Isothermic surfaces as solutions of Calapso PDE

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Abstract. The authors explore the solutions of Pasquale Calapso PDE with the symbolic computation in MAPLE. The main result is connected to solutions of particular forms which shows that this fourth order PDE is strongly connected to the Painleve ODEs and the theory of solitons.

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1 Isothermic surface PDE

The fourth order Pasquale Calapso PDE [1]

$$\left(\frac{z_{xy}}{z}\right)_{xx} + \left(\frac{z_{xy}}{z}\right)_{yy} + (z^2)_{xy} = 0 \qquad (PDE)$$

describes the isothermic surfaces $z = z(x, y)$ in R^3. As was shown in the paper [5], this PDE has solitonic sense. Also, this equation is strongly connected to the Painleve ODEs and we can study it from the point of view developed in [3]. To enlarge the interest of the reader in our point of view, we add the papers [4], [6], [7], [8].

Let us show that the symbolic computation in $MAPLE$ produces some solutions of Calapso PDE. For example, the *PDEtools package* is a collection of commands and routines for finding analytical solutions for PDEs based on the paper [2]. Though the package is an implementation of known methods for solving PDEs, it also allows us to look for solutions of equations not yet automatically solved by the package, or for different solutions when the package returns a solution which is not the most general one. For this purpose, we can make use of the *dchange command* and of the *HINT* option of the *pdsolve command* (especially the functional *hint*), both explained in more detail in the corresponding help pages of the last versions of MAPLE.

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2 Particular isothermic surfaces as solutions of Calapso PDE

A "lump" solution of the Calapso PDE is

$$z = \frac{4}{2 + x^2 + y^2}.$$

To find other isothermic surfaces as solutions of Pasquale Calapso PDE, we start with

$$> ansatz := z(x, y) = a/(f(x) * y);$$

We use pdetest to simplify the PDE with regard to this ansatz. It follows

$$> ans\_1 := pdetest(ansatz, PDE);$$

$$ans\_1 = \frac{1}{f(x)^3 y^3} \big(2f'(x)^3 y^2 - 3f'(x)f''(x)f(x)y^2 + f'''(x)f(x)^2 y^2 + 2f'(x)f(x)^2 + 4a^2 f'(x)\big)$$

The ansatz above separated the variables. The solution depends on the function $f(x)$.

$$> ans\_f := dsolve(ans_1, f(x));$$

$$ans\_f := f(x) = \_C_1,$$

$$\int^{f(x)} \frac{y}{\sqrt{-2ln(\_a)^2 \_a^2 - 2a^2 + 2\_C_1 \_a^2 y^2 - 2\_C_2 \_a^2 y^2 ln(\_a)}} d\_a - x - \_C_3 = 0,$$

$$\int^{f(x)} \frac{-y}{\sqrt{-2ln(\_a)^2 \_a^2 - 2a^2 + 2\_C_1 \_a^2 y^2 - 2\_C_2 \_a^2 y^2 ln(\_a)}} d\_a - x - \_C_3 = 0.$$

To build a (particular) solution to the PDE by substituting the result above in *ansatz*, we follow the following steps.

$$> ans := subs(ans_f, ansatz);$$

$$> pdetest(ans, PDE);$$

$$> ansatz1 := u(x, y) = f(1 + x^2 + y^2);$$

$$> ans_2 := pdetest(ansatz1, PDE);$$

Now, we use the HINT option to obtain a general solution, getting some inspiration from the solution above.

