Inner separation structures for topological spaces

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Abstract. In this paper we present new forms of the classical separation axioms on topological spaces. Our constructions generate a method to refine separation properties when passing to the quotient space and our results may be useful in the study of algebraic topological structures, such as topological groups and topological vector spaces.

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1 Introduction

The classical separation axioms have been generalized in several directions. In this work, we will investigate the $\mathcal{R}$-separated spaces, introduced in [3]. There, the author introduced the so called $\mathcal{R}$-separation properties on a topological space X on which a binary relation $\mathcal{R}$ is defined. The main idea is to replace the identity relation in the classical separation properties with the relation $\mathcal{R}$. So, for instance, X is said to be $\mathcal{R}$-Hausdorff iff given two $\mathcal{R}$-unrelated elements, say a and b, there exist $\mathcal{R}$-disjoint neighborhoods of a and b respectively (see [3]).
In Section 2, we briefly describe the main constructions and results in [3].
In Section 3 we present Kolmogorov's first relation on a topological space X, which is an order relation on T_0 (Kolmogorov) spaces. The main result in this section is Theorem 3.3, which shows that classical separability can be expressed in terms of separability with respect to Kolmogorov's relation.
In Section 4 we study separability with respect to Kolmogorov's second relation on a topological space, which as it turns out, is an equivalence relation. The central result in this section is Theorem 4.6. This theorem is actually the main result of the paper. It refines the separation properties of certain classes of topological spaces.We also show that the quotient space of Kolmogorov's second relation satisfies supplementary separation properties. We present concrete applications to topological groups and topological vector spaces.

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2 Notation and terminology

In consistency with the notation in [3], given a topological space (X, τ) together with a binary relation $\mathcal{R}$ on X we will denote by $\mathcal{V}_x$ the filter of neighborhoods of the element $x \in X$. For a set $A \subseteq X$, V_A will stand for an open subset of X containing A. As usual, the inverse of $\mathcal{R}$ will be written as $\mathcal{R}^{-1}$, whereas for $x \in X$ and $y \in X$ the notation $x\overline{\mathcal{R}}y$ will indicate that the pair (x, y) does not belong to the graph of $\mathcal{R}$. Analogously, for $A \subseteq X$ and $B \subseteq X$, we will write $A\overline{\mathcal{R}}B$ to indicate that the cross product $A \times B$ is contained in the complement of the graph of $\mathcal{R}$. We will abbreviate $\{x\}\mathcal{R}A$ as $x\mathcal{R}A$. For $x \in X$, the set $\{y : x\mathcal{R}y\}$ will be written as $\mathcal{R}(x)$. More generally, if $Y \subseteq X$, $\mathcal{R}(Y) = \bigcup_{x \in Y} \mathcal{R}(x)$.
The space X will be said to be:

- $T_0^{\mathcal{R}}$ iff whenever $x_1 \in X$, $x_2 \in X$ and $x_1\overline{\mathcal{R}}x_2$, there exists $V \in \mathcal{V}_{x_j}$ for $j = 1$ or $j = 2$ such that $x_i\overline{\mathcal{R}}V$ for $i \neq j$.

- $T_1^{\mathcal{R}}$ iff whenever $x_1 \in X$, $x_2 \in X$ and $x_1\overline{\mathcal{R}}x_2$, there exist $V_i \in \mathcal{V}_{x_i}$, $i = 1, 2$ such that $x_i\overline{\mathcal{R}}V_j$ for $i \neq j$.

- $T_2^{\mathcal{R}}$ iff for $x_1 \in X$, $x_2 \in X$ and $x_1\overline{\mathcal{R}}x_2$, there exist $V_i \in \mathcal{V}_{x_i}$, $i = 1, 2$ such that $V_1\overline{\mathcal{R}}V_2$.

- $T_3^{\mathcal{R}}$ or $\mathcal{R}$-regular iff for any closed subset $F \subset X$ and $x \in X$, $x\overline{\mathcal{R}}F$ implies the existence of neighborhoods V_x and V_F such that $V_x\overline{\mathcal{R}}V_F$, and $F\overline{\mathcal{R}}x$ implies $V_F\overline{\mathcal{R}}V_x$ for some neighborhoods V_x and V_F.

- $T_4^{\mathcal{R}}$ or$\mathcal{R}$-normal iff for each $A \subset X$ and $B \subset X$ such that $A\overline{\mathcal{R}}B$, there are neighborhoods V_A and V_B such that $V_A\overline{\mathcal{R}}V_B$.

