$. Then $3cx - 3b \neq 0$.

Therefore on $M = \mathbf{R}^n_+$, the metric g is

$$g(x^1, \ldots, x^n) = \begin{pmatrix} \dfrac{1}{(3cx^1 - 3b)^{\frac{4}{3}}} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \dfrac{1}{(3cx^n - 3b)^{\frac{4}{3}}} \end{pmatrix}$$

and $f(x^1, \ldots, x^n) = \dfrac{1}{2c}(3cx^1 - 3b)^{\frac{2}{3}} + \cdots + \dfrac{1}{2c}(3cx^n - 3b)^{\frac{2}{3}}$, $b < 0$ and $c > 0$.

3.2 Second step

Let us consider the Riemannian manifold $(\mathbf{R}^n_+, g)$, where

$$g(x^1, \ldots, x^n) = \begin{pmatrix} \dfrac{1}{g_1^2(x^1)} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \dfrac{1}{g_n^2(x^n)} \end{pmatrix},$$

$(x^1, \ldots, x^n) \in \mathbf{R}^n_+$ is of diagonal type and $g_i : \mathbf{R}_+ \to \mathbf{R} \backslash \{0\}$ are differentiable functions for all $i = \overline{1, n}$. We introduce a smooth separable function $f : \mathbf{R}^n_+ \to \mathbf{R}$,

$$f(x^1 \ \ldots, x^n) = f_1(x^1) + \cdots + f_n(x^n),$$

having the Hessian with respect to g non-degenerate and of constant signature.

We also suppose that the Hessian of f with respect to $h = \nabla^2_g f$ is non-degenerate and of constant signature.

In the following we study the system of PDEs determined by the conditions $\overline{\overline{\Gamma}}{}^p_{ij} = \Gamma^p_{ij}$ for all $i, j, p = \overline{1, n}$, where $\overline{\overline{\Gamma}}{}^p_{ij}$ are the Christoffel symbols produced by the pseudo-Riemannian Hessian metric $k = \nabla^2_h f$, where $h = \nabla^2_g f$.

Firstly, let us calculate the Christoffel symbols $\overline{\Gamma}{}^p_{ij}$ when f is a smooth separable function. We have already derived that the pseudo-Riemannian Hessian metric $h = \nabla^2_g f$ has the form

$$h(x^1,\dots,x^n)=\begin{pmatrix} \dfrac{\partial^2 f_1}{\partial(x^1)^2}+\dfrac{1}{g_1(x^1)}\dfrac{\partial g_1(x^1)}{\partial x^1}\dfrac{\partial f_1}{\partial x^1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \dfrac{\partial^2 f_n}{\partial(x^n)^2}+\dfrac{1}{g_n(x^n)}\dfrac{\partial g_n(x^n)}{\partial x^n}\dfrac{\partial f_n}{\partial x^n} \end{pmatrix}$$

and since h is of diagonal type metric, it follows that the components of the curvature tensor field $\bar{R}$ are 0, i. e., $\bar{R}^{\ell}_{ijk}=0$ for all $i,j,k,\ell=\overline{1,n}$.

We also deduced the Christoffel symbols $\overline{\Gamma}^p_{ij}=\Gamma^p_{ij}+\dfrac{1}{2}f,^{pk}f,_{ijk}$ produced by the metric h. The inverse metric h^{-1} is

$$\begin{aligned} h^{-1}&=\left(f,^{kp}\right)=\left(\frac{1}{f,_{kp}}\right)\\ &=\begin{pmatrix} \dfrac{1}{\dfrac{\partial^2 f_1}{\partial(x^1)^2}+\dfrac{1}{g_1(x^1)}\dfrac{\partial g_1(x^1)}{\partial x^1}\dfrac{\partial f_1}{\partial x^1}} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \dfrac{1}{\dfrac{\partial^2 f_n}{\partial(x^n)^2}+\dfrac{1}{g_n(x^n)}\dfrac{\partial g_n(x^n)}{\partial x^n}\dfrac{\partial f_n}{\partial x^n}} \end{pmatrix}. \end{aligned}$$

Hence, if $k\neq p$, then $f,^{kp}=0$. Thus $\overline{\Gamma}^p_{ij}=\Gamma^p_{ij}+\dfrac{1}{2}f,^{pp}f,_{ijp}$ for all $i,j,p=\overline{1,n}$.

