The Maxwell-Bloch equations on fractional Leibniz algebroids

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Abstract. Numerous Mircea Puta's papers were dedicated to the study of Maxwell - Bloch equations. The main purpose of this paper is to present several types of fractional Maxwell - Bloch equations.

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1 Introduction

It is well known that many dynamical systems turned out to be Hamilton - Poisson systems. Among other things, an important role is played by the Maxwell - Bloch equations from laser - matter dynamics. More details can be found in [1], [4] and M. Puta [5].

In this paper the revised and the dynamical systems associated to Maxwell - Bloch equations on a Leibniz algebroid are discussed.

Derivatives of fractional order have found many applications in recent studies in mechanics, physics and economics. Some classes of fractional differentiable systems have studied in [3]. In this paper we present some fractional Maxwell- Bloch equations associated to Hamilton - Poisson systems or defined on a fractional Leibniz algebroid.

2 The revised Maxwell - Bloch equations

Let $C^\infty(M)$ be the ring of smooth functions on a n - dimensional smooth manifold M. *A Leibniz bracket* on M is a bilinear map $[\cdot,\cdot] : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ such that it is a derivation on each entry, that is, for all $f, g, h \in C^\infty(M)$ the following relations hold:

$$(1) \qquad [fg, h] = [f, h]g + f[g, h] \quad \text{and} \quad [f, gh] = g[f, h] + [f, g]h.$$

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We will say that the pair $(M, [\cdot,\cdot])$ is a *Leibniz manifold.* If the bilinear map $[\cdot,\cdot]$ verify only the first equality in (1), we say that $[\cdot,\cdot]$ is a *left Leibniz bracket* and $(M, [\cdot,\cdot])$ is an *almost Leibniz manifold.*

Let P and g be two contravariant 2 - tensor fields on M. We define the map $[\cdot,(\cdot,\cdot)] : C^\infty(M) \times (C^\infty(M) \times C^\infty(M)) \to C^\infty(M)$ by:

(2) $\qquad [f,(h_1,h_2)] = P(df, dh_1) + g(df, dh_2), \quad$ for all $\quad f, h_1, h_2 \in C^\infty(M).$

We consider the map $[[\cdot,\cdot]] : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ defined by:

(3) $\qquad [[f,h]] = [f,(h,h)] = P(df, dh) + g(df, dh),$ for all $f, h \in C^\infty(M).$

It is easy to prove that $(M, P, g, [[\cdot,\cdot]])$ is a Leibniz manifold.

A Leibniz manifold $(M, P, g, [[\cdot,\cdot]])$ such that P and g is a skew - symmetric resp. symmetric tensor field is called *almost metriplectic manifold.*

Let $(M, P, g, [[\cdot,\cdot]]$ be an almost metriplectic manifold. If there exists $h_1, h_2 \in C^\infty(M)$ such that $P(df, dh_2) = 0$ and $g(df, dh_1) = 0$ for all $f \in C^\infty(M)$, then:

(4) $\qquad [[f, h_1 + h_2]] = [f,(h_1,h_2)], \quad$ for all $\; f \in C^\infty(M).$

In this case, we have:

(5) $\qquad [[f, h_1 + h_2]] = P(df, dh_1) + g(df, dh_2), \quad$ for all $\; f \in C^\infty(M).$

If $(x^i), i = \overline{1,n}$ are local coordinates on M, the differential system given by:

(6) $\qquad \dot{x}^i = [[x^i, h_1 + h_2]] = P^{ij}\frac{\partial h_1}{\partial x^j} + g^{ij}\frac{\partial h_2}{\partial x^j}, \; i, j = \overline{1,n}$

with $P^{ij} = P(dx^i, dx^j)$ and $g^{ij} = g(dx^i, dx^j)$, is called the *almost metriplectic system* on $(M, P, g, [[\cdot,\cdot]])$ associated to $h_1, h_2 \in C^\infty(M)$ which satisfies the conditions $P(df, dh_2) = 0$ and $g(df, dh_1) = 0$ for all $f \in C^\infty(M)$.