In [3], characterization theorems for the above separation properties were shown, that reduce to classical equivalent properties of separation when the relation under consideration is the identity on X. For example, X is $T_1^{\mathcal{R}}$ if and only if the sets $\mathcal{R}(x)$ and $\mathcal{R}^{-1}(x)$ are closed.

3 Kolmogorov's first relation

In this section we investigate the previous ideas in the particular case of the relation

$$x\mathcal{R}y \text{ iff } y \in \overline{\{x\}}, \tag{3.1}$$

called Kolmogorov's (first) relation (here $\overline{M}$ denotes the closure of the set M). Notice that the above condition is equivalent to

$$\overline{\{y\}} \subseteq \overline{\{x\}}. \tag{3.2}$$

More precisely (see Theorem 3.6 below), we show that $\mathcal{R}$-separation properties are equivalent to classical ones (i.e, separation corresponding to the identity relation on

X).
We start by pointing out the trivial facts that $\mathcal{R}$ is reflexive and transitive and that for $x \in X$, one has

$$\mathcal{R}(x) = \overline{\{x\}} \text{ and } \mathcal{R}^{-1}(x) = \bigcap_{A \in \mathcal{V}(x)} A$$

where $\mathcal{V}(x)$ denotes the filter of neighborhoods of x. Moreover, it is not hard to verify that for any subset A of X and any open subset Ω, it holds that $\mathcal{R}(\overline{A}) = \overline{\mathcal{R}(A)} = \overline{A}$ and $\mathcal{R}^{-1}(\Omega) = \Omega$. Notice also that any two closed (respectively, open) sets are disjoint if and only if they are $\mathcal{R}$-disjoint.

Lemma 3.1. *The separation axiom $T_0^{\mathcal{R}}$ holds for the Kolmogorov relation $\mathcal{R}$ on X. Moreover, X is T_0 if and only if $\mathcal{R}$ is an order relation.*

Proof. The first part is an immediate consequence of the definition of $\mathcal{R}$. Next, observe that $\mathcal{R}$ is an order relation if and only if it is antisymmetric. In that case, if a and b are two different elements of X, then the fact that at least one of the statements $a\mathcal{R}b$ or $b\mathcal{R}a$ is false, yields the validity of the T_0 axiom immediately. The converse follows easily. $\square$

Lemma 3.2. *If the space X is $T_1^{\mathcal{R}}$, then $\mathcal{R}$ is symmetric.*

Proof. If X is $T_1^{\mathcal{R}}$ and $a \in X$, $b \in X$ with $a\mathcal{R}b$, then the assumption $b\overline{\mathcal{R}}a$ yields the existence of a neighborhood V_b of b with $V_b\overline{\mathcal{R}}a$. This is impossible, since $a \in V_b$. $\square$

Theorem 3.3. *Let X be a topological space and $\mathcal{R}$ stand for Kolmogorov's first relation. Then the following statements are equivalent:*

(i) X *is* $T_1^{\mathcal{R}}$*;*

(ii) $\mathcal{R}$ *is the identity relation on* X*;*

(iii) X *is* T_1*.*

Proof. The implication $(i) \Rightarrow (ii)$ follows directly from Lemma 3.2 and the fact that, being $T_1^{\mathcal{R}}$, X is also $T_0^{\mathcal{R}}$, and hence antisymmetric.
The statement $(ii) \Rightarrow (iii)$ is an immediate consequence of the definition of $\mathcal{R}$.
Finally, if X is T_1 and $x_1\overline{\mathcal{R}}x_2$, then by definition of $\mathcal{R}$, there is an open neighborhood V_2 of x_2 such that $x_1 \notin V_2$. It is clear that no element of V_2 is in the closure of x_1, which yields $x_1\overline{\mathcal{R}}V_2$. On the other hand, since $x_1 \neq x_2$, we infer from (iii) the existence of a neighborhood V_1 of x_1 not containing x_2, which immediately yields the statement $x_2\overline{\mathcal{R}}V_1$. By definition, X is $T_1^{\mathcal{R}}$. $\square$

Lemma 3.4. *A topological space X is regular if and only if it is $\mathcal{R}$-regular*

Proof. If X is regular, $x \in X$ and $F \subset X$ is closed with $x\overline{\mathcal{R}}F$, it follows that $x \notin F$. Accordingly, there are disjoint open sets V_x and V_F containing x and F respectively. It follows now by definition of $\mathcal{R}$ that $V_x\overline{\mathcal{R}}V_F$. A similar reasoning shows that if $F\overline{\mathcal{R}}x$, one can find $\mathcal{R}$-disjoint open neighborhoods of F and x.
Conversely, assuming that X is $\mathcal{R}$-regular, given a closed subset $F \subseteq X$ and $x \notin F$, we consider an open neighborhood of x, V_x, disjoint with F. Clearly, $F\overline{\mathcal{R}}x$. Let W_x and W_F be open neighborhoods of x and F respectively such that $W_F\overline{\mathcal{R}}W_x$. Then it is clear that $W_x \cap W_F = \varnothing$. □