We use the formula $f,_{ijp}=\dfrac{\partial f,_{ij}}{\partial x^p}-\Gamma^{\ell}_{pi}f,_{\ell j}-\Gamma^{\ell}_{pj}f,_{\ell i}$. Since $f,_{\ell j}\neq 0$ only for $\ell=j$, $f,_{\ell i}\neq 0$ only for $\ell=i$ and $\dfrac{\partial f,_{ij}}{\partial x^p}\neq 0$ only for $i=j=p$, we have that $f,_{ijp}\neq 0$ only for $i=j=p$. Moreover $f,_{iii}=\dfrac{\partial f,_{ii}}{\partial x^i}-2\Gamma^i_{ii}f,_{ii}$. Then

$$\begin{aligned} \overline{\Gamma}^i_{ii}&=\Gamma^i_{ii}+\frac{1}{2}f,^{ii}f,_{iii}=\Gamma^i_{ii}+\frac{1}{2f,_{ii}}\left(\frac{\partial f,_{ii}}{\partial x^i}-2\Gamma^i_{ii}f,_{ii}\right)=\frac{1}{2f,_{ii}}\frac{\partial f,_{ii}}{\partial x^i}\\ &=\frac{1}{2}\frac{\partial}{\partial x^i}\left(\ln f,_{ii}\right). \end{aligned}$$

But we proved that $f,_{ii}=\dfrac{\dfrac{\partial}{\partial x^i}\left[\dfrac{\partial f_i}{\partial x^i}g_i(x^i)\right]}{g_i(x^i)}$. Therefore

$$\overline{\Gamma}^i_{ii}=\frac{1}{2}\frac{\partial}{\partial x^i}\left[\ln\frac{\dfrac{\partial}{\partial x^i}\left(\dfrac{\partial f_i}{\partial x^i}g_i(x^i)\right)}{g_i(x^i)}\right]$$

and $\overline{\Gamma}^p_{ij}=0$ in rest.

The Hessian of f with respect to h, $k=\nabla^2_h f$ has the components $k_{ij}=\bar{f},_{ij}=0$ if $i\neq j$ and

$$k_{ii} = \bar{f}_{,ii} = \frac{\partial^2 f_i}{\partial (x^i)^2} - \overline{\Gamma}^i_{ii} f_{,i} = \frac{\partial^2 f_i}{\partial (x^i)^2} - \frac{1}{2}\frac{\partial}{\partial x^i}\left[\ln \frac{\frac{\partial}{\partial x^i}\left(\frac{\partial f_i}{\partial x^i} g_i(x^i)\right)}{g_i(x^i)}\right]\frac{\partial f_i}{\partial x^i}.$$

The Hessian k is non-degenerate if and only if $\det(\mathrm{k_{ij}}) \neq 0$ or equivalent $k_{ii} \neq 0$ for all $i = \overline{1,n}$.

In order to study the system $\overline{\overline{\Gamma}}^p_{ij} = \Gamma^p_{ij}$ for all $i, j, p = \overline{1,n}$, we use the relations $\overline{\overline{\Gamma}}^i_{ii} = \overline{\Gamma}^i_{ii} + \frac{1}{2}\bar{f}_,{}^{ii}\bar{f}_{,iii}$ and $\overline{\Gamma}^i_{ii} = \Gamma^i_{ii} + \frac{1}{2} f_,{}^{ii} f_{,iii}$. Then the conditions $\overline{\overline{\Gamma}}^i_{ii} = \Gamma^i_{ii}$ lead us to

$$f_,{}^{ii} f_{,iii} = -\bar{f}_,{}^{ii}\bar{f}_{,iii}\,. \tag{3.4}$$

But $f_,{}^{ii} f_{,iii} = \frac{1}{f_{,ii}}\left(\frac{\partial f_{,ii}}{\partial x^i} - 2\Gamma^i_{ii} f_{,ii}\right) = \frac{1}{f_{,ii}}\frac{\partial f_{,ii}}{\partial x^i} - 2\Gamma^i_{ii}$ and also

$$\bar{f}_,{}^{ii}\bar{f}_{,iii} = \frac{1}{\bar{f}_{,ii}}\frac{\partial \bar{f}_{,ii}}{\partial x^i} - 2\overline{\Gamma}^i_{ii} = \frac{1}{\bar{f}_{,ii}}\frac{\partial \bar{f}_{,ii}}{\partial x^i} - 2\left(\Gamma^i_{ii} + \frac{1}{2}\frac{1}{f_{,ii}} f_{,iii}\right).$$

