On the Godbillon-Vey invariant and global holonomy of $\mathbb{R}\mathrm{P}^1$ - foliations

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Abstract. A transversely projective codimension one foliation $\mathcal{F}$ on a manifold M is defined by a triplet of 1-forms $(\omega, \omega_1, \omega_2)$ such that ω define $\mathcal{F}$, $d\omega = 2\omega \wedge \omega_1$, $d\omega_1 = \omega \wedge \omega_2$ and $d\omega_2 = 2\omega_2 \wedge \omega_1$. The cohomology class $gv(\mathcal{F})$ of the 3-form $\omega \wedge \omega_1 \wedge \omega_2$ is independent of the choice of the triplet $(\omega, \omega_1, \omega_2)$, it's called the Godbillon-Vey invariant of the foliation $\mathcal{F}$. Moreover, the same triplet defines a homomorphism group, $h : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{R})$ whose the conjugacy class of its image $\Gamma(\mathcal{F})$, called the global holonomy of the foliation $\mathcal{F}$, depends only on $\mathcal{F}$. In the present paper we prove that when $gv(\mathcal{F})$ is non zero then the orbits of $\Gamma(\mathcal{F})$ are dense in $\mathbb{R}\mathrm{P}^1$ and $\Gamma(\mathcal{F})$ is either discrete uniform or dense in $\mathrm{PSL}(2, \mathbb{R})$. In addition we prove the existence of transversely projective codimension one foliations with non zero Godbillon-Vey invariant and a dense global holonomy.

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1 Introduction

In [1], C.Godbillon and J.Vey introduced a characteristic class associated to codimension one foliations which lies in the third real cohomology group of the ambient manifold. Several authors studied the relation between this invariant and the structure of leaves.

In the present paper we study the relation between the Godbillon-Vey invariant and the global holonomy of a transversely projective codimension one foliations (or $\mathbb{R}\mathrm{P}^1$- foliations)(i.e. foliations which are modelled on the projective real line $\mathbb{R}\mathrm{P}^1$ with coordinate changes lying in the group $\mathrm{PSL}(2, \mathbb{R})$ of projective transformations of $\mathbb{R}\mathrm{P}^1$). See section 2 for more detailed discussion.

It is interesting to note that the first example of a foliation with non zero Godbillon-Vey invariant (due to Roussarie) is an $\mathbb{R}\mathrm{P}^1$- foliation with global holonomy a discrete uniform subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

Our main results are the following:

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Theorem 3.4. *Let $\mathcal{F}$ be a transversely projective foliation on a manifold M. If the Godbillon-Vey invariant $gv(\mathcal{F})$ is non zero, then the orbits of $\Gamma(\mathcal{F})$ in $\mathbb{R}\mathrm{P}^1$ are everywhere dense in $\mathbb{R}\mathrm{P}^1$. In particular, if the foliation is transverse to a fibration by circles, then every leaf of $\mathcal{F}$ is dense in M*

Theorem 3.5. *Let $\mathcal{F}$ be a transversely projective foliation on a manifold M. If $gv(\mathcal{F})$ is non zero, then $\Gamma(\mathcal{F})$ is discrete or everywhere dense in* $\mathrm{PSL}(2,\mathbb{R})$.

Theorem 4.2. *There exists a transversely projective foliations on a manifolds M ($dimM \geq 5$), having an invariant different from zero and a global holonomy dense in* $\mathrm{PSL}(2,\mathbb{R})$.

This paper is organized as follows: In section 1, we review the notion of transverse structures on foliations and recall some known results and definitions concerning $\mathbb{R}\mathrm{P}^1$-foliations. Section 3 is occupied by the proofs of the theorems (3.4) and (3.5) listed previously. In section 4, we expose a manner to construct $\mathbb{R}\mathrm{P}^1$-foliations with non zero GodbillonVey invariant and global holonomy dense in $\mathrm{PSL}(2,\mathbb{R})$ provided that the dimension of the ambient manifold is at least equal to five, this is Theorem 4.2.

2 Transverse structures on foliations

Recall that, a smooth codimension-k foliation on a smooth manifold M, can be defined by an open cover $\{U_\alpha\}_{\alpha\in I}$ of M and a family of maps $\{\varphi_\alpha\}_{\alpha\in I}$, where $\varphi_\alpha : U_\alpha \to \mathbb{R}^k$ is a submersion, with the following property: For each $\alpha, \beta \in I$, there exist a transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to diff(\mathbb{R}^k)$, locally constant, such that $\varphi_\alpha = g_{\alpha\beta} \circ \varphi_\beta$. When the family $\{(U_\alpha, \varphi_\alpha), \alpha \in I\}$ is maximal, we call it a transverse structure.

