A geometric construction of (para-)pluriharmonic maps into GL(2r)/Sp(2r)

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Abstract. In this work we use symplectic (para-)tt^*-bundles to obtain a geometric construction of (para-)pluriharmonic maps into the pseudo-Riemannian symmetric space $GL(2r,\mathbb{R})/Sp(\mathbb{R}^{2r})$. We prove, that these (para-)pluriharmonic maps are exactly the admissible (para-)pluriharmonic maps. Moreover, we construct symplectic (para-)tt^*-bundles from (para-)harmonic bundles and analyse the related (para-)pluriharmonic maps.

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1 Introduction

The first motivation of this work is the study of metric (para-)tt^*-bundles (E, D, S, g) over a (para-)complex manifold (M, J^ϵ) and their relation to admissible (para-)pluriharmonic maps from M into the space of (pseudo-)metrics. Roughly speaking there exists a correspondence between these objects. For metric tt^*-bundles (with positive definite metric) on the tangent bundle of a complex manifold this result was shown by Dubrovin [8]. In [17, 19] we generalised it to the case of metric tt^*-bundles on abstract vector bundles with metrics of arbitrary signature and to para-complex geometry. Solutions of (metric) (para-)tt^*-bundles are for example given by special (para-)complex and special (para-)Kähler manifolds (cf. [3, 19]) and by (para-)harmonic bundles [18, 22]. The related (para-)pluriharmonic maps are described in the given references. The analysis [20, 21] of tt^*-bundles $(E = TM, D, S)$ on the tangent bundle of an almost (para-)complex manifold (M, J^ϵ) shows that there exists a second interesting class of (para-)tt^*-bundles $(E = TM, D, S, \omega)$, carrying symplectic forms ω instead of metrics g. These will be called *symplectic (para-)tt^*-bundles*. Examples are given by Levi-Civita flat nearly (para-)Kähler manifolds (Here non-integrable (para-)complex structures appear.) and by (para-)harmonic bundles which are discussed later in this work. A constructive classification of Levi-Civita flat nearly (para-)Kähler manifolds

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is subject of [4, 5].
In the context of the above mentioned correspondence it arises the question if one can use these techniques to construct (para-)pluriharmonic maps out of symplectic (para-)tt^*-bundles and if one can characterise the obtained (para-)pluriharmonic maps. In this paper we answer positively to this question: We associate an admissible (cf. definition 4) (para-)pluriharmonic map from M into $GL(2r,\mathbb{R})/Sp(\mathbb{R}^{2r})$ to a symplectic (para-)tt^*-bundle and show that an admissible (para-)pluriharmonic map induces a symplectic (para-)tt^*-bundle on $E = M\times\mathbb{R}^{2r}$. This is the analogue of the correspondence discussed in the first paragraph. In other words we characterise in a geometric fashion the class of admissible (para-)pluriharmonic maps into $GL(2r,\mathbb{R})/Sp(\mathbb{R}^{2r})$. In the sequel we construct symplectic (para-)tt^*-bundles from (para-)harmonic bundles and analyse the relation between the (para-)pluriharmonic maps which are obtained from these symplectic (para-)tt^*-bundles and the (para-)pluriharmonic maps which were found in [18, 22]. We restrict to simply connected manifolds M, since the case of general fundamental group can be obtained like in [17, 19]. In the general case all (para-)pluriharmonic maps have to be replaced by twisted (para-)pluriharmonic maps.