Then (3.4) is equivalent to $\frac{1}{\bar{f}_{,ii}}\frac{\partial \bar{f}_{,ii}}{\partial x^i} = 2\Gamma^i_{ii}$ or $\frac{\frac{\partial \bar{f}_{,ii}}{\partial x^i}}{\bar{f}_{,ii}} = -2\frac{\frac{\partial g_i(x^i)}{\partial x^i}}{g_i(x^i)}$. By integration, we deduce that

$$\bar{f}_{,ii} g_i^2 = c, \tag{3.5}$$

where c is a real constant. Using the formula for $\bar{f}_{,ii}$ we obtain

$$\left\{\frac{\partial^2 f_i}{\partial (x^i)^2} - \frac{1}{2}\frac{\partial}{\partial x^i}\left[\ln \frac{\frac{\partial}{\partial x^i}\left(\frac{\partial f_i}{\partial x^i} g_i(x^i)\right)}{g_i(x^i)}\right]\frac{\partial f_i}{\partial x^i}\right\} g_i^2 = c. \tag{3.6}$$

Since it appears only the variable x^i, we replace x^i by x, g_i by G and f_i by F. The relation (3.6) takes the form $\left\{F'' - \frac{1}{2}\left[\ln \frac{(F'G)'}{G}\right]' F'\right\} G^2 = c$ or equivalent $F'\left\{\frac{F''}{F'} - \frac{1}{2}\left[\ln \frac{(F'G)'}{G}\right]'\right\} G^2 = c$. Since $\frac{F''}{F'} = (\ln F')'$ we obtain $F'G^2\left[\ln \frac{F'^2}{\frac{(F'G)'}{G}}\right]' = 2c$. But $\frac{F'^2}{\frac{(F'G)'}{G}} = \left[\frac{(F'G)'}{F'^2 G}\right]^{-1}$ and then we may write that

$$F'G^2\left[\ln \frac{(F'G)'}{F'^2 G}\right]' = -2c.$$

Introducing the function $P = F'G$, we deduce

$$GP\left(\ln \frac{P'G}{P^2}\right)' = -2c. \tag{3.7}$$

We also have to impose the condition $k_{ii} \neq 0$, which is equivalent to $\frac{P}{G}\left(\ln \frac{P'G}{P^2}\right)' \neq 0$.

In the following we shall find some solutions of this system.

1) We seek F such that $\ln \frac{P'G}{P^2} = \ln PG$. Then $P^3 = P'$, hence

$$P(x) = \pm \frac{1}{\sqrt{-2x-2a}}.$$

We recall that $x > 0$. We choose $P(x) = \frac{1}{\sqrt{-2x-2a}}$ and $a < 0$ such that $-x-a > 0$.

The relation (3.7) becomes $GP(\ln PG)' = -2c$ or equivalent $(PG)' = -2c$, hence $P(x)G(x) = -2cx+b$. Then $G(x) = (-2cx+b)\sqrt{-2x-2a}$. From the relation $F'(x) = \frac{P(x)}{G(x)} = \frac{1}{(2cx-b)(2x+2a)}$, it follows that

$$F(x) = \frac{1}{2(2ac+b)} \ln \left| \frac{2cx-b}{x+a} \right|.$$

But $-x-a > 0$, hence $|x+a| = -x-a$. We choose $c > 0$, $b < 0$ and then $F(x) = \frac{1}{2(2ac+b)} \ln \left(\frac{2cx-b}{-x-a} \right)$.

Therefore M becomes a hypercube

$$M = \left\{ (x^1, \ldots, x^n) \in \mathbf{R}^n_+ \mid 0 < x^i < -a,\ i = \overline{1,n},\ a < 0 \right\},$$

the initial metric g is

$$g(x^1, \ldots, x^n) = \begin{pmatrix} \frac{1}{(-2cx^1+b)^2(-2x^1-2a)} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{(-2cx^n+b)^2(-2x^n-2a)} \end{pmatrix}$$

and $f(x^1, \ldots, x^n) = \frac{1}{2(2ac+b)} \ln \left(\frac{2cx^1-b}{-x^1-a} \right) + \cdots + \frac{1}{2(2ac+b)} \ln \left(\frac{2cx^n-b}{-x^n-a} \right)$, where $a < 0$, $c > 0$ and $b < 0$ are real constants.

