# A framed $f(3, -1)$ structure on a $GL-$tangent manifold

Manuela Gîrţu

**Abstract.** A tangent manifold is a pair $(M, J)$ with $J$ a tangent structure ($J^2 = 0$, ker$J$ =im$J$) on the manifold $M$. One denotes by $HM$ any complement of im$J := TV$. Using the projections $h$ and $v$ on the two terms in the decomposition $TM = HM \oplus TV$ one defines the almost product structure $P = h - v$ on $M$. Adding to the pair $(M, J)$ a Riemannian metric $g$ in the bundle $TV$ one obtains what we call a $GL-$tangent manifold. One assumes that the $GL-$tangent manifold $(M, J, g)$ is of bundle-type, that is $M$ posses a globally defined Euler or Liouville vector field. This data allow us to deform $P$ to a framed $f(3, -1)-$structure $\mathcal{P}$. The later kind of structures have origin in the paper [6] by K. Yano. Then we show that $\mathcal{P}$ restricted to a submanifold that is similar to the indicatrix bundle in Finsler geometry, provides a Riemannian almost paracontact structure on the said submanifold. The present results extend to the framework of tangent manifold our previous results on framed structures of the tangent bundles of Finsler or Lagrange manifolds, see [2], [1].

**M.S.C. 2000**: 53C60.
**Key words**: framed f(3,-1)-structure, GL-tangent manifold.

## 1 Bundle-type tangent manifolds

Let $M$ be a smooth i.e. $C^\infty$ manifold. We denote by $\mathcal{F}(M)$ the ring of smooth functions $M$, by $TM$ the tangent bundle and by $\mathcal{X}(M) = \Gamma TM$ the $\mathcal{F}(M)-$module of vector fields on $M$ (sections in tangent bundle).

**Definition 1.1.** An *almost tangent structure* on $M$ is a tensor field J of type $(1, 1)$ on $M$ i.e. $J \in \Gamma\text{End}(TM)$ such that

$$J^2 = 0, \quad \text{im } J = \ker J. \tag{1.1}$$

It follows that the dimension of $M$ must be even, say $2n$ and rank$J = n$.

Balkan Journal of Geometry and Its Applications, Vol.13, No.1, 2008, pp. 47-54.

**Definition 1.2.** An almost tangent structure $J$ is called a *tangent structure* if there exists an atlas on $M$ with local coordinates $(x^i, y^i)$, $i, j, k... = 1, 2, ..., n$, such that

$$J = \left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i},\ J\left(\frac{\partial}{\partial y^i}\right) = 0. \tag{1.2}$$

A pair $(M, J)$ is called a *tangent manifold*. For the geometry of tangent manifolds we refer to [5]. The basic example is the tangent bundle $TM$, [3].

Let $(M, J)$ be a tangent manifold. The distribution im$J$ is integrable. It defines a **vertical** foliation $V$ with $TV$ =im$J$. Let us choose and fix a complement bundle $HM$ called also the horizontal bundle such that

$$TM = HM \oplus TV. \tag{1.3}$$

In the following we shall use bases adapted to the decomposition (1.3): $\left(\delta_i = \partial_i - N_i^j(x, y)\dot{\partial}_j, \dot{\partial}_i = \dfrac{\partial}{\partial y^i}\right)$, $\partial_i := \dfrac{\partial}{\partial x^i}$, such that $TV = \text{span}\{\dot{\partial}_j\}$, $HM = \text{span}\{\delta_i\}$.

The dual cobase is $(dx^i, \delta y^i = dy^i + N_j^i(x, y)dx^j)$, that is $(HM)^*$ =span$\{dx^i\}$ and $(TV)^*$ =span$\{\delta y^i\}$. Here $(N_i^j(x, y))$ are local functions. Notice that $J(\delta_i) = \dot{\partial}_i$, $J(\dot{\partial}_i) = 0$.

