About induced orthogonality and generalized reflections of affine coordinate plane $A(K)$ of odd order

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Abstract. The points of the affine coordinate plane $A(K)$ are identified with the elements of the ring $K[\alpha] = \{x + \alpha y \mid x, y \in K^2\}$, where $-\alpha$ is a root of a polynomial of second degree over the field K of odd order. Depending on the choice of that polynomial we introduce the induced orthogonality of lines in $A(K)$. The matrix formed of generalized reflections of $A(K)$ are given. Finally, we show that generalized reflections of $A(K)$ have entirely analogous properties to the ones of the reflections of the Euclidean plane.

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1 Introduction

Let K be a field. A *point* is defined as any ordered pair $(x, y) \in K^2$. A *line* is defined as a set of the points of the form $\{(x, y) \in K^2 \mid y = kx + l\}$ or $\{(x_0, y) \in K^2 \mid y \in K\}$, where k, l, x_0 are fixed elements of K. The line of the form $\{(x, y) \in K^2 \mid y = kx + l\}$ will be called "the line $y = kx + l$", and the line of the form $\{(x_0, y) \in K^2 \mid y \in K\}$ will be called "the line $x = x_0$".

Let $\mathcal{G}$ be the set of all lines. We shall say that a point $P \in K^2$ is *incident* to a line $g \in \mathcal{G}$ if $P \in g$. The incidence structure $A(K) := (K^2, \mathcal{G}, \in)$ will be called the *affine coordinate plane* over K. From now on, suppose K is a field of odd order.

Let $\lambda(x) = x^2 - ex - f \in K[x]$ be a polynomial with the discriminant $\Delta = e^2 + 4f \neq 0$.

The points of $A(K)$ can be identified with the elements of the ring $K[\alpha] = \{x + \alpha y \mid x, y \in K\}$, where $K[\alpha] \cong K[x]/(\lambda(x))$ and $-\alpha$ is a root of a polynomial $\lambda(x)$ ($\alpha \notin K$). So, we identify the element $x + \alpha y$ of $K[\alpha]$ with the point (x, y).

Moreover, we define the *squared length* of the vector $z = (x, y) \in K[\alpha]$ by

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$$d^{(2)}((x,y)) = \|(x,y)\|^2 = \|z\|^2 := z\overline{z} = (x+\alpha y)(x+\beta y),$$

where $-\alpha$ and $-\beta$ are the roots of $\lambda(x)$ and $\overline{z} = x + \beta y$ is the conjugated vector of $z = x + \alpha y$. It is easy to show that

$$d^{(2)}((x,y)) = \|(x,y)\|^2 = \|z\|^2 = z\overline{z} = x^2 - exy - fy^2.$$

The *squared distance* of the points $z_1 = x_1 + \alpha y_1$ and $z_2 = x_2 + \alpha y_2$ is defined by

$$\begin{aligned} d^{(2)}(z_1, z_2) &= d^{(2)}((x_1, y_1), (x_2, y_2)) = \|z_2 - z_1\|^2 = \\ &= (x_2 - x_1)^2 - e(x_2 - x_1)(y_2 - y_1) - f(y_2 - y_1)^2. \end{aligned}$$

The automorphisms of $A(K)$ preserving the squared distance of any two points are called *isometries*.

From now on, we suppose that the coefficients of $\lambda(x) = x^2 - ex - f$ are the elements of the prime subfield of the field K.

The matrix and the vector forms of isometries of $A(K) = AG(2, q)$, where K is the finite field $GF(q)$, are given in [2]. It can be shown that the following theorem, proven in [2] for the finite field K, holds for an arbitrary field K.

Theorem 1. *An automorphism of $A(K)$ is an isometry if and only if for each $(x, y) \in K[\alpha]$ its matrix form is one of the following*

$$(x,y) \to (x,y)\begin{bmatrix} k & l \\ fl & k-el \end{bmatrix} + (r,s) \tag{1.1}$$

or

$$(x,y) \to (x,y)\begin{bmatrix} k & l \\ -fl-ek & -k \end{bmatrix} + (r,s), \tag{1.2}$$

where $r, s, k, l \in K$, satisfying $k^2 - ekl - fl^2 = 1$.