2 Para-complex differential geometry

We shortly recall some notions and facts of para-complex differential geometry. For a more complete source we refer to [2].
In para-complex geometry one replaces the complex structure J with $J^2 = -\mathbb{1}$ (on a finite dimensional vector space V) by the para-complex structure $\tau \in End(V)$ satisfying $\tau^2 = \mathbb{1}$ and one requires that the ± 1-eigenspaces have the same dimension. An almost para-complex structure on a smooth manifold M is an endomorphism-field τ, which is a point-wise para-complex structure. If the eigen-distributions $T^{\pm}M$ are integrable τ is called para-complex structure on M and M is called a para-complex manifold. As in the complex case, there exists a tensor, also called Nijenhuis tensor, which is the obstruction to the integrability of the para-complex structure.
The real algebra, which is generated by 1 and by the para-complex unit e with $e^2 = 1$, is called the para-complex numbers and denoted by C. For all $z = x + ey \in C$ with $x, y \in \mathbb{R}$ we define the para-complex conjugation as $\bar{\cdot} : C \to C, x + ey \mapsto x - ey$ and the real and imaginary parts of z by $\mathcal{R}(z) := x,\ \Im(z) := y$. The free C-module C^n is a para-complex vector space where its para-complex structure is just the multiplication with e and the para-complex conjugation of C extends to $\bar{\cdot} : C^n \to C^n$, $v \mapsto \bar{v}$.
Note, that $z\bar{z} = x^2 - y^2$. Therefore the algebra C is sometimes called the hypercomplex numbers. The circle $\mathbb{S}^1 = \{z = x + iy \in \mathbb{C} \,|\, x^2 + y^2 = 1\}$ is replaced by the four hyperbolas $\{z = x + ey \in C \,|\, x^2 - y^2 = \pm 1\}$. We define $\tilde{\mathbb{S}}^1$ to be the hyperbola given by the one parameter group $\{z(\theta) = \cosh(\theta) + e\sinh(\theta) \,|\, \theta \in \mathbb{R}\}$.
A para-complex vector space (V, τ) endowed with a pseudo-Euclidean metric g is called para-hermitian vector space, if g is τ-anti-invariant, i.e. $\tau^* g = -g$. The para-unitary group of V is defined as the group of automorphisms

$$U^{\pi}(V) := \mathrm{Aut}(V, \tau, g) := \{L \in GL(V) | [L, \tau] = 0 \text{ and } L^* g = g\}$$

and its Lie-algebra is denoted by $\mathfrak{u}^{\pi}(V)$. For $C^n = \mathbb{R}^n \oplus e\mathbb{R}^n$ the standard para-hermitian structure is defined by the above para-complex structure and the metric

$g = \operatorname{diag}(\mathbb{1}, -\mathbb{1})$ (cf. Example 7 of [2]). The corresponding para-unitary group is given by (cf. Proposition 4 of [2]):

$$U^\pi(C^n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A, B \in \operatorname{End}(\mathbb{R}^n), A^T A - B^T B = \mathbb{1}_n,\ A^T B - B^T A = 0 \right\}.$$

There exist two bi-gradings on the exterior algebra: The one is induced by the splitting in $T^\pm M$ and denoted by $\Lambda^k T^* M = \bigoplus_{k=p+q} \Lambda^{p+,q-} T^* M$ and induces an obvious bi-grading on exterior forms with values in a vector bundle E. The second is induced by the decomposition of the para-complexified tangent bundle $TM^C = TM \otimes_{\mathbb{R}} C$ into the subbundles $T_p^{1,0} M$ and $T_p^{0,1} M$ which are defined as the $\pm e$-eigenbundles of the para-complex linear extension of τ. This induces a bi-grading on the C-valued exterior forms noted $\Lambda^k T^* M^C = \bigoplus_{k=p+q} \Lambda^{p,q}\, T^* M$ and finally on the C-valued differential forms on M $\Omega^k_C(M) = \bigoplus_{k=p+q} \Omega^{p,q}(M)$. In the case $(1,1)$ and $(1+,1-)$ the two gradings induced by τ coincide, in the sense that $\Lambda^{1,1}\, T^* M = (\Lambda^{1+,1-}\, T^* M) \otimes C$. The bundles $\Lambda^{p,q}\, T^* M$ are para-complex vector bundles in the following sense: A para-complex vector bundle of rank r over a para-complex manifold (M, τ) is a smooth real vector bundle $\pi : E \to M$ of rank $2r$ endowed with a fiber-wise para-complex structure $\tau^E \in \Gamma(\operatorname{End}(E))$. We denote it by (E, τ^E). In the following text we always identify the fibers of a para-complex vector bundle E of rank r with the free C-module C^r. One has a notion of para-holomorphic vector bundles, too. These were extensively studied in a common work with M.-A. Lawn-Paillusseau [14].