Let be another atlas on $M$ with local coordinates $(\tilde{x}^i, \tilde{y}^i)$ in which (1.2) also holds. Then necessarily one has

$$\tilde{x}^i = \tilde{x}^i(x),\ \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j}(x)y^j + b^i(x), \tag{1.4}$$

$$\frac{\partial}{\partial x^i} = \frac{\partial}{\partial \tilde{x}^j}\frac{\partial \tilde{x}^j}{\partial x^i} + \frac{\partial}{\partial \tilde{y}^j}\left(\frac{\partial^2 \tilde{x}^j}{\partial x^k \partial x^i}y^k + \frac{\partial b^j}{\partial x^i}\right), \frac{\partial}{\partial y^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{y}^j} \tag{1.4'}$$

$$\delta_i = \frac{\partial \tilde{x}^j}{\partial x^i}\delta_j. \tag{1.4''}$$

By (1.4") the functions $(N_j^i(x, y))$ change to the functions $(\widetilde{N}_h^k(\hat{x}, \hat{y}))$ given by

$$\tilde{N}_k^h \frac{\partial \tilde{x}^k}{\partial x^i} = \frac{\partial \tilde{x}^h}{\partial x^k}N_i^k - \left(\frac{\partial b^h}{\partial x^i} + \frac{\partial^2 \tilde{x}^h}{\partial x^k \partial x^i}y^k\right). \tag{1.5}$$

The projections on the two terms in (1.3) will be denoted by $h$ and $v$, respectively. Then $P = h - v$ is an almost product tensor structure that has the horizontal and vertical distribution as +1 (-1)-eigen distributions, respectively.

It is obvious that $J|_{HM}$ is an isomorphism $j : HM \to TV$ and $J = j \oplus 0$. Then $J' = 0 \oplus j^{-1}$ is an almost tangent structure, $Q = J' + J$ is an almost product structure and $F = J' - J$ is an almost complex structure.

In the adapted bases $(\delta_i, \dot{\partial}_i)$ we have:

$$J(\delta_i) = \dot{\partial}_i,\ J(\dot{\partial}_i) = 0, \tag{1.6$_1$}$$

$$J'(\delta_i) = 0,\ J'(\dot{\partial}_i) = \delta_i, \tag{1.6$_2$}$$

$$P(\delta_i) = \delta_i,\ P(\dot{\partial}_i) = -\dot{\partial}_i, \tag{1.6$_3$}$$

$$Q(\delta_i) = \dot{\partial}_i,\ Q(\delta_i) = \dot{\partial}_i, \tag{1.6$_4$}$$

$$F(\delta_i) = -\dot{\partial}_i,\ F(\dot{\partial}_i) = \delta_i. \tag{1.6$_5$}$$

Moreover, we have

$$PF = -FP = Q. \tag{1.7}$$

**Definition 1.3.** A pseudo-Riemannian metric $G$ on $M$ is said to be *compatible metric* if the subbundles $HM$ and TV are orthogonal with respect to $G$ and

$$G(JX, JY) = G(X, Y),\ \forall X, Y \in \Gamma HM. \tag{1.8}$$

In the adapted cobases $(dx^i, \delta y^i)$ we have

$$G(x, y) = a_{ij}(x, y)dx^i dx^j + a_{ij}(x, y)\delta y^i \delta y^j, \tag{1.9}$$

where $a_{ij} := G(\delta_i, \delta_j)$.

# 2 GL-tangent manifolds

Let $(M, J)$ be a tangent manifold.

**Definition 2.1.** A pseudo-Riemannian structure g in the vertical subbundle $TV = \mathrm{im}J$ will be called a *generalized Lagrange* $(GL)-$structure on M. We will say that g is a $GL-$metric and $(M, J, g)$ will be called a $GL-$tangent manifold.

**Remark 2.1.** The notion of GL-metric for tangent bundle $TM$ was defined by R. Miron. Properties of various classes of GL-metrics have been established in the monograph [3].