We can obtain finer transverse structures, if we put a special conditions on the transition functions $g_{\alpha\beta}$. Suppose that X is a manifold of dimension k, and G a group of diffeomorphisms of X, then $\mathcal{F}$ has a transverse structure (G, X) if the submersions φ_α can be defined from U_α to an open set V_α of X, and $g_{\alpha\beta}$ can be chosen belonging in G. More precisely, a transverse (G,X)-structure on $\mathcal{F}$ is a maximal atlas $\{(U_\alpha, \varphi_\alpha), \alpha \in I\}$ where $\{U_\alpha\}_{\alpha\in I}$ is an open cover of M, and $\varphi_\alpha : U_\alpha \to X$, are submersions with transition functions $g_{\alpha\beta}$ belonging in G. When G is a Lie-group and X is an homogeneous space of G, then the foliation obtained is said transversely homogeneous foliation. For $k = 1$, we have as examples, the following transverse structures:

1 - If G is the group of translations of $\mathbb{R}$, $\mathcal{F}$ is called transversely euclidian.

2 - If G is the group of the affine transformations of $\mathbb{R}$, $\mathcal{F}$ is called transversely affine.

3 - If G is the group of projective transformations $\mathrm{PSL}(2,\mathbb{R})$ of $\mathbb{R}\mathrm{P}^1$, $\mathcal{F}$ is called transversely projective or $\mathbb{R}\mathrm{P}^1$- foliation(for more details see [3]).

In terms of differential forms, a codimension-k foliation can be locally defined by a k differential 1-forms $\omega_1, \omega_2, ..., \omega_k$ linearly independent in each point, such that

$$d\omega_i = \sum \omega_{ij} \wedge \omega_j$$

for some 1-forms ω_{ij}. In particular, when $k = 1$, a transversely oriented codimension-1 foliation $\mathcal{F}$ is defined by a 1-form ω, without singularity, such that for some 1-form ω_1 we have

$$d\omega = \omega \wedge \omega_1.$$

It was shown, in this case, that the real cohomology class $[\Omega]$ of the closed form $\Omega = \omega_1 \wedge d\omega_1$ in $H^3(M, \mathbb{R})$ depends only on the given foliation. It is called the *Godbillon-Vey invariant* of the foliation and denoted by $gv(\mathcal{F})$ (for more details see [1]).

Now we are going to give some well known results.

Theorem 2.1. *Let G be a connected Lie group, $\mathfrak{g}$ the Lie algebra of G and π its Maurer-Cartan's form. Then,*

a) $d\pi + \frac{1}{2}[\pi, \pi] = 0$.

b) If $\Omega : T(M) \to \mathfrak{g}$ is a $\mathfrak{g}$- valued 1-form defined on M such that

$$d\Omega + \frac{1}{2}[\Omega, \Omega] = 0,$$

then each point $x \in M$ has a neighborhood U on which is defined a map $h : U \to G$ verifying $h^*\pi = \Omega_{|U}$. Moreover, any map $k : U \to G$ verifying $k^*\pi = \Omega_{|U}$, is of the form: $k = \gamma h$ where $\gamma \in G$. In addition if M is simply connected, then there exists a map $F : M \to G$, globally defined, such that $F^*\pi = \Omega$.

The Lie algebra $\mathcal{G}$ of the Lie group $\mathrm{PSL}(2, \mathbb{R})$ has a basis $(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$ such that $[\mathrm{X}, \mathrm{Z}] = \mathrm{Y}$, $[\mathrm{Z}, \mathrm{Y}] = 2\mathrm{Z}$, $[\mathrm{Y}, \mathrm{X}] = 2\mathrm{X}$, and in this basis every $\mathcal{G}$-valued 1-form:

$$\Omega : T(M) \to \mathcal{G}$$

can be written of the form

$$\Omega = \omega \mathrm{X} + \omega_1 \mathrm{Y} + \omega_2 \mathrm{Z}$$

where $\omega, \omega_1, \omega_2$ are real valued 1-forms called the *components* of Ω. Moreover the identity

$$d\Omega + \frac{1}{2}[\Omega, \Omega] = 0$$

is verified, if and only if, we have

$$d\omega = 2\omega \wedge \omega_1$$

$$d\omega_1 = \omega \wedge \omega_2$$

$$d\omega_2 = 2\omega_2 \wedge \omega_1.$$

These equalities are called the *Maurer-cartan's equations.*

Corollary 2.2. *A codimension one foliation is transversely projective, if and only if, it is defined by a 1-form, ω, such that, for some tow 1-forms ω_1 and ω_2, the triplet $(\omega, \omega_1, \omega_2)$ satisfies the Maurer-cartan's equations.*