Let us transfer some notions of hermitian linear algebra (cf. [22]) : A para-hermitian sesquilinear scalar product is a non-degenerate sesquilinear form $h : C^r \times C^r \to C$, i.e. it satisfies (i) h is non-degenerate: Given $w \in C^r$ such that for all $v \in C^r$ $h(v, w) = 0$, then it follows $w = 0$, (ii) $h(v, w) = \overline{h(w, v)}$, $\forall\ v, w \in C^r$, and (iii) $h(\lambda v, w) = \lambda h(v, w)$, $\forall\ \lambda \in C$; $v, w \in C^r$. The standard para-hermitian sesquilinear scalar product is given by

$$(z, w)_{C^r} := z \cdot \bar{w} = \sum_{i=1}^{r} z^i \bar{w}^i, \text{ for } z = (z^1, \ldots, z^r), w = (w^1, \ldots, w^r) \in C^r.$$

The para-hermitian conjugation is defined by $C \mapsto C^h = \bar{C}^t$ for $C \in End(C^r) = End_C(C^r)$ and C is called para-hermitian if and only if $C^h = C$. We denote by $\operatorname{herm}(C^r)$ the set of para-hermitian endomorphisms and by $\operatorname{Herm}(C^r) = \operatorname{herm}(C^r) \cap GL(r, C)$. We remark, that there is **no** notion of para-hermitian signature, since from $h(v, v) = -1$ for an element $v \in C^r$ we obtain $h(ev, ev) = 1$.

Proposition 1. *Given an element C of $End(C^r)$ then it holds $(Cz, w)_{C^r} = (z, C^h w)_{C^r}$, $\forall z, w \in C^r$. The set herm(C^r) is a real vector space. There is a bijective correspondence between Herm(C^r) and para-hermitian sesquilinear scalar products h on C^r given by $H \mapsto h(\cdot, \cdot) := (H\cdot, \cdot)_{C^r}$.*

A para-hermitian metric h on a para-complex vector-bundle E over a para-complex manifold (M, τ) is a smooth fiber-wise para-hermitian sesquilinear scalar product.

To unify the complex and the para-complex case we introduce some notations: First we note J^ϵ where ${J^\epsilon}^2 = \epsilon \mathbb{1}$ with $\epsilon \in \{\pm 1\}$. The ϵcomplex unit is denoted by $\hat{i}$, i.e. $\hat{i} := e$, for $\epsilon = 1$, and $\hat{i} = i$, for $\epsilon = -1$. Further we introduce $\mathbb{C}_\epsilon$ with $\mathbb{C}_1 = C$ and $\mathbb{C}_{-1} = \mathbb{C}$ and $\mathbb{S}^1_\epsilon$ with $\mathbb{S}^1_1 = \tilde{\mathbb{S}}^1$ and $\mathbb{S}^1_{-1} = \mathbb{S}^1$. In the rest of this work we extend our language by the following ϵ-notation: If a word has a prefix ϵ with $\epsilon \in \{\pm 1\}$, i.e. is of the form ϵX, this expression is replaced by

$$\epsilon X := \begin{cases} \text{X, for } \epsilon = -1, \\ \text{para-X, for } \epsilon = 1. \end{cases}$$

The ϵunitary group and its Lie-algebra are

$$U^\epsilon(p,q) := \begin{cases} U^\pi(C^r), \text{ for } \epsilon = 1, \\ U(p,q), \text{ for } \epsilon = -1 \end{cases} \quad \text{and } \mathfrak{u}^\epsilon(p,q) := \begin{cases} \mathfrak{u}^\pi(C^r), \text{ for } \epsilon = 1, \\ \mathfrak{u}(p,q), \text{ for } \epsilon = -1, \end{cases}$$

where in the complex case (p,q) for $r = p+q$ is the hermitian signature.
Further we use the notation