The GL-metric $g$ is determined by the local coefficients $g_{ij}(x, y) = g(\dot{\partial}_i, \dot{\partial}_j)$ with $\det(g_{ij}) \neq 0$ and the quadratic form $g_{ij}\xi^i\xi^j$, $(\xi \in \mathbb{R}^n)$, of constant signature. It is obvious that there exists a compatible metric $G$ on $M$ such that $G|_{TV} = g$. In the adapted cobase $(dx^i, \delta y^i)$ it has the form

$$G(x, y) = g_{ij}(x, y)dx^i dx^j + g_{ij}(x, y)\delta y^i \delta y^j. \tag{2.1}$$

From now on we assume that the GL-tangent manifold $(M, J, g)$ is of **bundle-type**, that is $C = y^i\dot{\partial}_i$ is a global vector field called Liouville or Euler vector field. Then in (1.4) we must have $b^i \equiv 0$.

# 3 A framed $f(3,-1)-$structure on a GL-tangent manifold of bundle-type

Let $(M,J,g)$ be a GL-tangent manifold of bundle-type such that $g$ is a Riemannian metric in $TV=\mathrm{im}S$.

We set $L=g_{ij}(x,y)y^iy^j$ and we get a positive function on $M$.

We call $L$ a Lagrangian on $M$ and if the matrix with the entries $\left(\frac{1}{2}\dot{\partial}_i\dot{\partial}_jL\right)$ is nonsingular, $L$ will be called a regular Lagrangian.

The condition "bundle-type" assures that the subset $O=\{(x^i,y^i)\mid y^i=0\}$ is a closed submanifold of $M$.We restrict our considerations to the open submanifold $\tilde{M}=M\setminus 0$ of $M$ and we keep the same notations for the geometrical objects involved.We notice that $(\tilde{M},J,g)$ is a GL-tangent manifold of bundle-type.

On $\tilde{M}$ we have $L>0$ and so we may consider the vector fields

$$\xi=\frac{1}{L}y^i\delta_i,\ \zeta=\frac{1}{L}y^i\dot{\partial}_i, \tag{3.1}$$

as well as the 1-forms

$$\omega=\frac{1}{L}y_idx^i,\ \eta=\frac{1}{L}y_i\delta y^i, \tag{3.2}$$

where $y_i=g_{ij}(x,y)y^j$.

It is immediately that

$$\begin{aligned}\omega(\xi)=1,&\quad\omega(\zeta)=0,\\ \eta(\xi)=0,&\quad\eta(\zeta)=1.\end{aligned} \tag{3.3}$$

Moreover, if $G$ is the compatible metric given by (2.1), then

$$G(\xi,\xi)=1,\ G(\xi,\zeta)=0,\ G(\zeta,\zeta)=1. \tag{3.4}$$

Recall that on $\tilde{M}$ we have the almost product structure $P$.

From $(1.6_3)$ it follows

$$P(\xi)=\xi,\ P(\zeta)=-\zeta, \tag{3.5}$$

and one checks

**Lemma 3.1.** *$\omega\circ P=\omega$, $\eta\circ P=-\eta$.*

Then (3.3) and (3.4) yield

**Lemma 3.2.** *$\omega(X)=G(X,\xi),\eta(X)=G(X,\zeta)$, $\forall X\in\mathcal{X}(\tilde{M})$.*

Now we set

$$\mathcal{P}=P-\omega\otimes\xi+\eta\otimes\zeta. \tag{3.6}$$

**Theorem 3.1.** *The triple $\mathcal{F}=(\mathcal{P},(\xi,\zeta),(\omega,\eta))$ is a framed $f(3,-1)-$ structure, that is*

$$\mathcal{P}(\xi)=\mathcal{P}(\zeta)=0,\ \omega\circ\mathcal{P}=\eta\circ\mathcal{P}=0,$$
(3.7)
$$\mathcal{P}^2=I-\omega\otimes\xi-\eta\otimes\zeta,$$

*where $I$ is the Krönecker tensor field.*

*Proof.* A direct calculation using (3.3), (3.5) and Lemma 3.1.

