A characterization of minimal surfaces in S^5 with parallel normal vector field

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Abstract. In this paper we proof that the Holomorphic angle for compact minimal surfaces in the sphere S^5 with constant Contact angle and with a parallel normal vector field must be constant.

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1 Introduction

The notion of Kähler angle was introduced by Chern and Wolfson in [3] and [12]; it is a fundamental invariant for minimal surfaces in complex manifolds. Using the technique of moving frames, Wolfson obtained equations for the Laplacian and Gaussian curvature for an immersed minimal surface in $\mathbb{CP}^n$. Later, Kenmotsu in [7], Ohnita in [10] and Ogata in [11] classified minimal surfaces with constant Gaussian curvature and constant Kähler angle.
A few years ago, Li in [14] gave a counterexample to the conjecture of Bolton, Jensen and Rigoli (see [2]), according to which a minimal immersion (non-holomorphic, non anti-holomorphic, non totally real) of a two-sphere in $\mathbb{CP}^n$ with constant Kähler angle would have constant Gaussian curvature.
In [8] we introduced the notion of Contact angle, that can be considered as a new geometric invariant useful to investigate the geometry of immersed surfaces in S^3. Geometrically, the Contact angle (β) is the complementary angle between the contact distribution and the tangent space of the surface. Also in [8], we deduced formulas for the Gaussian curvature and the Laplacian of an immersed minimal surface in S^3, and we gave a characterization of the Clifford Torus as the only minimal surface in S^3 with constant Contact angle.
We define α to be the angle given by $\cos\alpha = \langle ie_1, v\rangle$, where e_1 and v are defined in section 2. The Holomorphic angle α is the analogue of the Kähler angle introduced by Chern and Wolfson in [3].
Recently, in [9], we construct a family of minimal tori in S^5 with constant Contact and Holomorphic angle. These tori are parametrized by the following circle equation

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$$a^2 + \left(b - \frac{\cos\beta}{1+\sin^2\beta}\right)^2 = 2\frac{\sin^4\beta}{(1+\sin^2\beta)^2}, \tag{1.1}$$

where a and b are given in Section 3 (equation (3.7)). In particular, when $a = 0$ in (1.1), we recover the examples found by Kenmotsu, in [6]. These examples are defined for $0 < \beta < \frac{\pi}{2}$. Also, when $b = 0$ in (1.1), we find a new family of minimal tori in S^5, and these tori are defined for $\frac{\pi}{4} < \beta < \frac{\pi}{2}$. Also, in [9], when $\beta = \frac{\pi}{2}$, we give an alternative proof of this classification of a Theorem from Blair in [1], and Yamaguchi, Kon and Miyahara in [13] for Legendrian minimal surfaces in S^5 with constant Gaussian curvature.
In this paper, we will classify minimal surfaces in S^5 with constant Contact angle and with a parallel normal vector field. We suppose that e_3 (in equation (3.1)) is a parallel normal vector field, and we get the following

Theorem 1. *The Holomorphic angle* $(0 < \alpha < \frac{\pi}{2})$ *is constant for compact minimal surfaces in* S^5 *with constant Contact angle* β *and null principal curvatures* a, b

Remark 1. *The Theorem 1 implies a more general classification in [9] that gives a family of minimal flat tori in* S^5 *with constant Contact angle and constant Holomorphic angle*

2 Contact Angle for Immersed Surfaces in S^{2n+1}

Consider in $\mathbb{C}^{n+1}$ the following objects:

- the Hermitian product: $(z, w) = \sum_{j=0}^{n} z^j \bar{w}^j$;
- the inner product: $\langle z, w\rangle = Re(z, w)$;
- the unit sphere: $S^{2n+1} = \{z \in \mathbb{C}^{n+1} | (z, z) = 1\}$;
- the *Reeb* vector field in S^{2n+1}, given by: $\xi(z) = iz$;
- the contact distribution in S^{2n+1}, which is orthogonal to ξ:

$$\Delta_z = \{v \in T_z S^{2n+1} | \langle \xi, v\rangle = 0\}.$$

We observe that Δ is invariant by the complex structure of $\mathbb{C}^{n+1}$.
Let now S be an immersed orientable surface in S^{2n+1}.