We say that the foliation is defined by the triplet $(\omega, \omega_1, \omega_2)$. Clearly if $\mathcal{F}$ is an $\mathbb{R}\mathrm{P}^1$- foliation defined by a triplet $(\omega, \omega_1, \omega_2)$, then $gv(\mathcal{F}) = 4[\omega_1 \wedge \omega \wedge \omega_2]$.

Corollary 2.3. *Let $\mathcal{F}$ be an $\mathbb{R}\mathrm{P}^1$- foliation on a manifold M defined by a triplet $(\omega, \omega_1, \omega_2)$, and let $\tilde{p} : \widetilde{M} \to M$ be the universal covering of M. Then there exist a homomorphism group*

$$\Phi : Aut(\widetilde{M}) \to \mathrm{PSL}(2, \mathbb{R})$$

and a smooth function

$$F : \widetilde{M} \to \mathrm{PSL}(2, \mathbb{R})$$

such that for every $g \in Aut(\widetilde{M})$, $F \circ g = \Phi(g) F$.

The homomorphism Φ induces, via the natural isomorphism $\pi_1(M) \overset{\cong}{\to} Aut(\widetilde{M})$, a homomorphism group

$$h : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{R})$$

called the *homomorphism of global holonomy* . The subgroup

$$\Gamma(\mathcal{F}) = h(\pi_1(M))$$

of $\mathrm{PSL}(2, \mathbb{R})$ is called the *global holonomy group* of the foliation $\mathcal{F}$. Let

$$\widehat{M} = \widetilde{M}/ker(\Phi)$$

and $p : \widehat{M} \to M$ the canonical projection. Then we have the following basic result:

Proposition 2.4. *Let $\mathcal{F}$ be a transversely projective foliation on a manifold M. Then there exists a galoisian covering $p : \widehat{M} \to M$, an injective homomorphism $\varphi : Aut(\widehat{M}) \to \mathrm{PSL}(2, \mathbb{R})$ and a submersion $f : \widehat{M} \to \mathbb{R}\mathrm{P}^1$, satisfying the following properties*

a) $p^*\mathcal{F}$ is the foliation defined by the submersion f.

b) f is equivariant with respect to φ :

$$f(g(x)) = \varphi(g)(f(x))), \quad x \in \widehat{M}, \quad g \in Aut(\widehat{M}).$$

c) The holonomy group of a leaf L of $\mathcal{F}$ is isomorphic to the subgroup of the $Aut(\widehat{M})$ living $\widehat{L}$ invariant, where $\widehat{L}$ is any connected component of $p^{-1}(L)$.

It follows from Proposition 2.4 that to each leaf L of $\mathcal{F}$ correspond an orbit $\mathcal{O}(L) = f(p^{-1}(L))$ of $\Gamma(\mathcal{F})$ in $\mathbb{R}\mathrm{P}^1$ and,

Corollary 2.5. *a)If L is dense in M, then $\mathcal{O}(L)$ is dense in $f(\widehat{M})$ and, if $\mathcal{O}(L)$ is proper, then L is proper.*

b)Let $\theta \in f(\widehat{M})$, then the holonomy group of a leaf of $p(f^{-1}(\theta))$ is a subgroup of the isotropy group of θ.

If the foliation is transverse to a fibration by circles, then L is proper if and only if $\mathcal{O}(L)$ is proper and, L is dense if and only if $\mathcal{O}(L)$ is dense in $\mathbb{R}\mathrm{P}^1$.