Let be a Hamilton-Poisson system on M described by the Poisson tensor $P = (P^{ij})$ and the Hamiltonian $h_1 \in C^\infty(M)$ with the Casimir $h_2 \in C^\infty(M)$ (i.e. $P^{ij}\frac{\partial h_2}{\partial x^j} = 0$ for $\; i, j = \overline{1,n}$). The differential equations of the Hamilton-Poisson system are the following:

(7) $\qquad \dot{x}^i = P^{ij}\frac{\partial h_1}{\partial x^j}, \quad i, j = \overline{1,n}.$

We determine the matrix $g = (g^{ij})$ such that $\; g^{ij}\frac{\partial h_1}{\partial x^j} = 0 \;$ where:

(8) $$g^{ii}(x) = -\sum_{k=1,\; k\neq i}^{n} \left(\frac{\partial h_1}{\partial x^k}\right)^2, \qquad g^{ij}(x) = \frac{\partial h_1}{\partial x^i}\frac{\partial h_1}{\partial x^j} \quad \text{for } \; i \neq j.$$

The *revised system* of the Hamilton - Poisson system (7) is:

(9) $\qquad \dot{x}^i = P^{ij}\frac{\partial h_1}{\partial x^j} + g^{ij}\frac{\partial h_2}{\partial x^j}, \quad i, j = \overline{1,n}.$

The real valued 3- dimensional Maxwell-Bloch equations from laser - matter dynamics are usually written as:

$$(10) \qquad \dot{x}^1(t) = x^2(t), \quad \dot{x}^2(t) = x^1(t)x^3(t), \quad \dot{x}^3(t) = -x^1(t)x^2(t), \quad t \in R.$$

The dynamics (10) is described by the Poisson tensor P_1 and the Hamiltonian $h_{1,1}$ given by:

$$(11) \qquad P_1 = (P_1^{ij}) = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & 0 \\ -x^2 & 0 & 0 \end{pmatrix}, \quad h_{1,1}(x) = \frac{1}{2}(x^1)^2 + x^3,$$

or by the Poisson tensor P_2 and the Hamiltonian $h_{2,1}$ given by:

$$(12) \qquad P_2 = (P_2^{ij}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & x^1 \\ 0 & -x^1 & 0 \end{pmatrix}, \quad h_{2,1}(x) = \frac{1}{2}(x^2)^2 + \frac{1}{2}(x^3)^2.$$

The dynamics (10) can be written in the matrix form:

$$(13) \qquad \dot{x}(t) = P_1(x(t)) \cdot \nabla h_{1,1}(x(t)), \quad \text{or} \quad \dot{x}(t) = P_2(x(t)) \cdot \nabla h_{2,1}(x(t)),$$

where $\dot{x}(t) = (\dot{x}^1(t), \dot{x}^2(t), \dot{x}^3(t))^T$ and $\nabla h(x(t))$ is the gradient of h with respect to the canonical metric on R^3.

The dynamics (10) has the Hamilton-Poisson formulation $(R^3, P_1, h_{1,1})$, with the Casimir $h_{1,2} \in C^\infty(R^3)$ given by:

$$(14) \qquad h_{1,2}(x) = \frac{1}{2}[(x^2)^2 + (x^3)^2].$$

Applying (8) for $P = P_1, h_1(x) = h_{1,1}(x)$ and $h_2(x) = h_{1,2}(x)$ we obtain the symmetric tensor g_1 which is given by the matrix:

$$g_1 = \begin{pmatrix} -1 & 0 & x^1 \\ 0 & -(x^1)^2 - 1 & 0 \\ x^1 & 0 & -(x^1)^2 \end{pmatrix}.$$

Using (9) for the Hamilton- Poisson system $(R^3, P_1, h_{1,1})$, $h_{1,2}$ and g_1 we obtain the *revised Maxwell-Bloch equations associated to* $(P_1, h_{1,1}, h_{1,2})$:

$$(15) \qquad \dot{x}^1 = x^2 + x^1x^3, \quad \dot{x}^2 = x^1x^3 - (x^1)^2x^2 - x^2, \quad \dot{x}^3 = -x^1x^2 - (x^1)^2x^3.$$

