Locally isometric families of minimal surfaces

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Abstract. We consider a surface M immersed in $\mathbb{R}^3$ with induced metric $g = \psi\delta_2$ where δ_2 is the two dimensional Euclidean metric. We then construct a system of partial differential equations that constrain M to lift to a minimal surface via the Weierstrauss-Enneper representation, demanding the metric is of the above form. It is concluded that the associated surfaces connecting the prescribed minimal surface and its conjugate surface satisfy the system. Moreover, we find a non-trivial symmetry of the PDE which generates a one parameter family of surfaces isometric to a specified minimal surface.

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1 Introduction

Given two C^2 functions u and v that satisfy Laplace's equation, a complex valued harmonic function f is defined via the combination: $f = u+iv$. The Jacobian of such a function is given by $J_f = u_x v_y - u_y v_x$. We will only consider harmonic functions that are univalent (injective) with positive Jacobian on $\mathbb{D} = \{z : |z| < 1\}$. On a simply connected domain $D \subset \mathbb{C}$, a harmonic mapping f has a canonical decomposition $f = h + \overline{g}$ where h and g are analytic in D, which is unique up to a constant. The dilatation ω of a harmonic map f is defined by $\omega \equiv g'/h'$. The following theorem provides the link between harmonic univalent functions and minimal surfaces:

Theorem 1. *(Weierstrass-Enneper Representation). Every regular minimal surface locally has an isothermal parametric representation of the form*

$$\left(\mathrm{Re}\left\{\int^z p(1+q^2)dw\right\}, \mathrm{Im}\left\{\int^z p(1-q^2)dw\right\}, 2\mathrm{Im}\left\{\int^z pq\,dw\right\}\right), \tag{1.1}$$

in some domain $D \subset \mathbb{C}$, where p is analytic and q is meromorphic in D, with p vanishing only at the poles (if any) of q and having a zero of precise order $2m$ wherever q has a pole of order m. Conversely, each such pair of functions p and q analytic and meromorphic, respectively, in a simply connected domain D generate through the formulas (1.1) an isothermal parametric representation of a regular minimal surface.

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We will use (1.1) in the following form:

Corollary 1. *For a harmonic function $f = h + \overline{g}$, define the analytic functions h and g by $h = \int^z p d\zeta$ and $g = -\int^z pq^2 d\zeta$. Then the minimal surface representation (1.1) becomes*

$$\left(\mathrm{Re}\{h+g\}, \mathrm{Im}\{h-g\}, 2\mathrm{Im}\left\{\int^z \sqrt{h'g'}d\zeta\right\}\right). \tag{1.2}$$

See [6], [3] for a further introduction to harmonic mappings. For an introduction to minimal surfaces see [5].

2 The isometric condition

Let $\mathbf{x}(u,v)$ be a parametrization for a surface M immersed in $\mathbb{R}^3$. Set $z = u + iv$ and define $\phi = \partial\mathbf{x}/\partial z$. Let E, F, and G be the coefficients of the metric induced in $\mathbb{R}^3$ by $\mathbf{x}(u,v)$. We then have the relations

$$\phi^2 = \frac{1}{4}(E - G - 2iF), \tag{2.1}$$

$$\overline{\phi}^2 = \frac{1}{4}(E - G + 2iF), \tag{2.2}$$

$$|\phi|^2 = \frac{1}{4}(E + G), \tag{2.3}$$

where ϕ^2 is notation for $\phi \cdot \phi$. Inverting this system we find

$$E = \overline{\phi}^2 + \phi^2 + 2|\phi|^2, \tag{2.4}$$

$$F = i(\phi^2 - \overline{\phi}^2), \tag{2.5}$$

$$G = -\overline{\phi}^2 - \phi^2 + 2|\phi|^2. \tag{2.6}$$

Since the Weierstrass-Enneper representation theorem requires that M has an isothermal parametrization, we require $E = G$ and $F = 0$, which implies

$$\phi^2 = 0, \qquad \overline{\phi}^2 = 0, \qquad E = 2|\phi|^2. \tag{2.7}$$

The first two equations are identically satisfied. Expanding the constraint on E, and using the identity