Concerning transverse structures, an $\mathbb{R}\mathrm{P}^1$- foliation can admits several transverse projective structures and the triplet $(\omega, \omega_1, \omega_2)$ which defines $\mathcal{F}$ is not necessarily unique. On the other hand, it was shown that if M is compact, then every $\mathbb{R}\mathrm{P}^1$-foliation having holonomy has a unique transverse projective structure. In the general case we have the following

Proposition 2.6. *Let $\mathcal{F}$ be an $\mathbb{R}\mathrm{P}^1$- foliation on a manifold M. Then*

a)Any transverse projective structure on $\mathcal{F}$ is defined by a triplet $(\omega, \omega_1, \omega_2)$ of 1-forms satisfying the Maurer-cartan's equations.

b)Tow triplets $(\omega, \omega_1, \omega_2)$ and $(\varpi, \varpi_1, \varpi_2)$ define the same transverse projective structure on $\mathcal{F}$, if and only if, there exist tow functions $f, g : M \to \mathbb{R}$ with $f > 0$ and

i)$\varpi = f\omega$.

ii)$\varpi_1 = \omega_1 + g\omega + \frac{df}{2f}$.

iii)$\varpi_2 = \frac{1}{f}(g^2\omega - 2g\omega_1 + \omega_2 - dg)$.

Tow triplets $(\omega, \omega_1, \omega_2)$ and $(\varpi, \varpi_1, \varpi_2)$ satisfying the relations i), ii), iii) of Proposition 2.6 are said equivalent.

Now we are going to close this section by giving tow examples in order to illustrate Proposition 2.4.

Examples

1- *Example of Roussarie*

Let $(\alpha, \alpha_1, \alpha_2)$ be the components of the Maurer-Cartan's form of $\mathrm{PSL}(2, \mathbb{R})$ and H a discrete uniform subgroup of $\mathrm{PSL}(2, \mathbb{R})$ (see 3.1). The triplet of 1-forms $(\alpha, \alpha_1, \alpha_2)$ verifies the Maurer-Cartan's equations, it is left-invariant, (so invariant by the natural action of H on $V = \mathrm{PSL}(2, \mathbb{R})/H$) and consequently gives a triplet $(\beta, \beta_1, \beta_2)$ verifying the same equations on the quotient V and therefore defines a transversely projective foliation $\mathcal{R}$ on V. Note that the canonical projections $p : \mathrm{PSL}(2, \mathbb{R}) \to V$, and $q : \mathrm{PSL}(2, \mathbb{R}) \to \mathbb{R}\mathrm{P}^1$, are, respectively, a galoisian covering and a submersion satisfying the conditions of Proposition 2.4. The global holonomy group of $\mathcal{R}$ being, precisely, the subgroup H. On the other hand, the form $\beta \wedge \beta_1 \wedge \beta_2$ is a volume form on the compact manifold V is not be exact. This implies that $gv(\mathcal{F})$ is non zero.

2- *Transversely projective foliation transverse to a fibration by circles*

Let M be a manifold and $\varphi : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{R})$ be an homomorphism group. Consider the universal covering $\widetilde{M}$ of M. Then the group $\pi_1(M)$ acts on the product

$$\widetilde{V} = \widetilde{M} \times \mathbb{R}\mathrm{P}^1$$

by $\gamma : (x, t) \to (\gamma(x), \varphi(\gamma)(t))$. The foliation $\{\widetilde{M} \times \{t\},\ t \in \mathbb{R}\mathrm{P}^1\}$ is invariant under this action, and consequently gives, by dropping down, a transversely projective foliation $\mathcal{F}(\varphi)$ on the quotient manifold $V = \widetilde{V}/\pi_1(M)$. The galoisian covering $p : (\widetilde{M}/ker\varphi) \times \mathbb{R}\mathrm{P}^1 \to V$ and the canonical projection $p_2 : \widetilde{V} \to \mathbb{R}\mathrm{P}^1$ are satisfying the conditions of Proposition 2.4. The global holonomy of $\mathcal{F}(\varphi)$ is the group $\varphi(\pi_1(M))$.