2) The relation (3.7) is also equivalent to

$$GP \frac{\left(\frac{P'G}{P^2}\right)'}{\frac{P'G}{P^2}} = -2c \quad \text{or} \quad \frac{P^3}{P'} \left(\frac{P'G}{P^2} \right)' = -2c.$$

We seek F such that $\left(\frac{P'G}{P^2}\right)' = P^\alpha$, $\alpha > 0$. We have $\frac{P^3}{P'} P^\alpha = -2c$ or $\frac{P^{-\alpha-2}}{-\alpha-2} = -\frac{x}{2c} + a$. It follows that $P(x) = \left[(\alpha+2)\left(\frac{x}{2c} - a\right) \right]^{-\frac{1}{\alpha+2}}$. From the constraint $\left(\frac{P'G}{P^2}\right)' = P^\alpha$ we obtain that

$$\frac{P'(x)G(x)}{P^2(x)} = \int P^\alpha(x)dx = c\left[(\alpha+2)\left(\frac{x}{2c}-a\right)\right]^{\frac{2}{\alpha+2}},$$

hence

$$G(x) = -2c^2\left[(\alpha+2)\left(\frac{x}{2c}-a\right)\right]^{\frac{\alpha+3}{\alpha+2}}.$$

But $F'(x) = \dfrac{P(x)}{G(x)}$, hence

$$F(x) = \int \frac{P(x)}{G(x)}dx = -\frac{1}{c(\alpha+2)}\left[(\alpha+2)\left(\frac{x}{2c}-a\right)\right]^{-\frac{2}{\alpha+2}}.$$

We take $c<0$, $a>0$ and then it follows that $\dfrac{x}{2c}-a \neq 0$.

Therefore $M = \mathbf{R}^n_+$,

$$g(x^1,\ldots,x^n)=\begin{pmatrix} \dfrac{1}{4c^4\left[(\alpha+2)\left(\dfrac{x^1}{2c}-a\right)\right]^{\frac{2(\alpha+3)}{\alpha+2}}} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \dfrac{1}{4c^4\left[(\alpha+2)\left(\dfrac{x^n}{2c}-a\right)\right]^{\frac{2(\alpha+3)}{\alpha+2}}} \end{pmatrix}$$

and

$$\begin{aligned} f(x^1,\ldots,x^n) &= -\frac{1}{c(\alpha+2)}\left[(\alpha+2)\left(\frac{x^1}{2c}-a\right)\right]^{-\frac{2}{\alpha+2}} \\ &\quad +\cdots+\frac{-1}{c(\alpha+2)}\left[(\alpha+2)\left(\frac{x^n}{2c}-a\right)\right]^{-\frac{2}{\alpha+2}}. \end{aligned}$$

3) If $M = \mathbf{R}^n_+$ is endowed with the Euclidean metric, then $G(x)=1$. The relation (3.7) becomes $P\left(\ln\dfrac{P'}{P^2}\right)' = -2c$. We found two solutions of this equation.

3.1 It is easy to check that $P(x) = \operatorname{tg} x$ is a solution for $c=1$. Then $F'(x) = \dfrac{P(x)}{G(x)} = \operatorname{tg} x$ and hence $F(x) = \displaystyle\int \operatorname{tg} x dx = -\ln|\cos x|$. We choose $x \in \left(0, \dfrac{\pi}{2}\right)$, then $F(x) = -\ln(\cos x)$. Therefore M is the hypercube

$$M = \left\{(x^1,\ldots,x^n) \in \mathbf{R}^n_+ \mid 0 < x^i < \frac{\pi}{2},\ \forall i = \overline{1,n}\right\},$$

$$g(x^1,\ldots,x^n) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

and

$$f(x^1,\ldots,x^n) = -\ln(\cos x^1) - \cdots - \ln(\cos x^n).$$

3.2 The function $P(x) = -\operatorname{ctg} x$ is also a solution for $c=1$. Then

$$F'(x) = \frac{P(x)}{G(x)} = -\operatorname{ctg} x \quad \text{and} \quad F(x) = \int -\operatorname{ctg} x dx = -\ln|\sin x|.$$

We choose $x \in \left(0, \frac{\pi}{2}\right)$ and then $F(x) = -\ln(\sin x)$. That is why

$$M = \left\{(x^1, \dots, x^n) \in \mathbf{R}^n_+ \mid 0 < x^i < \frac{\pi}{2},\ \forall i = \overline{1,n}\right\},$$

$$g(x^1, \dots, x^n) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

and

$$f(x^1, \dots, x^n) = -\ln(\sin x^1) - \cdots - \ln(\sin x^n).$$

4 Hessian-harmonic functions

Let us consider an n-dimensional pseudo-Riemannian manifold (M, g). We suppose that there exists a function $f\colon M \to \mathbf{R}$ such that the Hessian $h = \nabla^2_g f$ is non-degenerate and of constant signature. Let us consider a smooth function $\phi\colon M \to \mathbf{R}$.