$$> struc2 := pdsolve(PDE, HINT = x + y + f(y));$$

$$struc2 := (z(x, y) = x + y + f(y))\&where[\{\frac{d}{dy} f(y) = -1\}]$$

$$> struc3 := pdsolve(PDE, HINT = x - y + f(x));$$

$$struc3 := (z(x,y) = x - y + f(x))\&where[\{\frac{d}{dx}f(x) = -1\}]$$

$$> struc4 := pdsolve(PDE, HINT = x + yf(y));$$

$$struc4 := (z(x,y) = x + yf(y))\&where[\{\frac{d}{dy}f(y) = -\frac{f(y)}{y}\}]$$

$$> struc5 := pdsolve(PDE, HINT = xf(x) - y);$$

$$struc5 := (z(x,y) = x + yf(y))\&where[\{\frac{d}{dx}f(x) = -\frac{f(x)}{x}\}]$$

$$> struc6 := pdsolve(PDE, HINT = f(x));$$

$$struc6 := (z(x,y) = f(x)\&where[\{f(x), are\, arbitrary\, functions\}]$$

$$> struc7 := pdsolve(PDE, HINT = \frac{f(x)}{xy});$$

$$struc7 := (z(x,y) = \frac{f(x)}{xy}\&where$$

$$[\{f''' = \frac{2f^3y^2 - 2f'^3x^3y^2 + 2f^3x^2 + 3ff'f''x^3y^2 - 2f^2f'x^3 - 4f^4f'x + 4f^5}{f^2x^3y^2}\}]$$

If we want to plot the solutions, we can follow the commands

$$> with(plots):$$

dsolve[numeric] solution using a range, so odeplot defaults to that range ; we take the case $y = 1$:

$$> p := dsolve(\{diff(f(x), `\$`(x,3)) = (2 * f(x)^3 - 2 * (diff(f(x),x))^3 * x^3$$
$$+2 * f(x)^3 * x^2 + 3 * (diff(f(x), `\$`(x,2))) * f(x) * (diff(f(x),x)) * x^3 - 2$$
$$*(diff(f(x),x)) * f(x)^2 * x^3 - 4 * f(x)^4 * (diff(f(x),x)) * x + 4 * f(x)^5)/(f(x)^2 * x^3),$$
$$f(1) = 1, D(f)(1) = 1, D(D(f))(1) = 1.1\}, type = numeric, range = 0..100):$$
$$> odeplot(p);$$
$$> ode := diff(f(x), `\$`(x,3)) = (2 * f(x)^3 - 2 * (diff(f(x),x))^3 * x^3 + 2 * f(x)^3 * x^2$$
$$+3 * (diff(f(x), `\$`(x,2))) * f(x) * (diff(f(x),x)) * x^3 - 2 * (diff(f(x),x)) * f(x)^2$$
$$*x^3 - 4 * f(x)^4 * (diff(f(x),x)) * x + 4 * f(x)^5)/(f(x)^2 * x^3);$$
$$> ans := dsolve(\{ode, f(1) = 1, D(f)(1) = 1, D(D(f))(1) = 1.1\},$$
$$f(x), type = series);$$
$$> plot(1 + (x-1) + 11/20 * (x-1)^2 + 11/20 * (x-1)^3 - 99/800 * (x-1)^4$$
$$-99/800 * (x-1)^5, x = -1..1);$$
$$> with(plots):$$
$$> y := 20;$$

$$> p1 := dsolve(\{diff(f(x), `\$`(x,3)) = (2*f(x)^3*y^2$$
$$-2*(diff(f(x),x))^3*x^3*y^2+2*f(x)^3*x^2+3*$$
$$(diff(f(x), `\$`(x,2)))*f(x)*(diff(f(x),x))*x^3*y^2$$
$$-2*(diff(f(x),x))*f(x)^2*x^3-4*f(x)^4*(diff(f(x),x))*x$$
$$+4*f(x)^5)/(f(x)^2*x^3*y^2),$$
$$f(1)=1, D(f)(1)=1, D(D(f))(1)=1.1\},$$
$$type = numeric, range = -10..10):$$
$$> odeplot(p1);$$
$$> with(DEtools):$$
$$> PDEtools[declare](f(x), prime = x);$$
$$> ODE := diff(f(x), `\$`(x,3)) = (2*f(x)^3*y^2$$
$$-2*(diff(f(x),x))^3*x^3*y^2+2*f(x)^3*x^2+3*(diff(f(x),$$
$$`\$`(x,2)))*f(x)*(diff(f(x),x))*x^3*y^2$$
$$-2*(diff(f(x),x))*f(x)^2*x^3-4*f(x)^4*(diff(f(x),x))*x$$
$$+4*f(x)^5)/(f(x)^2*x^3*y^2);$$
$$> ODE := f''' = \frac{1}{400f^2x^3}(800f^3-800f'^3x^3+2f^3x^2$$
$$+1200f''ff'x^3-2f^2f'x^3-4f^4f'x+4f^5)$$