$$|\phi|^2 = \frac{1}{4}|p|^2\left((1+q^2)(1+\overline{q}^2) + (1-q^2)(1-\overline{q}^2) + 4q\overline{q}\right), \tag{2.8}$$

we find $E = |h'|^2 + |g'|^2$. Defining $\mathrm{Re}\{h\} = {}_1h$, $\mathrm{Im}\{h\} = {}_2h$, $\mathrm{Re}\{g\} = {}_1g$, and $\mathrm{Im}\{g\} = {}_2g$, we have the Cauchy Riemann and isometric conditions:

$${}_1h_u - {}_2h_v = 0 \qquad {}_1h_v + {}_2h_u = 0 \tag{2.9}$$

$${}_1g_u - {}_2g_v = 0 \qquad {}_1g_v + {}_2g_u = 0 \tag{2.10}$$

$${}_1h_u^2 + {}_2h_u^2 + {}_1g_u^2 + {}_2g_u^2 - 2\sqrt{({}_1h_u^2 + {}_2h_u^2)({}_1g_u^2 + {}_2g_u^2)} = 0 \tag{2.11}$$

3 Symmetry analysis

We now proceed to calculate the symmetry group of (2.9)-(2.11). For an introduction to symmetry methods see [2], [7], and [8]. The infinitesimal generator of the symmetry group of the system is given by

$$\mathbf{v} = c^u\partial_u + c^v\partial_v + c^{1h}\partial_{1h} + c^{2h}\partial_{2h} + c^{1g}\partial_{1g} + c^{2g}\partial_{2g} + c^{\psi}\partial_{\psi}. \tag{3.1}$$

Since the system is first order, we need only consider the first prolongation

$$\begin{aligned}\mathrm{pr}^{(1)}\mathbf{v} = \mathbf{v} &+ {}_1h^u\partial_{1h_u} + {}_1h^v\partial_{1h_v} + {}_2h^u\partial_{2h_u} + {}_2h^v\partial_{2h_v}\\ &+ {}_1g^u\partial_{1g_u} + {}_1g^v\partial_{1g_v} + {}_2g^u\partial_{2g_u} + {}_2g^v\partial_{2g_v},\end{aligned} \tag{3.2}$$

where the c^i are functions of $u, v, \psi, {}_1h, {}_2h, {}_1g$, and ${}_2g$. Applying the first prolongation to the three PDE determines a system equations for the c^i. Integrating this system yields ten symmetries of (2.9)-(2.11).

$$\mathbf{v}_1 = \partial_u, \quad \mathbf{v}_2 = \partial_v, \quad \mathbf{v}_3 = \partial_{1h}, \quad \mathbf{v}_4 = \partial_{2h}, \quad \mathbf{v}_5 = \partial_{1g}, \quad \mathbf{v}_6 = \partial_{2g},$$
$$\mathbf{v}_7 = -v\partial_u + u\partial_v, \quad \mathbf{v}_8 = -{}_2h\partial_{1h} + {}_1h\partial_{2h}, \quad \mathbf{v}_9 = -{}_2g\partial_{1g} + {}_1g\partial_{2g},$$
$$\mathbf{v}_{10} = u\partial_u + v\partial_v + {}_1h\partial_{1h} + {}_2h\partial_{2h} + {}_1g\partial_{1g} + {}_2g\partial_{2g}.$$

Exponentiating these infinitesimal vector fields gives the symmetry transformations:

$$h \to h(z-s) \quad g \to g(z-s), \tag{3.3}$$

$$h \to h(z-is) \quad g \to g(z-is), \tag{3.4}$$

$$h \to h+s \quad g \to g, \tag{3.5}$$

$$h \to h+is \quad g \to g, \tag{3.6}$$

$$h \to h \quad g \to g+s, \tag{3.7}$$

$$h \to h \quad g \to g+is, \tag{3.8}$$

$$h \to h(e^{is}z) \quad g \to g(e^{is}z), \tag{3.9}$$

$$h \to e^{is}h \quad g \to g, \tag{3.10}$$

$$h \to h \quad g \to e^{is}g, \tag{3.11}$$

$$h \to e^s h(e^{-s}z) \quad g \to e^s g(e^{-s}z). \tag{3.12}$$

In [1] the minimal symmetry group for the real minimal surface equation

$$u_{xx}(1+u_y^2) + u_{yy}(1+u_x^2) - 2u_xu_yu_{xy} = 0, \tag{3.13}$$

was calculated. Many of the translational symmetries and an $e^s f(e^{-s}x, e^{-s}y)$ symmetry were found. We note that the analogue of $\mathbf{v}^{10}$ in [1] is similar but different, since is constrains a Weierstrauss-Enneper representation of a surface and not a graph.

4 Discussion and applications

Consider the transformation $h \to e^{i\theta}h$, $g \to e^{i\theta}g$ which preserves the metric $E = |h'|^2 + |g'|^2$. When $\theta = 0$, this is simply a minimal surface specified by defining ψ. When $\theta = \pi/2$ we get the conjugate surface. Thus all intermediate surfaces, called associated surfaces, are isometric. Since all minimal surfaces can be constructed from parts of a helicoid and catenoid [4], the following examples are of interest. First, we draw attention to the catenoid, given by $\psi = \cosh(v)^2$. It's conjugate surface is the helicoid and the associated surfaces between the two are plotted over $\mathbb{D}$ in Figure 1. Since all of the associated surfaces are isometric, geometrically they are equivalent. However, note the catenoid has topology $S^1 \times \mathbb{R}$ where as the helicoid has topology $\mathbb{R}^2$.

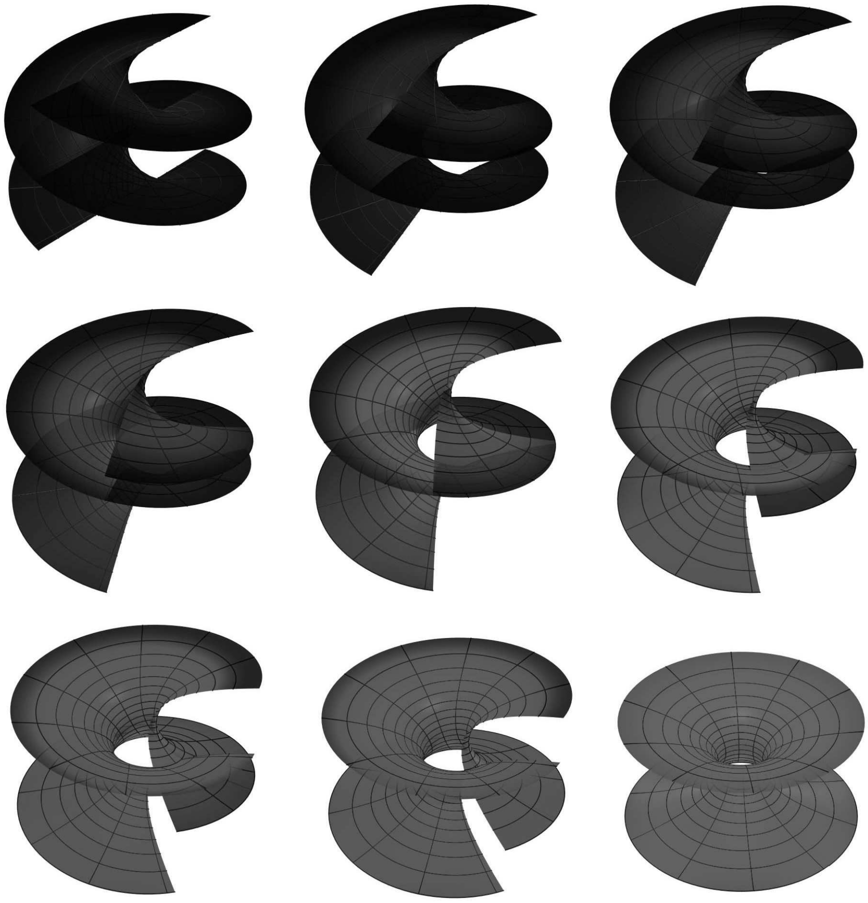

Figure 1: Helicoid to Catenoid Transformation.