Lemma 3.5. *A topological space X is normal if and only if it is $\mathcal{R}$-normal.*

Proof. If X is normal and F and G are $\mathcal{R}$-disjoint closed subsets of X, then $F \cap G = \varnothing$. Let V_F and V_G be disjoint open neighborhoods of F and G respectively. It is clear that V_F and V_G are $\mathcal{R}$-disjoint.
Conversely, assume that X is $\mathcal{R}$-normal and consider disjoint closed subsets F and G. It is clear that $F\overline{\mathcal{R}}G$. There are then $\mathcal{R}$-disjoint (hence , also disjoint) open neighborhoods V_F and V_G of F and G respectively.

□

Theorem 3.6. *For each $i, 1 \leq i \leq 4$, the separation axioms T_i and $T_i^{\mathcal{R}}$ are equivalent.*

Proof. The proof follows immediately from the above Lemmas. □

4 Kolmogorov's second relation

Let X be a topological space. We define the relation ρ on X as follows:

$$x\rho y \text{ if and only if } \overline{\{x\}} = \overline{\{y\}} \tag{4.1}$$

Notice that ρ is an equivalence relation on X and that $x\rho y \Leftrightarrow x\mathcal{R}y$ and $y\mathcal{R}x$, where $\mathcal{R}$ stands for Kolmogorov's First Relation defined in the previous section.
We also point out the fact that X is T_0 if and only if ρ is the identity on X. Therefore, setting $\hat{x}$ to be the equivalence class of $x \in X$, we have

$$\hat{x} = \mathcal{R}(x) \cap \mathcal{R}^{-1}(x) \tag{4.2}$$

for each $x \in X$. It is also clear that any topological space is T_0^{ρ} and that $\rho(\Omega) = \Omega$ for any open set Ω.
Therefore (see [3]) the following statement holds:

Theorem 4.1. *If X is T_i^{ρ}, $1 \leq i \leq 4$, then the quotient space $\hat{X}$ is T_i.*

From the previous observations it follows the well known result that the quotient topology on $\hat{X}$ is T_0.
We now present some basic characterizations of the axioms T_i^{ρ}, $1 \leq i \leq 4$.

Theorem 4.2. *The following conditions on any topological space X are equivalent:*

(i) X *is* T_1^ρ*;*

(ii) $\bigcap_{V_x \in \mathcal{V}_x} V_x = \overline{\{x\}}$.

Proof. If we assume (i), it is easy to see that $\overline{\{x\}} \subseteq \bigcap_{V_x \in \mathcal{V}_x} V_x$. Otherwise, there would exist $z \in \overline{\{x\}} \setminus \left(\bigcap_{V_x \in \mathcal{V}_x} V_x\right)$. In particular, $x\overline{\rho}z$, since $x \notin \overline{\{z\}}$. This would yield the existence of an open neighborhood of z, W_z such that $x\overline{\rho}W_z$. But this contradicts the fact that $z \in \overline{\{x\}}$. Conversely, if $z \in \left(\bigcap_{V_x \in \mathcal{V}_x} V_x\right) \setminus \overline{\{x\}}$, then $z\overline{\rho}x$, from which there must exist a neighoborhood U_x of x not containing z, which is again a contradiction. Conversely, consider x and y in X with $x\overline{\rho}y$. Then either $y \notin \overline{\{x\}}$ or $x \notin \overline{\{y\}}$. In the first case, there is an open neighborhood W_y of y, disjoint with $\{x\}$. It follows $W_y\overline{\rho}x$. Assuming (ii) we obtain an open neighborhood U_x of x, not containing y, so that $U_x\overline{\rho}y$. Te same argument applies to the case $x \notin \overline{\{y\}}$. Hence, X is T_1^ρ. □

Theorem 4.3. *The following statements are equivalent:*

(i) X *is* $T_2^{\mathcal{R}}$*.*

(ii) $\bigcap_{V_x \in \mathcal{V}_x} \overline{V_x} = \overline{\{x\}}$.