A pair of symmetry infinitesimals for this ODE is given by *symgen*

$$> infinitesimals := symgen(ODE);$$

From these infinitesimals, a related integrating factor for ODE is obtained via *intfactor*

$$> mu := intfactor(ODE);$$
$$> dsolve(ODE, Lie);$$

$$> with(PDEtools);$$

Here is a boundary value problem with y=c like an unknown parameter.

$$> c := 15;$$

$$> dsys4 := diff(f(x), `\$`(x,3)) = (2*f(x)^3*c^2$$
$$-2*(diff(f(x),x))^3*x^3*c^2+2*f(x)^3*x^2+3*(diff(f(x),$$
$$`\$`(x,2)))*f(x)*(diff(f(x),x))*x^3*c^2$$
$$-2*(diff(f(x),x))*f(x)^2*x^3-4*f(x)^4*(diff(f(x),x))*x$$
$$+4*f(x)^5)/(f(x)^2*x^3*c^2),$$

$$f(1) = 1, D(f)(1) = 1, D(D(f))(1) = 1.1;$$

$$> dsol4 := dsolve(dsys4, numeric);$$

$$> dsol4(0.1);$$

$$> dsol4(0.15);$$

$$> plots[odeplot](dsol4);$$

$$> dsys5 := diff(f(x), `\$`(x, 3)) = (2 * f(x)^3 * c^2$$

$$-2 * (diff(f(x), x))^3 * x^3 * c^2 + 2 * f(x)^3 * x^2 + 3 * (diff(f(x),$$

$$`\$`(x, 2))) * f(x) * (diff(f(x), x)) * x^3 * c^2$$

$$-2 * (diff(f(x), x)) * f(x)^2 * x^3 - 4 * f(x)^4 * (diff(f(x), x)) * x$$

$$+4 * f(x)^5)/(f(x)^2 * x^3 * c^2);$$

$$> dsolve(dsys5);$$

$$> dsolve(dsys5, f(x), useint);$$

$$> dsolve(dsys5, f(x), parametric);$$

$$> with(DEtools) :> PDEtools[declare](f(x), prime = x);$$

Here is an ODE depending on the parameter y.

$$> ODE1 := diff(f(x), `\$`(x, 3)) = (2 * f(x)^3 * y^2$$

$$-2 * (diff(f(x), x))^3 * x^3 * y^2 + 2 * f(x)^3 * x^2 + 3 * (diff(f(x),$$

$$`\$`(x, 2))) * f(x) * (diff(f(x), x)) * x^3 * y^2$$

$$-2 * (diff(f(x), x)) * f(x)^2 * x^3 - 4 * f(x)^4 * (diff(f(x), x)) * x$$

$$+4 * f(x)^5)/(f(x)^2 * x^3 * y^2);$$

A pair of symmetry infinitesimals for this ODE is given by

$$> infinitesimals := symgen(ODE1);$$

$$> sym3 := symgen(ODE1, way = 5);$$

$$> sym31 := symgen(ODE1, way = abaco2);$$

From these infinitesimals, we obtain two related integrating factors for ODE.