Definition 4.1 *ϕ is called Hessian-harmonic function if it has the property that the Laplacian*

$$\Delta_h \phi = 0.$$

In a coordinate chart $(U, x^1, \dots, x^n)$ on M, the condition $\Delta_h \phi = 0$ becomes

$$f^{ij}_{,} \left(\frac{\partial^2 \phi}{\partial x^i \partial x^j} - \bar{\Gamma}^k_{ij} \frac{\partial \phi}{\partial x^k} \right) = 0,$$

where $\bar{\Gamma}^k_{ij}$ are the Christoffel components of connection ∇_h. Using the theorem 2.1 we may write

$$f^{ij}_{,} \left\{ \frac{\partial^2 \phi}{\partial x^i \partial x^j} - \left[\Gamma^k_{ij} + \frac{1}{2} f^{kp}_{,} \left(f_{,ijp} + \left(R^m_{ipj} + R^m_{jpi} \right) f_{,m} \right) \right] \frac{\partial \phi}{\partial x^k} \right\} = 0$$

or, equivalent,

$$f^{ij}_{,} \phi_{,ij} - \frac{1}{2} f^{ij}_{,} f^{kp}_{,} \left[f_{,ijp} + \left(R^m_{ipj} + R^m_{jpi} \right) f_{,m} \right] \phi_{,k} = 0. \tag{4.1}$$

Definition 4.2 *f is called Hessian-selfharmonic if $\Delta_h f = 0$.*

Thus, if $\phi = f$, then (4.1) becomes

$$n - \frac{1}{2} f^{ij}_{,} f^{kp}_{,} \left[f_{,ijp} + \left(R^m_{ipj} + R^m_{jpi} \right) f_{,m} \right] f_{,k} = 0. \tag{4.2}$$

We notice that all indices i, j, k, p are indices of sum.

Particular case. If $M = \mathbf{R}^n_+$, the initial metric g is of diagonal type, i.e.

$$g(x^1,\dots,x^n) = \begin{pmatrix} \dfrac{1}{g_1^2(x^1)} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \dfrac{1}{g_n^2(x^n)} \end{pmatrix}$$

and

$$f\colon \mathbf{R}_+^n \to \mathbf{R}, \quad f(x^1,\dots,x^n) = f_1(x^1) + \cdots + f_n(x^n)$$

is a smooth separable functions, then $\Gamma^i_{ii} = -\dfrac{1}{g_i(x^i)}\dfrac{\partial g_i(x^i)}{\partial x^i}$, $\forall i = \overline{1,n}$, and 0 in rest, $R^\ell_{ijk} = 0$, $\forall i,j,k = \overline{1,n}$, $f_{,i} = \dfrac{\partial f_i}{\partial x^i}$,

$$f_{,ij} = \begin{cases} 0, & \text{if } i \neq j, \\ \dfrac{\partial^2 f}{\partial (x^i)^2}, & \text{if } i = f, \end{cases} \qquad f_{,}^{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ \dfrac{1}{f_{,ii}}, & \text{if } i = f. \end{cases}$$

The relation (4.2) takes the form $n - \dfrac{1}{2}\displaystyle\sum_{i,p} f_{,}^{ii} f_{,}^{pp} f_{,iip} f_{,p} = 0$. But $f_{,iip} \neq 0$ only if $i = p$, hence we obtain $n - \dfrac{1}{2}\displaystyle\sum_i (f_{,}^{ii})^2 f_{,iii} f_{,i} = 0$ or, equivalent,

$$\sum_i \left[\frac{f_{,iii} f_{,i}}{(f_{,ii})^2} - 2\right] = 0. \tag{4.3}$$

If

$$\frac{f_{,iii} f_{,i}}{(f_{,ii})^2} - 2 = 0, \tag{4.4}$$

for all $i = \overline{1,n}$, then $f(x^1,\dots,x^n) = f_1(x^1) + \cdots + f_n(x^n)$ satisfies relation (4.3).