3 The Godbillon-Vey invariant and the global holonomy

In this section we identify $\mathrm{PSL}(2, \mathbb{R})$ with $\mathrm{SL}(2, \mathbb{R})/\{\pm id\}$ and $\mathbb{R}\mathrm{P}^1$ with $\mathbb{R} \cup \{\infty\}$. Under this identification, a projective transformation of $\mathbb{R}\mathrm{P}^1$ is of the form

$$t \to \frac{at+b}{ct+d}\ , \quad ad-bc=1.$$

We denote it by

$$\epsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where ϵ means $\pm$. The projective transformations which are fixing ∞ are of the form

$$\epsilon\begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix}.$$

Their set is a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ called the affine group, and denoted by $\mathcal{A}$. The projective transformations which, globally, fixe the pair $\{0,\infty\}$ are of the form

$$\epsilon\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \text{ or } \epsilon\begin{pmatrix} 0 & b \\ \frac{1}{b} & 0 \end{pmatrix}.$$

Their set is a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. We denote it by $\mathcal{P}$. The projective transformations of the form

$$\epsilon\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

are without fixed point and called rotations. Their set is a subgroup denoted by $\mathcal{R}$. The elements

$$\epsilon\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

fixe only the point ∞ and called translations. Their set is a subgroup denoted by $\mathcal{T}$. The elements

$$\epsilon\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$$

of $\mathrm{PSL}(2,\mathbb{R})$ admit exactly tow fixed points, 0 and ∞, are called homotheties. Their set is a subgroup denoted by $\mathcal{H}$.

Remark 3.1. *A projective transformation different from the identity can not have more than tow distinct fixed points.*

Definition. A discrete subgroup H of $\mathrm{PSL}(2,\mathbb{R})$ is said uniform if the quotient $\mathrm{PSL}(2,\mathbb{R})/H$ is a compact manifold.

Lemma 3.2. *Let $\alpha, \alpha_1, \alpha_2$ be the components of the Maurer-Cartan's form of* $\mathrm{PSL}(2,\mathbb{R})$*. If H is one of the subgroups $\mathcal{A}$, $\mathcal{P}$ or a non uniform discrete subgroup of* $\mathrm{PSL}(2,\mathbb{R})$*, then there exists a 2-form ϖ on* $\mathrm{PSL}(2,\mathbb{R})$ *invariant by the group H and such that:*

$$d\varpi = \alpha \wedge \alpha_1 \wedge \alpha_2$$

Proof. Let $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm id\}$ and denote by $X = \epsilon\begin{pmatrix} x & u \\ y & v \end{pmatrix}$ its generic element. Then we have

$$\alpha = xdy - ydx, \quad \alpha_1 = xdv - ydu = udy - vdx, \quad \alpha_2 = vdu - udv.$$

On the other hand, by a simple calculation, we prove that the 2-form

$$\beta = (vdy - ydv) \wedge d\left(\frac{xy + uv}{y^2 + v^2}\right)$$

is invariant by the affine group $\mathcal{A}$ and verifies the equality

$$d\beta = \alpha \wedge \alpha_1 \wedge \alpha_2.$$

In this case we take $\varpi = \beta$. If $H = \mathcal{P}$ we consider the matrix $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the left translation

$$\tau_\sigma : \mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{R})$$

defined by $\tau_\sigma(X) = \sigma X$. Because $\tau_\sigma^2 = id$, the 2-form

$$\beta' = \frac{1}{2}(\beta + \tau_\sigma^* \beta)$$

is invariant by the subgroup $\mathcal{P}$ and verifies the equality

$$d\beta' = \alpha \wedge \alpha_1 \wedge \alpha_2.$$

In this case we take $\varpi = \beta'$.

Now Suppose that H is a discrete non uniform subgroup of $\mathrm{PSL}(2, \mathbb{R})$, and consider the canonical projection

$$q : \mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{R})/H.$$

The volume form $\alpha \wedge \alpha_1 \wedge \alpha_2$ is left-invariant on $\mathrm{PSL}(2, \mathbb{R})$. Then it drops down to the quotient giving a volume form ν on $\mathrm{PSL}(2, \mathbb{R})/H$. Because this quotient is not compact, ν is exact. Let η be a 2-form on $\mathrm{PSL}(2, \mathbb{R})/H$ such that

$$d\eta = \nu,$$

then the 2-form $\varpi = q^* \eta$ is invariant under H on $\mathrm{PSL}(2, \mathbb{R})$ and satisfies

$$d\varpi = \alpha \wedge \alpha_1 \wedge \alpha_2.$$

■

Theorem 3.3. *Let $\mathcal{F}$ be a transversely projective foliation with global holonomy group $\Gamma(\mathcal{F})$. Each of the flowing conditions implies that the GodbillonVey invariant $gv(\mathcal{F})$ of $\mathcal{F}$ is zero.*

i) $\Gamma(\mathcal{F})$ *is conjugated to a subgroup of the affine group* $\mathcal{A}$.

ii) $\Gamma(\mathcal{F})$ *is conjugated to a subgroup of the group* $\mathcal{P}$.

iii) $\Gamma(\mathcal{F})$ *is a non uniform discrete subgroup of* $\mathrm{PSL}(2, \mathbb{R})$.

iv) $\Gamma(\mathcal{F})$ *is conjugated to a subgroup of the group* $\mathcal{R}$.