Also, the Hamilton-Poisson formulation $(R^3, P_2, h_{2,1})$ of the dynamics (10) has the Casimir $h_{2,2} \in C^\infty(R^3)$ given by:

$$(16) \qquad h_{2,2}(x) = \frac{1}{2}(x^1)^2 + x^3$$

and its associated symmetric tensor g_2 given by the matrix:

$$g_2 = \begin{pmatrix} -(x^2)^2 - (x^3)^2 & 0 & 0 \\ 0 & -(x^3)^2 & x^2x^3 \\ 0 & x^2x^3 & -(x^2)^2 \end{pmatrix}.$$

In this case, for the Hamilton- Poisson system $(R^3, P_2, h_{2,1})$, $h_{2,2}$ and g_2 we obtain the *revised Maxwell-Bloch equations associated to* $(P_2, h_{2,1}, h_{2,2})$:

$$(17) \qquad \dot{x}^1 = x^2 - x^1(x^2)^2 - x^1(x^3)^2, \quad \dot{x}^2 = x^1x^3 + x^2x^3, \quad \dot{x}^3 = -x^1x^2 - (x^2)^2.$$

3 The dynamical system associated to Maxwell - Bloch equations on a Leibniz algebroid

In this section we refer to the dynamical systems on Leibniz algebroids. For more details can be consult the paper [2].

A *Leibniz algebroid structure* on a vector bundle $\pi : E \to M$ is given by a bracket (bilinear operation) $[\cdot,\cdot]$ on the space of sections $Sec(\pi)$ and two vector bundle morphisms $\rho_1, \rho_2 : E \to TM$ (called the *left* resp. *right anchor*) such that for all $\sigma_1, \sigma_2 \in Sec(\pi)$ and $f, g \in C^\infty(M)$, we have:

$$(18) \qquad [f\sigma_1, g\sigma_2] = f\rho_1(\sigma_1)(g)\sigma_2 - g\rho_2(\sigma_2)(f)\sigma_1 + fg[\sigma_1, \sigma_2].$$

A vector bundle $\pi : E \to M$ endowed with a Leibniz algebroid structure on E , is called *Leibniz algebroid* over M and denoted by $(E, [\cdot,\cdot], \rho_1, \rho_2)$.

In the paper [2], it proved that a Leibniz algebroid structure on a vector bundle $\pi : E \to M$ is determined by a linear contravariant 2- tensor field on manifold E^* of the dual vector bundle $\pi^* : E^* \to M$. More precisely , if Λ is a linear 2 - tensor field on E^* then the bracket $[\cdot,\cdot]_\Lambda$ of functions is given by:

$$(19) \qquad [f, g]_\Lambda = \Lambda(df, dg).$$

Let $(x^i), i = \overline{1,n}$ be a local coordinate system on M and let $\{e_1, \ldots, e_m\}$ be a basis of local sections of E. We denote by $\{e^1, \ldots, e^m\}$ the dual basis of local sections of E^* and (x^i, y^a) (resp., (x^i, ξ_a)) the corresponding coordinates on E (resp., E^*). Locally, the linear 2 - tensor Λ has the form:

$$(20) \qquad \Lambda = C_{ab}^d \xi_d \frac{\partial}{\partial \xi_a} \otimes \frac{\partial}{\partial \xi_b} + \rho_{1a}^i \frac{\partial}{\partial \xi_a} \otimes \frac{\partial}{\partial x^i} - \rho_{2a}^i \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial \xi_a},$$

with $C_{ab}^d, \rho_{1a}^i, \rho_{2a}^i \in C^\infty(M)$, $i = \overline{1,n}$, $a, b, d = \overline{1,m}$.