In the following we determine a class of functions which satisfy relation (4.4). We use

$$f_{,ii} = \frac{\dfrac{\partial}{\partial x^i}\left[g_i(x^i)\dfrac{\partial f_i(x^i)}{\partial x^i}\right]}{g_i(x^i)}, \qquad f_{,iii} = \frac{\dfrac{\partial}{\partial x^i}\left[g_i(x^i)\dfrac{\partial}{\partial x^i}\left(g_i(x^i)\dfrac{\partial f_i(x^i)}{\partial x^i}\right)\right]}{g_i^2(x^i)}.$$

Since it appears only the variable x^i, we replace x^i by x, g_i by G and f_i by F. Then the relation (4.4) has the form $\dfrac{F'[G(F'G)']'}{[(F'G)']^2} = 2$. Denote $P = F'G$, hence $F' = \dfrac{P}{G}$. Therefore we have $\dfrac{P}{P'}\dfrac{(GP')'}{GP'} = 2$ or, equivalent, $\dfrac{(GP')'}{GP'} = 2\dfrac{P'}{P}$. By integration, we obtain $GP' = aP^2$ ($a > 0$ is a real constant) or, equivalent, $G(F'G)' = a(F'G)^2$.

Multiplying with F', we have $F'G(F'G)' = aF'(F'G)^2$ or $\dfrac{(F'G)'}{F'G} = aF'$. By integration, we deduce $\ln(F'(x)G(x)) = aF(x) + b$ or $F'(x)e^{-aF(x)} = e^b\dfrac{1}{G(x)}$. By integration we have

$$e^{-aF(x)} = \int \frac{-ae^b}{G(x)}dx \quad \text{or} \quad F(x) = \frac{-1}{a} \ln \int \frac{-ae^b}{G(x)}dx.$$

Finally we may write $F(x) = \frac{-1}{a} \ln \int \frac{1}{G(x)}dx + c$, where a, c are real constants, $a > 0$.

Proposition 4.1. For an arbitrary initial metric of diagonal type

$$g(x^1, \ldots, x^n) = \begin{pmatrix} \frac{1}{g_1^2(x^1)} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{g_n^2(x^n)} \end{pmatrix},$$

a local solution of equation (4.3) is

$$f(x^1, \ldots, x^n) = -\frac{1}{a} \ln \int \frac{1}{g_1(x^1)}dx^1 + \cdots + \frac{-1}{a} \ln \int \frac{1}{g_n(x^n)}dx^n + c,$$

where $a > 0$ and c are arbitrary constants.

5 The PDEs determined by associativity of the deformation algebra

We recall that if M is an n-dimensional C^∞ manifold, then we denote by $\mathcal{F}(\mathcal{M})$ the ring of C^∞-real functions defined on M. We also denote by $\mathcal{X}(\mathcal{M})$ the $\mathcal{F}(\mathcal{M})$-module of vector fields.

We suppose that the manifold M is endowed with two linear connections $(\nabla, \overline{\nabla})$. If $X, Y \in \mathcal{X}(\mathcal{M})$, then one can define the product between X and Y by $X * Y = \overline{\nabla}_X Y - \nabla_X Y$. Thus $\mathcal{X}(\mathcal{M})$ becomes an $\mathcal{F}(\mathcal{M})$-algebra. This algebra is called *the algebra of deformation of pair of connections* $(\nabla, \overline{\nabla})$ and it is denoted $\mathcal{U}(\mathcal{M}, \nabla, \overline{\nabla})$ [6].

We also introduce the $(1,2)$-tensor field $A = \overline{\nabla} - \nabla$. Let $(U, x^1, \ldots, x^n)$ be a coordinate chart on M. If we denote A^k_{ij} the components of A, then the condition of associativity of the algebra $\mathcal{U}(M, \nabla, \overline{\nabla})$ is $A^i_{sk} A^s_{j\ell} - A^i_{s\ell} A^s_{jk} = 0$, for all $i, j, \ell, k = \overline{1, n}$.

In our case, let us consider the Riemannian manifold $(\mathbf{R}^n_+, g, \nabla)$, where

$$g(x^1, \ldots, x^n) = \begin{pmatrix} \frac{1}{g_1^2(x^1)} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{g_n^2(x^n)} \end{pmatrix}$$

is of diagonal type and $g_i \colon \mathbf{R}_+ \to \mathbf{R} \setminus \{0\}$ are differentiable functions for all $i = \overline{1, n}$.