**Theorem 3.2.** *The tensor field $\mathcal{P}$ is of rank $2n-2$ and satisfies*

$$\mathcal{P}^3-\mathcal{P}=0. \tag{3.8}$$

*Proof.* The equation (3.8) easily follows from (3.7). We show that ker$\mathcal{P}$ is spanned by $\xi$ and $\zeta$, that is ker$\mathcal{P}$ =span$\{\xi,\zeta\}$. The inclusion "$\supset$" follows from (3.7). For proving the inclusion "$\subset$" let be $Z=X^i\delta_i+Y^i\dot{\partial}_i\in$ker$\mathcal{P}$. Then by (3.6), $\mathcal{P}(Z)=X^i\delta_i-Y^i\dot{\partial}_i-(\omega_iX^i)\xi+(\eta_iY^i)\zeta$ and $\mathcal{P}(Z)=0$ gives $X^i=\dfrac{1}{L}(\omega_iX^i)y^i$ and $Y^i=\dfrac{1}{L}(\eta_iY^i)y^i$ and so $Z=(\omega_iX^i)\xi+(\eta_iY^i)\zeta$. Hence $Z\in$span$\{\xi,\zeta\}$, q.e.d.

The study of structures on manifold defined by tensor field $f$ satisfying $f^3\pm f=0$ has the origin in a paper by K. Yano, [6]. Later on, these structures have generically called $f-$structures. They have been extended and can be encountered under various names. We refer to the book [4].

**Theorem 3.3.** *The Riemannian metric $G$ defined by (2.1) satisfies*

$$G(\mathcal{P}X,\mathcal{P}Y)=G(X,Y)-\omega(X)\omega(Y)-\eta(X)\eta(Y),\ \forall X,Y\in\mathcal{X}(\tilde{M}). \tag{3.9}$$

*Proof.* First, we notice that from Lemma 3.2 it follows $G(PX,\xi)=\omega(X)$ and $G(PX,\zeta)=-\eta(X)$ for all $X\in\mathcal{X}(\tilde{M})$. Then we have

$$G(PX-\omega(X)\xi+\eta(X)\zeta,PY-\omega(Y)\xi+\eta(Y)\zeta)=$$
$$=G(PX,PX)-\omega(Y)G(PX,\xi)+\eta(Y)G(PX,\zeta)-\omega(X)G(PY,\xi)+$$
$$+\omega(X)\omega(Y)+\eta(X)G(PY,\zeta)+\eta(X)\eta(Y)=G(X,Y)-\omega(X)\omega(Y)-\eta(X)\eta(Y),$$

because of $G(PX,PY)=G(X,Y)$, q.e.d.

Theorem 3.3 says that $(\mathcal{P},G)$ is a Riemannian framed $f(3,-1)-$ structure on $\tilde{M}$.

# 4 A Riemannian almost paracontact structure

Let be $IL=\{(x,y)\in\tilde{M}\mid L(x,y)=1\}$. This set is a $(2n-1)-$ dimensional submanifold of $\tilde{M}$. It will be called the indicatrix of $L$. We are interested to study the restriction of the Riemannian framed $f(3,-1)-$ structure to $IL$.

We shall see that in certain hypothesis on $L$, the said restriction is a Riemannian almost paracontact structure.

We consider $\tilde{M}$ endowed with the Riemannian metric $G$ given by (2.1) and we try to find a unit normal vector field to $IL$.

Let be

$$\begin{aligned}x^i&=x^i(u^\alpha),\\ y^i&=y^i(u^\alpha),\end{aligned}\quad rank\left(\frac{\partial x^i}{\partial u^\alpha},\frac{\partial y^i}{\partial u^\alpha}\right)=2n-1,\ \alpha=1,2,...,2n-1, \tag{4.1}$$

a parametrization of the submanifold $IL$.

The local vector fields $\left(\dfrac{\partial}{\partial u^\alpha}\right)$ that form a base of the tangent space to $IL$, take the form

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial x^i}{\partial u^\alpha}\delta_i + \left(\frac{\partial y^i}{\partial u^\alpha} + N^i_j \frac{\partial x^j}{\partial u^\alpha}\right)\dot{\partial}_i \tag{4.2}$$

and it comes out that $\zeta$ is normal to $IL$ if and only if

$$G\left(\frac{\partial}{\partial u^\alpha}, \zeta\right) = \frac{1}{L}\left(\frac{\partial y^i}{\partial u^\alpha} + N^i_j \frac{\partial x^j}{\partial u^\alpha}\right) y_i = 0. \tag{4.3}$$