We call a *dynamical system on Leibniz algebroid* $\pi : E \to M$, the dynamical system associated to vector field X_h with $h \in C^\infty(M)$ given by:

$$(21) \qquad X_h(f) = \Lambda(df, dh), \quad \text{for all } f \in C^\infty(M).$$

Locally, the dynamical system (21) is given by:

$$(22) \qquad \dot{\xi}_a = [\xi_a, h]_\Lambda = C_{ab}^d \xi_d \frac{\partial h}{\partial \xi_b} + \rho_{1a}^i \frac{\partial h}{\partial x^i}, \qquad \dot{x}^i = [x^i, h]_\Lambda = -\rho_{2a}^i \frac{\partial h}{\partial \xi_a}.$$

Let the vector bundle $\pi : E = R^3 \times R^3 \to R^3$ and $\pi^* : E^* = R^3 \times (R^3)^* \to R^3$ its dual. We consider on E^* the linear 2 - tensor field Λ , the anchors $\rho_1, \rho_2 : Sec(\pi) \to$

$T(R^3)$ and the function h given by:

$$(23)\qquad P = \begin{pmatrix} 0 & -\xi_3 x^3 & \xi_2 x^2 \\ \xi_3 x^3 & 0 & -\xi_1 x^1 \\ -\xi_2 x^2 & \xi_1 x^1 & 0 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & 0 \\ -x^2 & 0 & 0 \end{pmatrix}$$

$$(24)\qquad \rho_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -x^1 \\ 0 & x^1 & 0 \end{pmatrix} \quad \text{and} \quad h(x,\xi) = x^2\xi_2 + x^3\xi_3.$$

Proposition 3.1.([2]) *The dynamical system* (22) *on the Leibniz algebroid* $(R^3 \times R^3, P, \rho_1, \rho_2)$ *associated to function h, where* P, ρ_1, ρ_2, h *are given by* (23) *and* (24) *is:*

$$(25)\qquad \begin{cases} \dot{\xi}_1 &= x^3(x^2-1)\xi_2 - x^2(x^3-1)\xi_3 \\ \dot{\xi}_2 &= -x^3x^1\xi_1 \\ \dot{\xi}_3 &= x^1x^2\xi_1 \end{cases}, \quad \begin{cases} \dot{x}^1 &= x^2 \\ \dot{x}^2 &= x^1x^3 \\ \dot{x}^3 &= -x^1x^2 \end{cases}$$

The dynamical system (25) is called the *Maxwell - Bloch equations on the Leibniz algebroid* $\pi : E = R^3 \times R^3 \to R^3$.

4 The fractional Maxwell - Bloch equations

Let $f : [a,b] \to R$ and $\alpha \in R, \alpha > 0$. The *Riemann - Liouville fractional derivative at to left of a* is the function $f \to D_t^\alpha f$, where:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}(-\frac{d}{dt})^m \int_a^t (t-s)^{m-\alpha-1}(f(s)-f(a))ds,$$

with $m \in N^*$ such that $m-1 \leq \alpha \leq m, \Gamma$ is the Euler gamma function and $(\frac{d}{dt})^m = \frac{d}{dt} \circ \frac{d}{dt} \circ ... \circ \frac{d}{dt}$. Clearly, if $\alpha \to 1$ then $D_t^\alpha f(t) = \frac{df}{dt}$.

We have (see, [3]):

(i) If $f(t) = c, (\forall)t \in [a,b], \ D_t^\alpha f(t) = 0.$

(ii) If $f_1(t) = t^\gamma, (\forall)t \in [a,b]$, then $D_t^\alpha f_1(t) = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}$.

(iii) $D_t^\alpha(uf_1(t) + vf_2(t)) = uD_t^\alpha f_1(t) + vD_t^\alpha f_2(t)$, for all $u, v \in R$.

For $\alpha \in R, \alpha > 0$ and a manifold M, let $(T^\alpha(M), \pi^\alpha, M)$ the fractional tangent bundle to M (see [3]). Locally, if $x_0 \in U$ and $c : I \to M$ is a curve given by $x^i = x^i(t), (\forall)t \in I$, on $(\pi^\alpha)^{-1}(U) \in T^\alpha(M)$, the coordinates of the class $([c]_{x_0}^\alpha) \in T^\alpha(M)$ are $(x^i, y^{i(\alpha)})$, where:

$$(26)\qquad x^i = x^i(0), \qquad y^{i(\alpha)} = \frac{1}{\Gamma(1+\alpha)} D_t^\alpha x^i(t), i = \overline{1,n}.$$