Proof. If $\Gamma(\mathcal{F})$ verifies $i)$, $ii)$ or $iii)$ then there exist, by Lemma.3.2, a 2-form ϖ on $\mathrm{PSL}(2,\mathbb{R})$ invariant by $\Gamma(\mathcal{F})$ with $d\varpi = \alpha\wedge\alpha_1\wedge\alpha_2$. Let $\tilde{p}:\widetilde{M}\to M$ be the universal covering of M. Then by Corollary 2.3, there exists a map $F:\widetilde{M}\to\mathrm{PSL}(2,\mathbb{R})$, such that $F^*(\alpha,\alpha_1,\alpha_2) = \tilde{p}^*(\omega,\omega_1,\omega_2)$ and for every $g\in Aut(\widetilde{M}), F\circ g = \varphi(g)F$. It follows that, the 2-form $F^*\varpi$ is invariant by the group of $Aut(\widetilde{M})$ and $d(F^*\varpi) = \tilde{p}^*(\omega\wedge\omega_1\wedge\omega_2)$. Consequently the 2-form $F^*\varpi$ drops down to M giving a 2-form λ such that $d\lambda = \omega\wedge\omega_1\wedge\omega_2$. This means that the GodbillonVey invariant $gv(\mathcal{F})$ is zero. This proves the parts $i)$, $ii)$ and $iii)$ of the Theorem.

In the case $iv)$, the elements of Γ are without fixed point and consequently, by Corollary 2.5, $\mathcal{F}$ is without holonomy, therefore $gv(\mathcal{F})$ is zero. ■

Theorem 3.4. *Let $\mathcal{F}$ be a transversely projective foliation on a manifold M. If the GodbillonVey invariant $gv(\mathcal{F})$ is non zero, then the orbits of $\Gamma(\mathcal{F})$ in $\mathbb{R}\mathrm{P}^1$ are everywhere dense in $\mathbb{R}\mathrm{P}^1$. In particular, if the foliation is transverse to a fibration by circles, then every leaf of $\mathcal{F}$ is dense in M.*

Proof. Let $\Gamma(\mathcal{F}) = \Gamma$ be the global holonomy group of $\mathcal{F}$. Suppose at first that Γ has a proper orbit $\mathcal{O}$. Then:

a)If $\mathcal{O}$ is reduced to a single point $\{\theta_0\}$, Γ is conjugated to a subgroup of the affine group $\mathcal{A}$.

b)If $\mathcal{O}$ is reduced to a pair $\{\theta_1,\theta_2\}$, then Γ is conjugated to a subgroup of the group $\mathcal{P}$.

c) If $\mathcal{O}$ contains more than two points, then for any point $\theta_0\in\mathcal{O}$, and any $\gamma\in\Gamma-\{id\}$ we have $\gamma(\theta_0)\neq\theta_0$. Indeed, suppose that $\gamma\neq id$ and $\gamma(\theta_0)=\theta_0$. Then there exists $\theta\in\mathcal{O}$ such that $\gamma(\theta)\neq\theta$ and therefore the sequence $\gamma^n(\theta)$ (or $\gamma^{-n}(\theta)$) converges to the fixed point θ_0, this implies that $\mathcal{O}$ is not proper which is a contradiction. So $\gamma(\theta_0)$ is an other point of $\mathcal{O}$. It follows that if θ_0,θ_1 are tow consecutive points in $\mathcal{O}$ and $A =]\theta_0,\theta_1[$, then $\gamma(A)\cap A = \varnothing$ hence A dos not contain two distinct points equivalent by Γ, and so $q^{-1}(A)$ ($q:\mathrm{PSL}(2,\mathbb{R})\to\mathbb{R}\mathrm{P}^1$ is the canonical projection) is a non relatively compact open set of $\mathrm{PSL}(2,\mathbb{R})$ having no distinct points equivalent by Γ . Consequently, Γ is a non uniform discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