It is obvious that all isometries of $A(K)$ form the subgroup $\mathcal{I}(A(K))$ of the group of all automorphisms of $A(K)$. An isometry different from the identity and fixes all the points belonging to some line (the axis) will be called the *generalized reflection* of $A(K)$. Also, an isometry of the form (1.1) will be called the *generalized rotation* of $A(K)$, and it can be easily proven that all generalized rotations of $A(K)$ form a subgroup of $\mathcal{I}(A(K))$.

From Theorem 1 it easily follows that the group $\mathcal{I}(A(K))$ is a semidirect product of subgroups $\mathcal{T}$ and $(\mathcal{I}(A(K)))_0$, where $\mathcal{T}$ is a group of all translations $(x, y) \mapsto (x + r, y + s)$ of $A(K)$ and $(\mathcal{I}(A(K)))_0$ is the stabilizer of the point $0 = (0, 0)$.

Theorem 1 leads to the following characterization of the group $(\mathcal{I}(A(K)))_0$.

Corollary 2. *The group $(\mathcal{I}(A(K)))_0$ consists exactly of mappings*

$$(x,y) \to (x,y)\begin{bmatrix} k & l \\ fl & k-el \end{bmatrix} \tag{1.3}$$

or

$$(x,y) \to (x,y)\begin{bmatrix} k & l \\ -fl-ek & -k \end{bmatrix}, \tag{1.4}$$

where $k, l \in K$, satisfying $k^2 - ekl - fl^2 = 1$.

2 Generalized orthogonality

Let K be a field of odd order.

The *squared length* of the vector $u = (x_1, x_2)$ of K^2 is defined by

$$d^{(2)}(u) = Q(u) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2,$$

where Q is a quadratic form $Q : K^2 \to K$. The corresponding polar bilinear (symmetric) form $\overline{f} : K^2 \times K^2 \to K$ is defined by

$$\overline{f}(u, v) = \frac{1}{2}[Q(u + v) - Q(u) - Q(v)],$$

where $u = (x_1, x_2)$, $v = (y_1, y_2)$ and $u + v = (x_1 + y_1, x_2 + y_2)$. We obtain

$$\overline{f}(u, v) = a_{11}x_1y_1 + a_{12}(x_1y_2 + x_2y_1) + a_{22}x_2y_2$$

and $\overline{f}(u, u) = Q(u)$.

We say that the vectors $u, v \in K^2$ are $\overline{f}-$*orthogonal* if $\overline{f}(u, v) = 0$. In this case, we write $u \perp v$.

Let p_1, p_2 be the lines of $A(K)$ containing the point $S \equiv (x_S, y_S)$ and let $M_i \equiv (x_i, y_i) \neq S$ be arbitrary points from p_i, where $i = 1, 2$. Hence, $\overrightarrow{SM_i} \equiv (x_i - x_S, y_i - y_S)$, for $i = 1, 2$. We say that the lines p_1, p_2 are $\overline{f}-$*orthogonal* if $\overline{f}(\overrightarrow{SM_1}, \overrightarrow{SM_2}) = 0$.

Proposition 3. *If* $p_1 \equiv y - y_S = k_1(x - x_S)$ *and* $p_2 \equiv y - y_S = k_2(x - x_S)$, *then*

$$\overline{f}(\overrightarrow{SM_1}, \overrightarrow{SM_2}) = 0 \Leftrightarrow a_{11} + a_{12}(k_1 + k_2) + a_{22}k_1k_2 = 0. \tag{2.1}$$