Definition 1. The *Contact angle* β is the complementary angle between the contact distribution Δ and the tangent space TS of the surface.

Let (e_1, e_2) be a local frame of TS, where $e_1 \in TS \cap \Delta$. Then $\cos\beta = \langle \xi, e_2\rangle$. Finally, let v be the unit vector in the direction of the orthogonal projection of e_2 on Δ, defined by the following relation

$$e_2 = \sin\beta v + \cos\beta\xi. \tag{2.1}$$

3 Equations for Gaussian curvature and Laplacian of a minimal surface in S^5

In this section, we deduce the equations for the Gaussian curvature and for the Laplacian of a minimal surface in S^5 in terms of the Contact angle and the Holomorphic angle. Consider the normal vector fields

$$\begin{array}{rcl} e_3 &=& i\csc\alpha e_1 - \cot\alpha v \\ e_4 &=& \cot\alpha e_1 + i\csc\alpha v \\ e_5 &=& \csc\beta\xi - \cot\beta e_2 \end{array} \tag{3.1}$$

where $\beta \neq 0, \pi$ and $\alpha \neq 0, \pi$. We will call $(e_j)_{1\leq\ j\leq 5}$ an *adapted frame*.

Using (2.1) and (3.1), we get

$$\begin{array}{rr} v = \sin\beta e_2 - \cos\beta e_5, & iv = \sin\alpha e_4 - \cos\alpha e_1 \\ & \xi = \cos\beta e_2 + \sin\beta e_5 \end{array} \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\begin{array}{rcl} ie_1 &=& \cos\alpha\sin\beta e_2 + \sin\alpha e_3 - \cos\alpha\cos\beta e_5 \\ ie_2 &=& -\cos\beta z - \cos\alpha\sin\beta e_1 + \sin\alpha\sin\beta e_4 \end{array} \tag{3.3}$$

Consider now the dual basis (θ^j) of (e_j). The connection forms (θ^j_k) are given by

$$De_j = \theta^k_j e_k,$$

and the second fundamental form with respect to this frame are given by

$$II^j \quad = \quad \theta^j_1\theta^1 + \theta^j_2\theta^2; \quad j = 3, ..., 5.$$

Using (3.3) and differentiating v and ξ on the surface S, we get

$$\begin{array}{rcl} D\xi &=& -\cos\alpha\sin\beta\theta^2 e_1 + \cos\alpha\sin\beta\theta^1 e_2 + \sin\alpha\theta^1 e_3 + \sin\alpha\sin\beta\theta^2 e_4 \\ && -\cos\alpha\cos\beta\theta^1 e_5, \\ Dv &=& (\sin\beta\theta^1_2 - \cos\beta\theta^1_5)e_1 + \cos\beta(d\beta - \theta^2_5)e_2 + (\sin\beta\theta^3_2 - \cos\beta\theta^3_5)e_3 \\ && +(\sin\beta\theta^2_4 - \cos\beta\theta^4_5)e_4 + \sin\beta(d\beta + \theta^5_2)e_5. \end{array} \tag{3.4}$$