We derive the identity $L^2(x(u^\alpha), y(u^\alpha)) \equiv 1$ with respect to $u^\alpha$ and we obtain

$$(\delta_i L^2)\frac{\partial x^i}{\partial u^\alpha} + \left(\frac{\partial y^i}{\partial u^\alpha} + N^i_j \frac{\partial x^i}{\partial u^\alpha}\right)(\dot{\partial}_i L^2) \equiv 0. \tag{4.4}$$

Looking at (4.4) and (4.3) it comes out that (4.3) holds if $L$ satisfies the following two conditions:

$(H_1)$ $\delta_i L^2 = 0$,

$(H_2)$ $\dot{\partial}_i L^2 = f y_i$, for $f \neq 0$ any smooth function on $\tilde{M}$.

If $(H_1)$ and $(H_2)$ hold, then $\zeta$ is the unit normal vector to $IL$. We restrict to $IL$ the element from $\mathcal{F}$ and we point out this by a bar over those elements. Thus we have:

- $\bar{\xi} = \xi$ since $\xi$ is tangent to $IL$,
- $\bar{\eta} = 0$ since $\eta(X) = G(X, \xi) = 0$ for any vector field tangent to $IL$
- $\bar{\mathcal{P}} = \mathcal{P} - \omega \otimes \xi$, because of $G(\bar{\mathcal{P}}X, \xi) = G(PX, \zeta) = \eta(PX) = -\eta(X) = 0$.

for any vector field $X$ tangent to $IL$. Now we state

**Theorem 4.1.** *The triple $(\bar{\mathcal{P}}, \bar{\xi}, \bar{\omega})$ defines a Riemannian almost paracontact structure on $IL$, that is*

*(i) $\bar{\omega}(\bar{\xi}) = 1$, $\bar{\mathcal{P}}(\bar{\xi}) = 0$, $\bar{\omega} \circ \bar{\mathcal{P}} = 0$,*

*(ii) $\bar{\mathcal{P}}^2 = I - \bar{\omega} \otimes \bar{\xi}$ on $IL$,*

*(iii) $G(\bar{\mathcal{P}}X, \bar{\mathcal{P}}Y) = G(X, Y) - \bar{\omega}(X)\bar{\omega}(Y)$ for any vector fields tangent to $IL$. Moreover, we have*

*(iv) $\tilde{\mathcal{P}}^2 - \tilde{\mathcal{P}} = 0$ and $\mathrm{rank}\tilde{\mathcal{P}} = 2\mathrm{n} - 1$.*

*Proof.* All assertions easily follow from Theorems 3.1 - 3.3.

We end with a discussion on the hypothesis $(H_1)$ and $(H_2)$. More precisely we show that these hypothesis can be replaced with a weaker one $(H)$ that is referring to $(g_{ij})$ only.

$(H)$ The functions $g_{ij}(x, y)$ are 0-homogeneous in $(y^i)$ and the functions $C_{ijk} = \dfrac{1}{2}\dot{\partial}_k g_{ij}$ are symmetrical in the indices $i, j, k$.

First, $(H)$ implies $C_{ijk}y^k = C_{ijk}y^i = C_{ijk}y^j = 0$. Using these we compute: $\dot{\partial}_j L^2 = 2C_{ijk}y^i y^j + 2g_{jk}y^k = 2y_j$. Thus $(H)$ implies $(H_2)$. A new derivation with respect to $(y^i)$ gives $\dfrac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2 = g_{ij}$. This says that $L$ is a regular Lagrangian.

In order to show that $(H)$ implies also $(H_1)$ we need to find a set of local coefficients $(N^i_j(x,y)$ depending only on $(g_{ij})$.