If $f \colon \mathbf{R}^n_+ \to \mathbf{R}$ is a smooth function having the Hessian with respect to g non-degenerate and with constant signature, then we consider the pseudo-Riemannian Hessian metric $h = \nabla^2_g f$ and the pseudo-Riemannian manifold $(\mathbf{R}^n_+, h, \overline{\nabla})$. So we may introduce the algebra of deformation $\mathcal{U}(\mathbf{R}^n_+, \nabla, \overline{\nabla})$ and the $(1,2)$-tensor field $A = \overline{\nabla} - \nabla$. In a local chart, A has the components

$$A^p_{ij} = \overline{\Gamma}^p_{ij} - \Gamma^p_{ij} = \frac{1}{2} f,^{pk} f,_{ijk} .$$

Thus the condition of associativity of the algebra $\mathcal{U}(\mathbf{R}^n_+, \nabla, \overline{\nabla})$ is

$$\frac{1}{4} f,^{ip} f,^{sr} f,_{skp} f,_{j\ell r} - \frac{1}{4} f,^{ip} f,^{sr} f,_{slp} f,_{jkr} = 0$$

or equivalent $f,^{ip} f,^{sr} (f,_{skp} f,_{j\ell r} - f,_{s\ell p} f,_{jkr}) = 0$. In these relations s, p and r are indices of sum. Finally we may write

$$f,^{sr} (f,_{ski} f,_{j\ell r} - f,_{s\ell i} f,_{jkr}) = 0. \tag{5.1}$$

Since the matrix $(f,_{rs})$ is symmetric, it follows that the inverse of this matrix $(f,^{rs})$ is also symmetric. Thus we may change r with s in the last term:

$$f,^{sr} f,_{s\ell i} f,_{jkr} = f,^{rs} f,_{r\ell i} f,_{jks} = f,^{sr} f,_{r\ell i} f,_{jks} .$$

Therefore the relation (5.1) is equivalent to

$$f,^{sr} (f,_{ski} f,_{j\ell r} - f,_{r\ell i} f,_{jks}) = 0. \tag{5.2}$$

It is known (see for example [5] and [11]) that if $M = \mathbf{R}^n$ is endowed with the Euclidean metric and $f: \mathbf{R}^n \to \mathbf{R}$ is a smooth strongly convex function, then $h_{ij}(x) = \dfrac{\partial^2 f}{\partial x^i \partial x^j}(x)$ defines a Riemannian structure which has the Riemannian curvature tensor given by

$$\begin{aligned} \bar{R}_{k\ell ij}(x) \; &= \; -\frac{1}{4} h^{sr}(x) \left[\frac{\partial^3 f}{\partial x^k \partial x^i \partial x^s}(x) \frac{\partial^3 f}{\partial x^r \partial x^j \partial x^\ell}(x) \right. \\ & \left. - \frac{\partial^3 f}{\partial x^k \partial x^j \partial x^s}(x) \frac{\partial^3 f}{\partial x^r \partial x^i \partial x^\ell}(x) \right] . \end{aligned}$$

In our case if $M = \mathbf{R}^n_+$ is endowed with the Euclidean metric

$$\delta(x^1, \dots, x^n) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

then $f,_{ij} = \dfrac{\partial^2 f}{\partial x^i \partial x^j}$, $f,_{ijk} = \dfrac{\partial^3 f}{\partial x^i \partial x^j \partial x^k}$.

Hence the condition (5.2) is equivalent to

$$\bar{R}_{k\ell ij} = 0, \quad \text{for all } k, \ell, i, j = \overline{1, n}.$$

Therefore we obtained the following result:

Proposition 5.1 *If $\mathbf{R}^n_+$ is endowed with the Euclidean metric, then the algebra of deformation $\mathcal{U}(\mathbf{R}^n_+, \nabla, \overline{\nabla})$ is associative if and only if the curvature tensor field of the metric $h = \nabla^2_\delta f$ identically vanishes ($\bar{R} = 0$).*

Remark 5.1 *As we already have seen, if $f: \mathbf{R}^n_+ \to \mathbf{R}$ is a separable function, then $\bar{R} = 0$. Hence for a separable function, the algebra $\mathcal{U}(\mathbf{R}^n_+, \nabla, \overline{\nabla})$ is associative.*

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