Differentiating e_3, e_4 and e_5, we have

$$
\begin{aligned}
\theta_3^1 &= -\theta_1^3 \\
\theta_3^2 &= \sin\beta(d\alpha+\theta_4^1)-\cos\beta\sin\alpha\theta^1 \\
\theta_3^4 &= \csc\beta\theta_1^2-\cot\alpha(\theta_1^3+\csc\beta\theta_2^4) \\
\theta_3^5 &= \cot\beta\theta_2^3-\csc\beta\sin\alpha\theta^1 \\
\theta_4^1 &= -d\alpha-\csc\beta\theta_2^3+\sin\alpha\cot\beta\theta^1 \\
\theta_4^2 &= -\theta_2^4 \\
\theta_4^3 &= \csc\beta\theta_2^1+\cot\alpha(\theta_1^3+\csc\beta\theta_2^4) \\
\theta_4^5 &= \cot\beta\theta_2^4-\sin\alpha\theta^2 \\
\theta_5^1 &= -\cos\alpha\theta^2-\cot\beta\theta_2^1 \\
\theta_5^2 &= d\beta+\cos\alpha\theta^1 \\
\theta_5^3 &= -\cot\beta\theta_2^3+\csc\beta\sin\alpha\theta^1 \\
\theta_5^4 &= -\cot\beta\theta_2^4+\sin\alpha\theta^2
\end{aligned} \tag{3.5}
$$

The conditions of minimality and of symmetry are equivalent to the following equations:

$$\theta_1^\lambda\wedge\theta^1+\theta_2^\lambda\wedge\theta^2=0=\theta_1^\lambda\wedge\theta^2-\theta_2^\lambda\wedge\theta^1. \tag{3.6}$$

On the surface S, we consider

$$\theta_1^3 = a\theta^1+b\theta^2$$

It follows from (3.6) that

$$
\begin{aligned}
\theta_1^3 &= a\theta^1+b\theta^2 \\
\theta_2^3 &= b\theta^1-a\theta^2 \\
\theta_1^4 &= d\alpha+(b\csc\beta-\sin\alpha\cot\beta)\theta^1-a\csc\beta\theta^2 \\
\theta_2^4 &= d\alpha\circ J-a\csc\beta\theta^1-(b\csc\beta-\sin\alpha\cot\beta)\theta^2 \\
\theta_1^5 &= d\beta\circ J-\cos\alpha\theta^2 \\
\theta_2^5 &= -d\beta-\cos\alpha\theta^1
\end{aligned} \tag{3.7}
$$

where J is the complex structure of S is given by $Je_1=e_2$ and $Je_2=-e_1$. Moreover, the normal connection forms are given by:

$$
\begin{aligned}
\theta_3^4 &= -\sec\beta d\beta\circ J-\cot\alpha\csc\beta d\alpha\circ J+a\cot\alpha\cot^2\beta\theta^1 \\
&\quad +(b\cot\alpha\cot^2\beta-\cos\alpha\cot\beta\csc\beta+2\sec\beta\cos\alpha)\theta^2 \\
\theta_3^5 &= (b\cot\beta-\csc\beta\sin\alpha)\theta^1-a\cot\beta\theta^2 \\
\theta_4^5 &= \cot\beta(d\alpha\circ J)-a\cot\beta\csc\beta\theta^1+ \\
&\quad (-b\csc\beta\cot\beta+\sin\alpha(\cot^2\beta-1))\theta^2,
\end{aligned} \tag{3.8}
$$

while the Gauss equation is equivalent to the equation:

$$d\theta_2^1+\theta_k^1\wedge\theta_2^k = \theta^1\wedge\theta^2. \tag{3.9}$$

Therefore, using equations (3.7) and (3.9), we have

$$
\begin{aligned}
K &= 1-|\nabla\beta|^2-2\cos\alpha\beta_1-\cos^2\alpha-(1+\csc^2\beta)(a^2+b^2)\\
&\quad +2b\sin\alpha\csc\beta\cot\beta+2\sin\alpha\cot\beta\alpha_1-|\nabla\alpha|^2\\
&\quad +2a\csc\beta\alpha_2-2b\csc\beta\alpha_1-\sin^2\alpha\cot^2\beta\\
&= 1-(1+csc^2\beta)(a^2+b^2)-2b\csc\beta(\alpha_1-\sin\alpha\cot\beta)+2a\csc\beta\alpha_2\\
&\quad -|\nabla\beta+\cos\alpha e_1|^2-|\nabla\alpha-\sin\alpha\cot\beta e_1|^2
\end{aligned}\tag{3.10}
$$