Now suppose that Γ has an exceptional minimal set $\mathfrak{M}$, and let $]\theta_1,\theta_2[$ be an interval(i.e: a connected component) of $\mathbb{R}\mathrm{P}^1-\mathfrak{M}$. If γ is an element of Γ which fixes θ_1, it fixes also θ_2, because otherwise one of the sequences $\gamma^n(\theta_2)$ or $\gamma^{-n}(\theta_2)$ converges to θ_1 and so the interval $]\theta_1,\theta_2[$ would contain points of the orbit of θ_2 which are points of $\mathfrak{M}$. On the other hand the isotropy group of θ_1 is cyclic, because otherwise, the orbits would be locally dense in a neighborhood of θ_1. Consequently, an element which does not fixe θ_1, sends $]\theta_1,\theta_2[$ to another connected component of $\mathbb{R}\mathrm{P}^1-\mathfrak{M}$, this implies that the orbit of any point of $]\theta_1,\theta_2[$ is proper and infinite. We conclude as previously that Γ is a non uniform discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Now the proof is an immediate consequence of Theorem 3.3. ■

Theorem 3.5. *Let $\mathcal{F}$ be a transversely projective foliation on a manifold M. If $gv(\mathcal{F})$ is non zero then $\Gamma(\mathcal{F})$ is discrete uniform or everywhere dense in* $\mathrm{PSL}(2,\mathbb{R})$.

Proof. If Γ is neither discrete nor everywhere dense in $\mathrm{PSL}(2,\mathbb{R})$, then the closure $\overline{\Gamma}$ of Γ in $\mathrm{PSL}(2,\mathbb{R})$ is a Lie-subgroup of dimension one or tow of $\mathrm{PSL}(2,\mathbb{R})$.

If $\overline{\Gamma}$ is of dimension one, the connected component Γ_0 of $\bar{\Gamma}$ containing the identity element, is a normal subgroup of $\overline{\Gamma}$, so only the three following cases are possible.
i) Γ_0 is conjugated to $\mathcal{R}$, and in this case $\overline{\Gamma} = \Gamma_0$.
ii) Γ_0 is conjugated to the group $\mathcal{T}$ of translations, also in this case $\overline{\Gamma} = \Gamma_0$.
iii) Γ_0 is conjugated to the group $\mathcal{H}$ of homotheties, and therefore Γ is conjugated to a subgroup of $\mathcal{P}$.

If $\overline{\Gamma}$ is of dimension tow, then Γ_0 is conjugated to the affine group $\mathcal{A}$. Consequently, Γ is conjugated to a subgroup of the affine group $\mathcal{A}$. Now the conclusion is an easy consequence of Theorem 3.2. ∎

4 An existence theorem

The example of Roussarie is a transversely projective foliation with a discrete uniform global holonomy and non zero GodbillonVey invariant. In this section we are going to construct a transversely projective foliation with a dense global holonomy, and a non zero GodbillonVey invariant. For this we need the next Proposition where $\mathrm{S}^1 = \mathbb{R}/\mathbb{Z}$.

Proposition 4.1. *Let $\mathcal{F}$ be a transversely projective foliation on a manifold M. Then there exists a transversely projective foliation $\mathcal{H}$ on $M \times \mathrm{S}^1$ transverse to the factor S^1 and $\mathcal{H}_{\ |M\times\{0\}} = \mathcal{F}$.*

Proof. Let $(\omega, \omega_1, \omega_2)$ be a triplet of 1-forms defining the foliation $\mathcal{F}$ on M and denote by $d\theta$ the canonical volume form of S^1. Then the triplet $(\varpi, \varpi_1, \varpi_2)$ given by

$$\varpi = (\cos^2\theta)\omega - (\sin 2\theta)\omega_1 - (\sin^2\theta)\omega_2 + d\theta$$

$$\varpi_1 = \frac{1}{2}(\sin 2\theta)\omega + \frac{1}{2}(\sin 2\theta)\omega_2 + (\cos 2\theta)\omega_1$$

$$\varpi_2 = (\cos^2\theta)\omega_2 - (\sin^2\theta)\omega - (\sin 2\theta)\omega_1 - d\theta$$

where θ is the generic element of S^1, defines a transversely projective foliation $\mathcal{H}$ on $M \times \mathrm{S}^1$ verifying the intended conditions.