We denote by $(g^{jk})$ the inverse of the matrix $(g_{ij})$ and consider the functions $G^i(x,y)$ given by

$$4G^i(x,y) = g^{ik}[(\dot{\partial}_k \partial_h L^2)y^h - \partial_k L^2], \tag{4.5}$$

and define the local coefficients $N^i_j(x,y)$ as

$$N^i_j(x,y) = \frac{\partial G^i}{\partial y^j}. \tag{4.6}$$

When we replace the adapted coordinates $(x^i, y^i)$ with the adapted coordinates $(\tilde{x}^i, \tilde{y}^i)$, a direct calculation says that the new functions $\tilde{G}^i$ are related to $G^i$ by

$$\tilde{G}^i(\tilde{x},\tilde{y}) = G^i(x,y) - \frac{1}{2}\frac{\partial \tilde{x}^i}{\partial x^k \partial x^h} y^k y^h. \tag{4.7}$$

And as a consequence of (4.7) easily follows that the functions $(N^i_j)$ are related to $(\tilde{N}^i_j)$ by (1.5) with $b^i \equiv 0$. We recall that $\tilde{M}$ is a tangent manifold of bundle-type.

Now we are preparing for the computation of $\delta_i L^2$.

First, we write (4.5) in the form

$$\begin{aligned} 4g_{jk}G^k = \partial_h(2y_j)y^h - \partial_j L^2 = 2(\partial_h g_{jk})y^k y^h - \partial_j L^2 = \\ = (2\partial_h g_{jk} - \partial_j g_{kh})y^k y^h \end{aligned}$$

and derive the both members with respect to $(y^i)$.

We get the equation

$$8C_{jki}G^k + 4g_{jk}N^k_i = 2(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik})y^k.$$

Equivalently,

$$N^h_i = \frac{1}{2} g^{hj}(\partial_k g_{ij} + \partial_i g_{kj} - \partial_j g_{ik})y^k$$

which, by a contraction with $(y^j)$ yields

$$2y_k N^k_i = (\partial_i g_{jk})y^j y^k. \tag{4.8}$$

We continue computing

$$\delta_i L^2 = \partial_i L^2 - N^k_i \dot{\partial}_k L^2 = \partial_i(g_{jk})y^j y^k - 2N^k_i y^k \overset{(4.8)}{=} \partial_i(g_{jk})y^j y^k - (\partial_i g_{jk})y^j y^k = 0.$$

Thus $(H)$ implies $(H_1)$, too.

A simple case when the hypothesis (H) holds is when the functions $(g_{ij})$ depend on $x$ only. Then $g_{ij}$ are homogeneous of any degree in $(y^i)$ and $C_{ijk} \equiv 0$.

# Conclusions

It is well-known that a tangent bundle is a bundle-type tangent manifold. The results of this paper generalize those from our papers [2], [1] first from Finsler setting to $GL-$metrics and then from tangent bundles framework to bundle-type tangent manifolds.

## References


[1] M. Gîrţu, *A framed $f(3,-1)$ structure on the tangent bundle of a Finsler manifold*, Studii Cercet. Ştiinţ., Ser. Mat.Univ. Bacău, 11 (2001), 91-97.

[2] M. Gîrţu, *A framed $f(3,-1)$ structure on the tangent bundle of a Lagrange space*, Demonstratio Math. 37, 4 (2004), 955-961.

[3] R. Miron R., M. Anastasiei, *The Geometry of Lagrange Spaces. Theory and Applications*, Kluwer Academic Publishers, FTPH 59, 1994.

[4] I. Mihai, R. Roşca, L. Verstraelen, *Some aspects of the differential geometry of vector fields. On skew symmetric Killing and conformal vector fields, and their relations to various geometrical structures* (English), [B] Centre for Pure and Applied Differential Geometry (PADGE), 2, Leuven: Katholieke Univ. Leuven, Department of Mathematics, Brussel: Katholieke Univ. Brussel, Group of Exact Sciences, 1996.

[5] I. Vaisman, *Lagrange geometry on tangent manifolds*, Int. J. Math. Math. Sci. 2003 (51), 3241-3266.

[6] Yano, K., *On a structure defined by a tensor field $f$ satisfying $f^3+f=0$*, Tensor N.S. 14 (1963), 99-109.



*Author's address:*

Manuela Gîrţu
Faculty of Sciences, University of Bacău, Romania.
E-mail: girtum@yahoo.com