Let $\mathcal{D}^\alpha(U)$ the module of 1 - forms on U. The fractional exterior derivative $d^\alpha : C^\infty(U) \to \mathcal{D}^\alpha(U), \ f \to d^\alpha(f)$ (see [3]), is given by:

$$(27)\qquad d^\alpha(f) = d(x^i)^\alpha D_{x^i}^\alpha(f), \quad \text{where}$$

$$(28)\qquad D_{x^i}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^{x^i} \frac{\partial f(x^1,...,x^{i-1},s,x^{i+1},...,x^n)}{\partial x^i} \frac{1}{(x^i-s)^\alpha} ds.$$

We denote by $\mathcal{X}^{\alpha}(U)$ the module of fractional vector fields generated by $\{D^{\alpha}_{x^i},\ i=\overline{1,n}\}$. The fractional differentiable equations associated to $\overset{\alpha}{X} \in \mathcal{X}^{\alpha}(U)$, where $\overset{\alpha}{X} = \overset{\alpha}{X}{}^{i} D^{\alpha}_{x^i}$ with $\overset{\alpha}{X}{}^{i} \in C^{\infty}(U)$ is defined by:

$$(29) \qquad D^{\alpha}_t x^i(t) = \overset{\alpha}{X}{}^{i}(x(t)),\ i=\overline{1,n}.$$

Let $\overset{\alpha}{P}$ resp. $\overset{\alpha}{g}$ be a skew-symmetric resp. symmetric fractional 2− tensor field on M. We define the bracket $[\cdot,(\cdot,\cdot)]^{\alpha} : C^{\infty}(M)\times(C^{\infty}(M)\times C^{\infty}(M)) \to C^{\infty}(M)$ by:

$$(30) \qquad [f,(h_1,h_2)]^{\alpha} = \overset{\alpha}{P}(d^{\alpha}f, d^{\alpha}h_1) + \overset{\alpha}{g}(d^{\alpha}f, d^{\alpha}h_2),\ (\forall) f, h_1, h_2 \in C^{\infty}(M).$$

The fractional vector field $\overset{\alpha}{X}_{h_1h_2}$ defined by

$$(31) \qquad \overset{\alpha}{X}_{h_1h_2} = [f,(h_1,h_2)]^{\alpha}, \quad (\forall) f \in C^{\infty}(M).$$

is called the *fractional almost Leibniz vector field.*

Locally, the *fractional almost Leibniz system associated to* $(\overset{\alpha}{P}, \overset{\alpha}{g}, h_1, h_2)$ *on* M is the differential system associated to $\overset{\alpha}{X}_{h_1h_2}$, that is:

$$(32) \qquad D^{\alpha}_t x^i(t) = \overset{\alpha}{P}{}^{ij} D^{\alpha}_{x^j} h_1 + \overset{\alpha}{g}{}^{ij} D^{\alpha}_{x^j} h_2.$$

Proposition 4.1. *The fractional almost Leibniz system associated to* $(\overset{\alpha}{P}, \overset{\alpha}{g}, \overset{\alpha}{h}_1, \overset{\alpha}{h}_2)$ *on* R^3*, where* $\overset{\alpha}{P} = P_1$, $\overset{\alpha}{g} = g_1$, $\overset{\alpha}{h}_1 = \frac{1}{2}(x^1)^{1+\alpha} + (x^3)^{\alpha}$ *and* $\overset{\alpha}{h}_2 = \frac{1}{2}(x^2)^{1+\alpha} + \frac{1}{2}(x^3)^{1+\alpha}$ *is:*

$$(33) \qquad \begin{cases} D^{\alpha}_t x^1 &=& \Gamma(1+\alpha)x^2 + \frac{1}{2}\Gamma(2+\alpha)x^1x^3 \\ D^{\alpha}_t x^2 &=& \frac{1}{2}\Gamma(2+\alpha)[x^1x^3 - (x^1)^2x^2 - x^2] \\ D^{\alpha}_t x^3 &=& \frac{1}{2}\Gamma(2+\alpha)[-x^1x^2 - (x^1)^2x^3] \end{cases}$$