Observe that $\Gamma(\mathcal{H}) = \Gamma(\mathcal{F})$ and $gv(\mathcal{H})$ is non zero if and only if $gv(\mathcal{F})$ is non zero. In addition, the projective structure induced by the foliation $\mathcal{H}$ on the factor S^1 is the canonical projective structure of S^1 defined by the triplet $(d\theta,\ 0,\ -d\theta)$. ∎

Theorem 4.2. *There exists a transversely projective foliation on a manifold M ($\dim M \geq 5$), having an invariant different from zero, and a dense global holonomy in $\mathrm{PSL}(2,\mathbb{R})$.*

Proof. By Proposition 4.1, we can easily construct two transversely projective foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ on manifolds M_0 and M_1, respectively, having the following properties

1)The invariant $gv(\mathcal{F}_0)$ is different from zero.

2)$\Gamma(\mathcal{F}_1)$ is not discrete.

3)There exists two closed transversal τ_0 and τ_1 to $\mathcal{F}_0$ and $\mathcal{F}_1$, respectively, such that the projective structures induced by $\mathcal{F}_0$ on τ_0 and by $\mathcal{F}_1$ on τ_1 are identical.

Denote by $V_\alpha,\ \alpha = 1, 2$ a tubular foliated neighborhood of τ_α diffeomorphic to $S^1 \times \mathbb{R}^{n-1}$, and by M'_α the manifold $M_\alpha - S^1 \times D^{n-1}$ with boundary $\partial M'_\alpha = S^1 \times S^{n-2}$ and by $\mathcal{F}'_\alpha$ the restriction of $\mathcal{F}_\alpha$ to M'_α. We can suppose, by changing if necessary the triplets defining $\mathcal{F}_0$ and $\mathcal{F}_1$ by equivalent triplets, that the two foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ define the same structures on a neighborhood of $\partial M'_0 = \partial M'_1 = S^1 \times S^{n-2}$. Consequently, by sticking the two foliations $\mathcal{F}'_0$ and $\mathcal{F}'_1$, we obtain a transversely projective foliation $\mathcal{F}$ on the manifold

$$M = M'_0 \bigcup_{S^1 \times S^{n-2}} M'_1.$$

Now the proof is a immediate consequence of the next lemmas. ■

Lemma 4.3. *The global holonomy* $\Gamma(\mathcal{F})$ *of* $\mathcal{F}$ *is everywhere dense in* $\mathrm{PSL}(2, \mathbb{R})$.

Proof. - We prove by the Van-Kampen's theorem that $\Gamma(\mathcal{F}'_\alpha) = \Gamma(\mathcal{F}_\alpha)$, $\alpha = 0, 1$ and $\Gamma(\mathcal{F})$ is generated by $\Gamma(\mathcal{F}'_0)$ and $\Gamma(\mathcal{F}'_1)$. Since $gv(\mathcal{F}_0) \neq 0$, then $\Gamma(\mathcal{F}_0)$ is either discrete uniform or dense in $\mathrm{PSL}(2, \mathbb{R})$. Now $\Gamma(\mathcal{F}_1)$ is not discrete, so in both cases, $\Gamma(\mathcal{F})$ is everywhere dense in $\mathrm{PSL}(2, \mathbb{R})$. ■

Lemma 4.4. *If the dimension of* M_0 *is superior or equal to 5, then* $gv(\mathcal{F}) \neq 0$.

Proof. - Let $i : M'_0 \to M_0$ be the canonical injection, then

$$i^* : H^3(M_0) \to H^3(M'_0)$$

is injective. Indeed, we have

$$M_0 = M'_0 \bigcup_{S^1 \times S^{n-2}} S^1 \times D^{n-1}$$

and an exact sequence

$$H^2(S^1 \times S^{n-2}) \to H^3(M_0) \to H^3(M'_0) \oplus H^3(S^1 \times D^{n-1}).$$

Since $n \geq 5$, we have $H^2(S^1 \times S^{n-2}) = 0$ and so i^* is injective. Consider the canonical injection

$$j : M'_0 \to M'_0 \bigcup_{S^1 \times S^{n-2}} M'_1 = M,$$

we have that

$$j^* gv(\mathcal{F}) = gv(\mathcal{F}'_0) = i^* gv(\mathcal{F}_0).$$

Since i^* is injective and $gv(\mathcal{F}_0)$ is non zero, it follows that $gv(\mathcal{F}) \neq 0$. This proves Lemma 4.4. ■

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