Using (3.5) and the complex structure of S, we get

$$
\theta_2^1 = \tan\beta(d\beta\circ J-2\cos\alpha\theta^2)\tag{3.11}
$$

Differentiating (3.11), we conclude that

$$
\begin{aligned}
d\theta_2^1 &= (-(1+\tan^2\beta)|\nabla\beta|^2-\tan\beta\Delta\beta-2\cos\alpha(1+2\tan^2\beta)\beta_1\\
&\quad +2\tan\beta\sin\alpha\alpha_1-4\tan^2\beta\cos^2\alpha)\theta^1\wedge\theta^2
\end{aligned}
$$

where $\Delta = tr\nabla^2$ is the Laplacian of S. The Gaussian curvature is therefore given by:

$$
\begin{aligned}
K &= -(1+\tan^2\beta)|\nabla\beta|^2-\tan\beta\Delta\beta-2\cos\alpha(1+2\tan^2\beta)\beta_1\\
&\quad +2\tan\beta\sin\alpha\alpha_1-4\tan^2\beta\cos^2\alpha.
\end{aligned}\tag{3.12}
$$

From (3.10) and (3.12), we obtain the following formula for the Laplacian of S:

$$
\begin{aligned}
\tan\beta\Delta\beta &= (1+\csc^2\beta)(a^2+b^2)+2b\csc\beta(\alpha_1-\sin\alpha\cot\beta)-2a\csc\beta\alpha_2\\
&\quad -\tan^2\beta(|\nabla\beta+2\cos\alpha e_1|^2-|\cot\beta\nabla\alpha+\sin\alpha(1-\cot^2\beta)e_1|^2)\\
&\quad +\sin^2\alpha(1-\tan^2\beta)
\end{aligned}\tag{3.13}
$$

4 Gauss-Codazzi-Ricci equations for a minimal surface in S^5 with constant Contact angle β

In this section, we will compute Gauss-Codazzi-Ricci equations for a minimal surface in S^5 with constant Contact angle β.
Using the connection form (3.7) and (3.8) in the Codazzi-Ricci equations, we have

$$
d\theta_1^3+\theta_2^3\wedge\theta_1^2+\theta_4^3\wedge\theta_1^4+\theta_5^3\wedge\theta_1^5 = 0
$$

This implies that

$$
\begin{aligned}
&(b_1-a_2)+(a^2+b^2)\cot\alpha\csc\beta\cot^2\beta-a\cot\alpha(\csc^2\beta+\cot^2\beta)\alpha_2\\
&+b(cot\alpha(\csc^2\beta+\cot^2\beta)\alpha_1-\cos\alpha\cot\beta(\csc^2\beta+\cot^2\beta-3\sec^2\beta(1+\sin^2\beta)))\\
&-\cos\alpha\csc\beta(2(\cot\beta-\tan\beta)\alpha_1-\sin\alpha(\cot^2\beta-3))+\cot\alpha\csc\beta|\nabla\alpha|^2=0
\end{aligned}\tag{4.1}
$$

Replacing the following (3.8) in the Codazzi-Ricci equations

$$
\begin{aligned}
d\theta_2^3+\theta_1^3\wedge\theta_2^1+\theta_4^3\wedge\theta_2^4+\theta_5^3\wedge\theta_2^5 &= 0\\
d\theta_1^4+\theta_2^4\wedge\theta_1^2+\theta_3^4\wedge\theta_1^3+\theta_5^4\wedge\theta_1^5 &= 0\\
d\theta_3^5+\theta_1^5\wedge\theta_3^1+\theta_2^5\wedge\theta_3^2+\theta_4^5\wedge\theta_3^4 &= 0
\end{aligned}
$$