Proof. Note that

$$\begin{aligned}\overline{f}(\overrightarrow{SM_1}, \overrightarrow{SM_2}) &= a_{11}(x_1 - x_S)(x_2 - x_S) + a_{12}[(x_1 - x_S)k_2(x_2 - x_S) + \\ &+ (x_2 - x_S)k_1(x_1 - x_S)] + a_{22}k_1k_2(x_1 - x_S)(x_2 - x_S) = \\ &= (x_1 - x_S)(x_2 - x_S)[a_{11} + a_{12}(k_1 + k_2) + a_{22}k_1k_2].\end{aligned}$$

So, we have $\overline{f}(\overrightarrow{SM_1}, \overrightarrow{SM_2}) = 0$ if and only if $a_{11} + a_{12}(k_1 + k_2) + a_{22}k_1k_2 = 0$. □

In this way the "condition of the orthogonality" (2.1) is obtained. I°. t can be easily proven that the condition of orthogonality of the lines $p_1 \equiv y - y_S = k_1(x - x_S)$ and $p_2 \equiv x = x_S$ is $a_{12} + a_{22}k_1 = 0$. Furthermore, we find the lines $y = y_0$ and $x = x_0$ to be $\overline{f}-$orthogonal if and only if $a_{12} = 0$. I°. n this paper we consider the case when the points of $A(K) \equiv K[\alpha]$ are $u = (x_1, x_2) = z = x_1 + x_2\alpha$, where $-\alpha$ is a root of polynomial $\lambda(x) = x^2 - ex - f \in K[x]$ $(e^2 + 4f \neq 0)$. Therefore we have $d^{(2)}(u) = Q(u) = \|z\|^2 = x_1^2 - ex_1x_2 - fx_2^2$. If $\lambda(x)$ is an irreducible polynomial over the field K, then $d^{(2)} = Q$ is the squared Euclidean length, since $\ker Q = \{0\}$. If $\lambda(x)$ is a reducible polynomial over K, then $\ker Q$ consists of two different lines from $A(K)$ and $d^{(2)} = Q$ is the squared Minkowskian length. For both cases, the condition of the orthogonality is

$$p_1 \perp p_2 \Leftrightarrow 1 - \frac{e}{2}(k_1 + k_2) - fk_1k_2 = 0.$$

This is regarded as "induced orthogonality" in $A(K)$.

Example 4. **a)** *For all* $u \in A(K)$, *let us take the squared Euclidean length* $d^{(2)}(u) = Q(u) = Q_E(u) = x_1^2 + x_2^2$. *The corresponding bilinear form* $\overline{f}$ *is* $\overline{f}(u,v) = \overline{f}_E(u,v) = x_1y_1 + x_2y_2$, *where* $u = (x_1, x_2)$ *and* $v = (y_1, y_2)$. *Note that this is the standard scalar product by coordinates. In this case, the condition of the orthogonality is* $p_1 \perp p_2 \Leftrightarrow k_1k_2 = -1$ *which is well known for the real affine coordinate plane.*

b) *For all* $u \in A(K)$, *let us take the squared Minkowskian length* $d^{(2)}(u) = Q(u) = Q_M(u) = x_1^2 - x_2^2$. *The corresponding bilinear form* $\overline{f}$ *is* $\overline{f}(u,v) = \overline{f}_M(u,v) = x_1y_1 - x_2y_2$. *The condition of the orthogonality is* $p_1 \perp p_2 \Leftrightarrow k_1k_2 = 1$.

3 Generalized reflections of $A(K)$

Our intention is to find all generalized reflections of $A(K)$ and to establish their properties.

Theorem 5. *An isometry of* $A(K)$ *is a generalized reflection if and only if for each* $(x,y) \in K[\alpha]$ *it is an involution of the form*

$$(x,y) \to (x,y) \begin{bmatrix} k & l \\ -fl - ek & -k \end{bmatrix} + (r,s), \tag{2}$$

where $r, s, k, l \in K$, *satisfying* $k^2 - ekl - fl^2 = 1$.