Proof. The equations (32) are written in the following matrix form:

$$(34) \qquad \begin{pmatrix} D^{\alpha}_t x^1 \\ D^{\alpha}_t x^2 \\ D^{\alpha}_t x^3 \end{pmatrix} = \overset{\alpha}{P} \begin{pmatrix} D^{\alpha}_{x^1}\overset{\alpha}{h}_1 \\ D^{\alpha}_{x^2}\overset{\alpha}{h}_1 \\ D^{\alpha}_{x^3}\overset{\alpha}{h}_1 \end{pmatrix} + \overset{\alpha}{g} \begin{pmatrix} D^{\alpha}_{x^1}\overset{\alpha}{h}_2 \\ D^{\alpha}_{x^2}\overset{\alpha}{h}_2 \\ D^{\alpha}_{x^3}\overset{\alpha}{h}_2 \end{pmatrix}.$$

We have $D^{\alpha}_{x^1}\overset{\alpha}{h}_1 = \frac{1}{2}\Gamma(2+\alpha)x^1$, $D^{\alpha}_{x^2}\overset{\alpha}{h}_1 = 0$, $D^{\alpha}_{x^3}\overset{\alpha}{h}_1 = \Gamma(1+\alpha)$, $D^{\alpha}_{x^1}\overset{\alpha}{h}_2 = 0$, $D^{\alpha}_{x^2}\overset{\alpha}{h}_2 = \frac{1}{2}\Gamma(2+\alpha)x^2$, $D^{\alpha}_{x^3}\overset{\alpha}{h}_2 = \frac{1}{2}\Gamma(2+\alpha)x^3$,

With P_1 given by (11) and $g_1, \overset{\alpha}{h}_1, \overset{\alpha}{h}_2$, the system (34) becomes:

$$\begin{pmatrix} D^{\alpha}_t x^1 \\ D^{\alpha}_t x^2 \\ D^{\alpha}_t x^3 \end{pmatrix} = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & 0 \\ -x^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\Gamma(2+\alpha)x^1 \\ 0 \\ \Gamma(1+\alpha) \end{pmatrix} +$$

$$+\left(\begin{array}{ccc} -1 & 0 & x^1 \\ 0 & -(x^1)^2-1 & 0 \\ x^1 & 0 & -(x^1)^2 \end{array}\right)\left(\begin{array}{c} 0 \\ \frac{1}{2}\Gamma(2+\alpha)x^2 \\ \frac{1}{2}\Gamma(2+\alpha)x^3 \end{array}\right).$$

By direct computation we obtain the equations (33). □

Similarly we prove the following proposition.

Proposition 4.2. *The fractional almost Leibniz system associated to* $(\overset{\alpha}{P}, \overset{\alpha}{g}, \overset{\alpha}{h}_1, \overset{\alpha}{h}_2)$ *on* R^3, *where* $\overset{\alpha}{P} = P_2$, $\overset{\alpha}{g} = g_2$, $\overset{\alpha}{h}_1 = \frac{1}{2}(x^2)^{1+\alpha} + \frac{1}{2}(x^3)^{1+\alpha}$ *and* $\overset{\alpha}{h}_2 = \frac{1}{2}(x^1)^{1+\alpha} + (x^3)^\alpha$ *is:*

$$(35) \qquad \left\{ \begin{array}{lcl} D_t^\alpha x^1 & = & \frac{1}{2}\Gamma(2+\alpha)[x^2 - x^1(x^2)^2 - x^1(x^3)^2] \\ D_t^\alpha x^2 & = & \frac{1}{2}\Gamma(2+\alpha)x^1x^3 + \Gamma(1+\alpha)x^2x^3 \\ D_t^\alpha x^3 & = & -\frac{1}{2}\Gamma(2+\alpha)x^1x^2 - \Gamma(1+\alpha)(x^2)^2 \end{array} \right.$$

The differential system (33) resp.(35) is called the *revised fractional Maxwell-Bloch equations associated to Hamilton-Poisson realization* $(R^3, P_1, h_{1,1})$ resp. $(R^3, P_2, h_{2,1})$.

If in (33) resp. (35), we take $\alpha \to 1$, then one obtain the revised Maxwell-Bloch equations (15) resp. (17).