$$> mu := intfactor(ODE1);$$

$$\mu := \frac{1}{f}, \ \frac{-ln\,(x) + ln\,(f)}{f}$$

$$> odeadvisor(mu[1] * ODE1);$$

$$\#after\,multiplying\,by\,mu[1]\,or\,mu[2], ODE1\,becomes\,exact$$

$$odeadvisor[[\_3rd\_order, \_exact, \_nonlinear]]$$

$$> firint(mu[1] * ODE1); \#a\,first\,integral\,related\,to\,the\,mu[1]$$

$$\frac{2f^2}{x^2y^2}+\frac{2ln(f)}{y^2}-\frac{f'^2}{f^2}+\frac{f''}{f}+\frac{1}{x^2}-\frac{2ln(x)}{y^2}+\_C_1=0$$

$$> firint(mu[2]*ODE1); \#a\, first\, integral\, related\, to\, the\, mu[2]$$

$$\frac{2f^2ln(x)}{x^2y^2}-\frac{ln(f)^2}{y^2}+\frac{2ln(x)ln(f)}{y^2}-\frac{2f^2ln(f)}{x^2y^2}+\frac{f^2}{x^2y^2}-\frac{ln(f)}{x^2}$$
$$-\frac{f'^2ln(x)}{f^2}+\frac{f'^2ln(f)}{f^2}-\frac{f'}{xf}+\frac{f'^2}{2f^2}+\left(\frac{ln(x)}{f}-\frac{ln(f)}{f}\right)f''+\frac{ln(x)}{x^2}$$
$$+\frac{1}{2x^2}-\frac{ln(x)^2}{y^2}+\_C_1=0$$

Eliminating 'f'', 'f''' from these two first integrals and renaming $\_C1->\_C2$ in one of them we obtain the ODE solution; this process is performed internally by *dsolve* to arrive at:

$$> dsolve(ODE1);$$

$$f=RootOf\left(-x+\int^{\_Z}\frac{y}{F(\_f)}d\_f+\_C_3\right)x,$$

where

$$F(\_f)=\sqrt{-2\_f^2-2\_C_2y^2-2ln(\_f)^2-2ln(\_f)\_C_1y^2\_f}.$$

Summarizing a part of the main results we obtain

Theorem. *The Calapso PDE admit solutions of the form* $z(x,y)=\dfrac{f(x)}{xy}$, *where the function* $f(x)$ *is a solution of the ODE*

$$f'''=\frac{2f^3y^2-2f'^3x^3y^2+2f^3x^2+3ff'f''x^3y^2-2f^2f'x^3-4f^4f'x+4f^5}{f^2x^3y^2},$$

depending on the parameter y. *This ODE admits the integrant factors*

$$\mu:=\frac{1}{f},\ \frac{-ln\,(x)+ln\,(f)}{f}$$

and the first integrals

$$\frac{2f^2}{x^2y^2}+\frac{2ln(f)}{y^2}-\frac{f'^2}{f^2}+\frac{f''}{f}+\frac{1}{x^2}-\frac{2ln(x)}{y^2}+\_C_1=0,$$

$$\frac{2f^2ln(x)}{x^2y^2}-\frac{ln(f)^2}{y^2}+\frac{2ln(x)ln(f)}{y^2}-\frac{2f^2ln(f)}{x^2y^2}+\frac{f^2}{x^2y^2}-\frac{ln(f)}{x^2}$$
$$-\frac{f'^2ln(x)}{f^2}+\frac{f'^2ln(f)}{f^2}-\frac{f'}{xf}+\frac{f'^2}{2f^2}+\left(\frac{ln(x)}{f}-\frac{ln(f)}{f}\right)f''+\frac{ln(x)}{x^2}$$
$$+\frac{1}{2x^2}-\frac{ln(x)^2}{y^2}+\_C_2=0.$$

Open problem. A very important PDE-constrained optimization problem is: *does a isothermic surface have a minimal total Gaussian curvature?*

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