Proof. Suppose $A = \begin{bmatrix} k & l \\ -fl - ek & -k \end{bmatrix}$ and $(k,l) \neq (1,0)$.
If ω is an involution of the form (1.2), i.e. $\omega((x,y)) = (x,y)A + (r,s)$, we obtain $(1-k)s + lr = 0$. Also, by Theorem 1, ω is an isometry. From $(x,y)A + (r,s) = (x,y)$ follows that the isometry ω fixes all the points of some line in $A(K)$. In case $(k,l) = (-1,0)$ this line is $ey = 2x - r$, otherwise the line is $(k+1)y = lx + s$ (we use $(1-k)s + lr = 0$). Hence, ω is a generalized reflection.
To prove the reverse, suppose ω_1 is a generalized reflection. Since ω_1 is an isometry, by Theorem 1, ω_1 has the form (1.1) or (1.2). It can be obtained that isometries of the form (1.1) (rotations), which are different from identity, fix only one single point of $A(K)$. So we can conclude ω_1 is of the form (1.2), i.e. $\omega_1((x,y)) = (x,y)A + (r,s)$. Since, ω_1 fixes all the points belonging to some line (axis), then from $(x,y)A + (r,s) = (x,y)$ follows $(1-k)s + lr = 0$. Also, if $(k,l) = (-1,0)$ the axis is the line $ey = 2x - r$, otherwise the axis is the line $(k+1)y = lx + s$. It can be verified that ω_1 is an involution (we use $(1-k)s + lr = 0$).
The proof in the case $(k,l) = (1,0)$ is similar to the previous proof. □

From the proof of Theorem 5, it follows

Proposition 6. *Let* ω *be a generalized reflection of the form (1.2). If* $(k,l) = (-1,0)$ *the corresponding axis is the line* $ey = 2x - r$, *otherwise the axis is the line* $(1+k)y = lx + s$.

A°. ll the properties of the generalized reflections of $A(K)$ are entirely analogous to the properties of reflections of the Euclidean plane. Here the orthogonality is the

"induced orthogonality".
For example, if we take the generalized reflection $(x,y) \to (x,y)A + (r,s)$, where $A = \begin{bmatrix} k & l \\ -fl - ek & -k \end{bmatrix}$ and $(k,l) \neq (\pm 1, 0)$, then by Proposition 6 and the proof of Theorem 5, the axis is the line $y = \frac{l}{k+1}x + \frac{s}{k+1}$ and $r = \frac{k-1}{l}s$. Let us denote $K_1 = \frac{l}{k+1}$. The axis contains the midpoint of the segment with the end points $(x,y)A+(r,s)$ and (x,y). The slope of the line containing the point (x,y) and its picture $(x,y)A+(r,s)$, is

$$K_2 = \frac{lx - (k+1)y + s}{(k-1)x + (fl - ek)y + \frac{k-1}{l}s} = \cdots = \frac{l}{k-1}.$$

It is seen that the condition of orthogonality $1 - \frac{e}{2}(K_1 + K_2) - fK_1K_2 = 0$ is fulfilled.

4 The elements of $(\mathcal{I}(A(K)))_0$

Finally, for some choices of $\lambda(x) = x^2 - ex - f$, we will find the elements of the group $(\mathcal{I}(A(K)))_0$. Also, the Lorentz transformations of $A(K)$ will be obtained.

Corollary 7. *Generalized reflections of $(\mathcal{I}(A(K)))_0$ are exactly all isometries of the form*

$$(1.4) \qquad (x,y) \to (x,y) \begin{bmatrix} k & l \\ -fl - ek & -k \end{bmatrix}$$

where $k, l \in K$ satisfying $k^2 - ekl - fl^2 = 1$.

Proof. The claim follows from Theorem 5, since each isometry of the form (1.4) is an involution. □

Corollary 8. *If $(k,l) = (-1,0)$ the axis of the generalized reflection*

$$(x,y) \to (x,y) \begin{bmatrix} k & l \\ -fl - ek & -k \end{bmatrix}$$

is the line $ey = 2x$, otherwise the axis is the line $(1+k)y = lx$, where $k, l \in K$, satisfying $k^2 - ekl - fl^2 = 1$.