We now turn our attention to the other symmetries found in the analysis for the half catenoid. We will see that the symmetries generate surfaces that are topologically distinct from the catenoid, but geometrically identical as in the above example. Let

f be the harmonic mapping $f = h + \overline{g}$ where

$$h = \frac{1}{2}\left(\frac{1}{2}\log\left[\frac{1+z}{1-z}\right] + \frac{z}{1-z^2}\right), \tag{4.1}$$

and

$$g = \frac{1}{2}\left(\frac{1}{2}\log\left[\frac{1+z}{1-z}\right] - \frac{z}{1-z^2}\right), \tag{4.2}$$

which lifts to the catenoid. We make the transformation in equation (3.12) by letting $h \to e^s h(e^{-s}z)$ and $g \to e^s g(e^{-s}z)$. Figure 2 gives several plots of this transformation for various s values. The topology of the half catenoid is $\mathbb{R}^2$ for all s up to some value between $(0.3, 0.4)$ where it changes to a punctured cylinder. Note as $s \to \infty$ that the minimal surfaces eventually degenerate to a line, in a manner peculiarly similar to neckpinch singularities of the Ricci Flow.

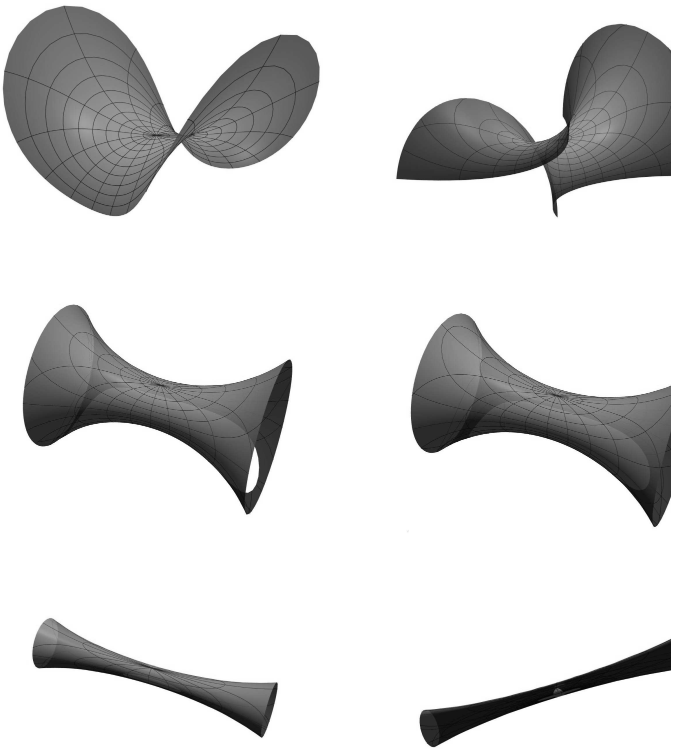

Figure 2: Symmetry (3.12) for $s = \{-1.2, -0.5, 0, 0.3, 0.4, 0.5, 1, 1.5, 3\}$.

When (3.12) is applied to the helicoid, we find that the number of rotations of the helicoid about its axis are scaled. Thus we have:

Theorem 2. *Let S be the helicoid over $\mathbb{D}$ parameterized isothermally by* $\mathbf{x} = (\sinh u \sin v, \sinh u \cos v, -v)$. *For helicoids S_1 given by $u \in (0, 2\pi)$, $v \in (v_0, v_1)$, and S_2 by $u \in (0, 2\pi)$, $v \in (v_2, v_3)$ where $v_i \in \mathbb{R}$ then S_1 and S_2 are locally isometric.*

It would be interesting to generalize the symmetry methods of this paper to higher dimensional Riemannian or Lorentizian manifolds. One would need a generalized Weierstrauss-Enneper. Moreover, we believe there are potential topological theorems coming from symmetry (3.12). For instance, if one calculates the one parameter family of minimal surfaces given by symmetry (3.12) and a simply connected minimal surface, does the topology always change from the plane to $S^1 \times \mathbb{R}^1$ or some variant thereof?

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