Proof. Clearly, $\overline{\{x\}} \subseteq \bigcap_{V_x \in \mathcal{V}_x} \overline{V_x}$. If X is $T_2^{\mathcal{R}}$, then the equality in (ii) must hold; in fact, if $z \in \left(\bigcap_{V_x \in \mathcal{V}_x} \overline{V_x}\right) \setminus \overline{\{x\}}$, then let W_z and W_x be neighborhoods of z and z respectively such that

$$W_z\overline{\rho}W_x. \tag{4.3}$$

It follows that $z \notin \overline{W_x}$, otherwise W_z and W_x would have non-empty intersection, which contradicts (4.3).
On the other hand, if (ii) holds and x and y are elements in X such that $x\overline{\rho}y$, then either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. Obviously, it is necessary to handle only one case, say the first. From (ii), we conclude that there exists a neighborhood V_y of y such that

$$x \notin \overline{V_y} \tag{4.4}$$

and therefore, there exists an open set $V_x \in \mathcal{V}_x$ with $V_x \cap \overline{V_y} = \varnothing$. It is now clear that

$$V_x\overline{\rho}V_y.$$

This completes the proof. □

Lemma 4.4. *The topological space* X *is regular if and only if it is* ρ*-regular.*

Let X be a regular topological space, choose $x \in X$ and a closed set $F \subset X$ such that $x\overline{\rho}F$. Then $\{x\} \cap F = \varnothing$ and there exists open sets Ω and V_x such that $F \subset \Omega$, $x \in V_x$ and $\Omega \cap V_x = \varnothing$, which implies $\Omega\overline{\rho}V_x$.

Assuming now that x is ρ-regular, if $x \in X$ and F is a closed subset of X with $\{x\} \cap F = \varnothing$, one can find a neighborhood V_x of x with $V_x \cap F = \varnothing$. Therefore, $V_x \overline{\rho} F$, from which one obtains $x \overline{\rho} F$. Let $x \in U_x$ and $F \subseteq \Omega$ where V_x and Ω are open sets such that $U_x \overline{\rho} \Omega$. It is easy to see that $U_x \cap \Omega = \varnothing$. The proof is now complete. □

Lemma 4.5. *A topological space X is normal if and only if it is ρ-normal.*

Proof. The proof is an immediate consequence of the fact that any two open subsets of X (respectively, any two closed subsets of X) are disjoint if and only if they are ρ-disjoint. □

We are now in a position to prove the main Theorem of this Section, which is an immediate consequence of the above Lemmas:

Theorem 4.6. *Let X be a topological space such that for each $x \in X$, the equality*

$$\overline{\{x\}} = \bigcap_{V_x \in \mathcal{V}_x} V_x \tag{4.5}$$

holds. Let $\hat{X} = X/\rho$ be the quotient space, where ρ stands for Kolmogorov's Second Equivalence Relation.
Then any two of the following statements are equivalent:

(i) $\hat{X}$ is a T_1 space;

(ii) If X is regular, then $\hat{X}$ is T_3;

(iii) If X is normal, then $\hat{X}$ is T_4;

(iv) If $\overline{\{x\}} = \bigcap_{V_x \in \mathcal{V}_x} \overline{V_x}$, then $\hat{X}$ is T_2.

□

The following Remark shows that Theorem 4.6 provides a unifying framework for several otherwise isolated results in the theory of topological vector spaces and topological groups.

Remark 4.7. *(i) Condition (4.4) in Theorem* 4.6 *holds true in any topological group or topological vector space.*

This is a consequence of the well known equality:

$$\overline{\{0\}} = \bigcap_{V \in \mathcal{V}_0} V,$$

where $\mathcal{V}_0$ stands for the filter of neighborhoods of the origin (see [4], Prop. 3.2, p. 32.)
(ii) Notice that, in the notation of this section, we have $x \rho y$ if and only if $x \equiv$

$y(\text{mod}\overline{\{0\}})s$. This results from the fact that $\overline{\{0\}}$ is a normal subroup of X if X is a topological group or a closed subspace of X, if X is a topological vector space (see [B], Proposition 1, p. 226). Therefore,

$$x\rho y \Leftrightarrow \overline{\{x\}} = \overline{\{y\}} \Leftrightarrow x + \overline{\{0\}} = y + \overline{\{0\}} \Leftrightarrow x - y \in \overline{\{0\}}.$$

The previous Remark allows us to state the following:

Corollary 4.8. *If X is a topological group, the quotient space $\hat{X} = X/\overline{\{0\}}$ is T_1. If X is a topological vector space, then $\hat{X}$ is T_3.*

Proof. The proof of the first assertion follows directly from part (i) of Theorem 4.6 and the previous Remark. The second proposition is a consequence of Theorem 8 (p.42) in [5] and (iii) in our Theorem 4.6. We underline the fact that Corollary 4.8 improves the corollary to Proposition 4.5 on page 34 of [4].

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