5 The fractional Maxwell - Bloch equations on a fractional Leibniz algebroid

If E is a Leibniz algebroid over M then, in the description of fractional Leibniz algebroid , the role of the tangent bundle is played by the fractional tangent bundle $T^\alpha M$ to M. For more details about this subject see [3].

A *fractional Leibniz algebroid structure* on a vector bundle $\pi : E \to M$ is given by a bracket $[\cdot,\cdot]^\alpha$ on the space of sections $Sec(\pi)$ and two vector bundle morphisms $\overset{\alpha}{\rho}_1, \overset{\alpha}{\rho}_2 : E \to T^\alpha M$ (called the *left* resp. *right fractional anchor*) such that for all $\sigma_1, \sigma_2 \in Sec(\pi)$ and $f, g \in C^\infty(M)$ we have:

$$(36) \qquad \left\{ \begin{array}{l} [e_a, e_b]^\alpha = C_{ab}^c e_c \\ [f\sigma_1, g\sigma_2]^\alpha = f\overset{\alpha}{\rho}_1(\sigma_1)(g)\sigma_2 - g\overset{\alpha}{\rho}_2(\sigma_2)(f)\sigma_1 + fg[\sigma_1,\sigma_2]^\alpha. \end{array} \right.$$

A vector bundle $\pi : E \to M$ endowed with a fractional Leibniz algebroid structure on E , is called *fractional Leibniz algebroid* over M and denoted by $(E, [\cdot,\cdot]^\alpha, \overset{\alpha}{\rho}_1, \overset{\alpha}{\rho}_2)$.

A fractional Leibniz algebroid structure on a vector bundle $\pi : E \to M$ is determined by a linear fractional 2 - tensor field $\overset{\alpha}{\Lambda}$ on the dual vector bundle $\pi^* : E^* \to M$ (see [3]). Then the bracket $[\cdot,\cdot]_{\overset{\alpha}{\Lambda}}$ is defined by:

$$(37) \qquad [f,g]_{\overset{\alpha\beta}{\Lambda}} = \overset{\alpha\beta}{\Lambda}(d^{\alpha\beta}f, d^{\alpha\beta}g), \ (\forall) \ f,g \in C^\infty(E^*),$$

where

$$(38) \qquad d^{\alpha\beta}f = d(x^i)^\alpha D_{x^i}^\alpha f + d(\xi_a)^\beta D_{\xi_a}^\beta f = d^\alpha(f) + d^\beta(f).$$

If (x^i), (x^i, y^a) resp., (x^i, ξ_a) for $i = \overline{1,n}$, $a = \overline{1,m}$ are coordinates on M, E resp. E^*, then the linear fractional tensor $\overset{\alpha\beta}{\Lambda}$ on E^* has the form:

$$\overset{\alpha\beta}{\Lambda} = C^d_{ab}\xi_d D^\beta_{\xi_a} \otimes D^\beta_{\xi_b} + \overset{\alpha i}{\rho}_{1a} D^\beta_{\xi_a} \otimes D^\alpha_{x^i} - \overset{\alpha i}{\rho}_{2a} D^\alpha_{x^i} \otimes D^\beta_{\xi_a}. \tag{39}$$

We call a *fractional dynamical system on* $(E, [\cdot,\cdot]^\alpha, \overset{\alpha}{\rho}_1, \overset{\alpha}{\rho}_2)$, the fractional system associated to vector field $\overset{\alpha\beta}{X}_h$ with $h \in C^\infty(E^*)$ given by:

$$\overset{\alpha\beta}{X}_h(f) = \overset{\alpha\beta}{\Lambda}(d^{\alpha\beta}f, d^{\alpha\beta}h), \quad \text{for all} \quad f \in C^\infty(E^*). \tag{40}$$

Locally, the dynamical system (40) reads:

$$\begin{cases} D^\alpha_t \xi_a = [\xi_a, h]_{\overset{\alpha\beta}{\Lambda}} = C^d_{ab}\xi_d D^\beta_{\xi_b} h + \overset{\alpha i}{\rho}_{1a} D^\alpha_{x^i} h \\ D^\alpha_t x^i = [x^i, h]_{\overset{\alpha\beta}{\Lambda}} = -\overset{\alpha i}{\rho}_{2a} D^\beta_{\xi_a} h \end{cases} \tag{41}$$