Proof. The assertion follows from Corollary7 and Proposition 6. □

Isometries of the form (1.3) are the generalized rotations around 0.

Proposition 9. *The generalized rotations (around 0) form a subgroup of $(\mathcal{I}(A(K)))_0$. The product of two generalized reflections in lines through 0 is a generalized rotation (around 0). The product of a generalized rotation (around 0) and a generalized reflection in a line through 0 is a generalized reflection in a line through 0.*

Proof. The assertion is trivial to prove. □

Example 10. **(A)** *Let $\lambda(x) = x^2 + 1$, i.e. $d^{(2)}(u) = Q_E(u) = x_1^2 + x_2^2$, where $u = (x_1, x_2) \in A(K)$, $e = 0$ and $f = -1$.*
In this case, isometries of the Euclidean plane $A(K)$ are

$$\begin{array}{lcl}(x,y) & \to & (x,y)\begin{bmatrix} k & l \\ -l & k \end{bmatrix} + (r,s) \\ & or & \\ (x,y) & \to & (x,y)\begin{bmatrix} k & l \\ l & -k \end{bmatrix} + (r,s),\end{array}$$

where $k,l,r,s \in K$, satisfying $k^2+l^2=1$. Also, the isometries

$$(x,y) \to (x,y)\begin{bmatrix} k & l \\ -l & k \end{bmatrix}$$

are the "Euclidean" rotations around 0 *and the isometries*

$$(x,y) \to (x,y)\begin{bmatrix} k & l \\ l & -k \end{bmatrix}$$

are the "Euclidean" reflections in the lines through 0*. If $k \neq -1$ the corresponding axis is the line $y = \frac{l}{k+1}x$ and if $k=-1$ the axis is y-axis.*

(B) *Let $\lambda(x) = x^2-1$, i.e. $d^{(2)}(u) = Q_M(u) = x_1^2 - x_2^2$, where $u = (x_1,x_2) \in A(K)$, $e=0$ and $f=1$.*
In this case, the elements of $(\mathcal{I}(A(K)))_0$ are the Lorentz rotations around 0

$$(x,y) \to (x,y)\begin{bmatrix} k & l \\ l & k \end{bmatrix}$$

or the Lorentz reflections in the lines through 0

$$(x,y) \to (x,y)\begin{bmatrix} k & l \\ -l & -k \end{bmatrix},$$

where $k,l \in K$, satisfying $k^2-l^2=1$. If $k \neq -1$ the corresponding axis is $y = \frac{l}{k+1}x$ and if $k=-1$ the axis is y-axis.I°. n case of the real affine coordinate plane $A(\mathbb{R})$, the elements of $(\mathcal{I}(A(K)))_0$ are the Lorentz rotations

$$(x,y) \to (x,y)\begin{bmatrix} \frac{1}{\pm\sqrt{1-\frac{v^2}{c^2}}} & \frac{-v}{\pm c\sqrt{1-\frac{v^2}{c^2}}} \\ \frac{-v}{\pm c\sqrt{1-\frac{v^2}{c^2}}} & \frac{1}{\pm\sqrt{1-\frac{v^2}{c^2}}} \end{bmatrix}$$

$(\det = k^2-l^2=1$*; where $k = \frac{1}{\pm\sqrt{1-\frac{v^2}{c^2}}}$, $l = \frac{-v}{\pm c\sqrt{1-\frac{v^2}{c^2}}}$) and the involutions*

$$(x,y) \to (x,y)\begin{bmatrix} \frac{1}{\pm\sqrt{1-\frac{v^2}{c^2}}} & \frac{-v}{\pm c\sqrt{1-\frac{v^2}{c^2}}} \\ \frac{v}{\pm c\sqrt{1-\frac{v^2}{c^2}}} & \frac{-1}{\pm\sqrt{1-\frac{v^2}{c^2}}} \end{bmatrix}; \qquad (\det = -1)$$

which are the Lorentz reflections in the lines $y = \frac{-v}{c\pm\sqrt{c^2-v^2}}x$.

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