We get

$$(4.2)\qquad \begin{aligned}&(a_1+b_2)+b\cot\alpha\alpha_2+a(\cot\alpha\alpha_1+6\tan\beta\cos\alpha)\\&-2\sec\beta\cos\alpha\alpha_2=0\end{aligned}$$

Using the connection form (3.8) in the Codazzi-Ricci equations

$$\begin{aligned}d\theta_2^4+\theta_1^4\wedge\theta_2^1+\theta_3^4\wedge\theta_2^3+\theta_5^4\wedge\theta_2^5 &= 0\\ d\theta_4^5+\theta_1^5\wedge\theta_4^1+\theta_2^5\wedge\theta_4^2+\theta_3^5\wedge\theta_4^3 &= 0\\ d\theta_3^4+\theta_1^4\wedge\theta_3^1+\theta_2^4\wedge\theta_3^2+\theta_5^4\wedge\theta_3^5 &= 0\end{aligned}$$

We have

$$(4.3)\qquad \begin{aligned}&(a_2-b_1)-(a^2+b^2)\cot\alpha\sin\beta\cot^2\beta+a\cot\alpha\alpha_2\\&+b(-cot\alpha\alpha_1+2\cos\alpha(\cot\beta-3\tan\beta))+2\cos\alpha\sin\beta(\cot\beta-\tan\beta)\alpha_1\\&+\sin\alpha\cos\alpha\sin\beta(5-\cot^2\beta)+\sin\beta\Delta\alpha=0\end{aligned}$$

Codazzi-Ricci equations

$$\begin{aligned}d\theta_1^2+\theta_3^2\wedge\theta_1^3+\theta_4^2\wedge\theta_1^4+\theta_5^2\wedge\theta_1^5 &= \theta^2\wedge\theta^1\\ d\theta_1^5+\theta_2^5\wedge\theta_1^2+\theta_3^5\wedge\theta_1^3+\theta_4^5\wedge\theta_1^4 &= 0\end{aligned}$$

give the following equation

$$(4.4)\qquad \begin{aligned}&(a^2+b^2)(1+\csc^2\beta)+2b\csc\beta(\alpha_1-\cot\beta\sin\alpha)-2a\csc\beta\alpha_2\\&+|\nabla\alpha|^2+2\sin\alpha(\tan\beta-\cot\beta)\alpha_1-4\tan^2\beta\cos^2\alpha\\&-\sin^2\alpha(1-\cot^2\beta)=0\end{aligned}$$

The following Codazzi equation is automatically verified

$$d\theta_2^5+\theta_1^5\wedge\theta_2^1+\theta_3^5\wedge\theta_2^3+\theta_4^5\wedge\theta_2^4 = 0$$

5 Proof of the Theorem 1

In this section, we will give a proof of the theorem, using Gauss-Codazzi-Ricci equations for a minimal surface in S^5 with constant Contact angle and null principal curvatures a, b.
Suppose that a, b are nulls and the Contact angle β is constant, then using the Codazzi equation (4.1), we have

$$(5.1)\qquad \cos\alpha(2(\cot\beta-\tan\beta)\alpha_1-\sin\alpha(\cot^2\beta-3))-\cot\alpha|\nabla\alpha|^2=0$$

On the other hand, Codazzi equation (4.3) with a, b nulls and constant Contact angle implies

$$(5.2)\qquad 2\cos\alpha(\cot\beta-\tan\beta)\alpha_1+\sin\alpha\cos\alpha(5-\cot^2\beta)+\Delta\alpha=0$$

Using equations (5.1) and (5.2), we obtain a new Laplacian equation of α

$$\Delta\alpha = -\sin(2\alpha) - \cot\alpha|\nabla\alpha|^2 \tag{5.3}$$

Now suppose that $(0 < \alpha < \frac{\pi}{2})$. Using the Hopf's Lemma, we get the Theorem 1. □

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