If $P^\beta = (C^d_{ab}\xi_d)$, $\rho_1 = (\overset{\alpha i}{\rho}_{1a})$ and $\rho_2 = (\overset{\alpha i}{\rho}_{2a})$ then the dynamical system (41) can be written in the matrix form:

$$\begin{pmatrix} D^\beta_t \xi_1 \\ D^\beta_t \xi_2 \\ D^\beta_t \xi_1 \end{pmatrix} = P^\beta \begin{pmatrix} D^\beta_{\xi_1} h \\ D^\beta_{\xi_2} h \\ D^\beta_{\xi_3} h \end{pmatrix} + \rho_1 \begin{pmatrix} D^\alpha_{x^1} h \\ D^\alpha_{x^2} h \\ D^\alpha_{x^3} h \end{pmatrix}, \quad \begin{pmatrix} D^\alpha_t x^1 \\ D^\alpha_t x^2 \\ D^\alpha_t x^3 \end{pmatrix} = -\rho_2 \begin{pmatrix} D^\beta_{\xi_1} h \\ D^\beta_{\xi_2} h \\ D^\beta_{\xi_1} h \end{pmatrix}. \tag{42}$$

Proposition 5.1. *Let the dual* $\pi^* : E^* = R^3 \times (R^3)^* \to R^3$ *of the vector bundle* $\pi : E = R^3 \times R^3 \to R^3$ *and* $\alpha > 0, \beta > 0$. *Let* $\overset{\alpha}{\Lambda}$ *defined by the matrix* P^β *and* $\overset{\alpha}{\rho}_1, \overset{\alpha}{\rho}_2, h$ *given by:*

$$P^\beta = \begin{pmatrix} 0 & -\xi_3 x^3 & \xi_2 x^2 \\ \xi_3 x^3 & 0 & -\xi_1 x^1 \\ -\xi_2 x^2 & \xi_1 x^1 & 0 \end{pmatrix}, \quad \overset{\alpha}{\rho}_1 = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & 0 \\ -x^2 & 0 & 0 \end{pmatrix},$$

$$\overset{\alpha}{\rho}_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -x^1 \\ 0 & x^1 & 0 \end{pmatrix}, \quad h(x,\xi) = (x^2)^\alpha(\xi_2)^\beta + (x^3)^\alpha(\xi_3)^\beta.$$

The fractional dynamical system (41) *on the fractional Leibniz algebroid* $(R^3 \times R^3, P, \rho_1, \rho_2)$ *associated to the function* h *is :*

$$\begin{cases} D^\beta_t \xi_1 = \Gamma(1+\beta)(-\xi_3(x^2)^\alpha x^3 + \xi_2 x^2 (x^3)^\alpha) + \\ \qquad + \Gamma(1+\alpha)(-x^3(\xi_2)^\beta + x^2(\xi_3)^\beta) \\ D^\beta_t \xi_2 = -\Gamma(1+\beta) x^1 (x^3)^\alpha \xi_1, \\ D^\beta_t \xi_3 = \Gamma(1+\beta) x^1 (x^2)^\alpha \xi_1 \\ D^\alpha_t x^1 = \Gamma(1+\beta)(x^2)^\alpha \\ D^\alpha_t x^2 = \Gamma(1+\beta) x^1 (x^3)^\alpha \\ D^\alpha_t x^3 = -\Gamma(1+\beta) x^1 (x^2)^\alpha \end{cases} \tag{43}$$

The fractional dynamical system (43) is the $(\alpha, \beta)-$ *fractional dynamical system associated to fractional Maxwell - Bloch equations.*

If $\alpha \to 1, \beta \to 1$, the fractional system (43) reduces to the Maxwell - Bloch equations (25) on the Leibniz algebroid $\pi : E = R^3 \times R^3 \to R^3$. **Conclusion.** The numerical integration of the fractional Maxwell- Bloch systems presented in this paper will be discussed in future papers.

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