$$\mathrm{Herm}^\epsilon_{p,q}(\mathbb{C}^r_\epsilon) := \begin{cases} \mathrm{Herm}(C^r); \epsilon = 1, \\ \mathrm{Herm}_{p,q}(\mathbb{C}^r); \epsilon = -1, \end{cases} \quad \mathrm{herm}^\epsilon_{p,q}(\mathbb{C}^r_\epsilon) := \begin{cases} \mathrm{herm}(C^r); \epsilon = 1, \\ \mathrm{herm}_{p,q}(\mathbb{C}^r); \epsilon = -1, \end{cases}$$

where, for $p+q = r$, $\mathrm{Herm}_{p,q}(\mathbb{C}^r)$ are the hermitian matrices of hermitian signature (p,q) and $\mathrm{herm}_{p,q}(\mathbb{C}^r)$ are the hermitian matrices with respect to the standard hermitian product of hermitian signature (p,q) on $\mathbb{C}^r$. The standard ϵhermitian sesquilinear scalar product is $(z,w)_{\mathbb{C}^r_\epsilon} := z \cdot \bar{w} = \sum_{i=1}^r z^i \bar{w}^i$, for $z = (z^1, \ldots, z^r), w = (w^1, \ldots, w^r) \in \mathbb{C}^r_\epsilon$ and we note

$$\cos_\epsilon(x) := \begin{cases} \cos(x), \text{ for } \epsilon = -1, \\ \cosh(x), \text{ for } \epsilon = 1 \end{cases} \quad \text{and } \sin_\epsilon(x) := \begin{cases} \sin(x), \text{ for } \epsilon = -1, \\ \sinh(x), \text{ for } \epsilon = 1. \end{cases}$$

3 tt^*-bundles

For the convenience of the reader we recall the definition of an ϵtt^*-bundle given in [3, 17, 19] and the notion of a symplectic ϵtt^*-bundle [20, 21]:

Definition 1. An ϵtt^*-bundle (E, D, S) over an ϵcomplex manifold (M, J^ϵ) is a real vector bundle $E \to M$ endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \mathrm{End}\, E)$ satisfying the ϵtt^*-equation

$$R^\theta = 0 \quad \text{for all} \quad \theta \in \mathbb{R}\,, \tag{3.1}$$

where R^θ is the curvature tensor of the connection D^θ defined by

$$D^\theta_X := D_X + \cos_\epsilon(\theta) S_X + \sin_\epsilon(\theta) S_{J^\epsilon X} \quad \text{for all} \quad X \in TM\,. \tag{3.2}$$

A symplectic ϵtt^*-bundle (E, D, S, ω) is an ϵtt^*-bundle (E, D, S) endowed with the structure of a symplectic vector bundle[1] (E, ω), such that ω is D-parallel and S is ω-symmetric, i.e. for all $p \in M$

$$\omega(S_X \cdot, \cdot) = \omega(\cdot, S_X \cdot) \quad \text{for all} \quad X \in T_pM\,. \tag{3.3}$$

[1] see D. Mc Duff and D. Salamon [15]

Remark 1.
1) It is obvious that every ϵtt^*-bundle (E, D, S) induces a family of ϵtt^*-bundles (E, D, S^θ), for $\theta \in \mathbb{R}$, with

$$S^\theta := D^\theta - D = \cos_\epsilon(\theta)S + \sin_\epsilon(\theta)S_{J^\epsilon}\,. \tag{3.4}$$

The same remark applies to symplectic ϵtt^*-bundles.
2) Notice that a symplectic ϵtt^*-bundle (E, D, S, ω) of rank $2r$ carries a D-parallel volume given by $\underbrace{\omega \wedge \ldots \wedge \omega}_{\text{r times}}$.

The next proposition gives explicit equations for D and S, such that (E, D, S) is an ϵtt^*-bundle.

Proposition 2. *(cf. [17, 19]) Let E be a real vector bundle over an ϵcomplex manifold (M, J^ϵ) endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \operatorname{End} E)$. Then (E, D, S) is an ϵtt^*-bundle if and only if D and S satisfy the following equations:*

$$R^D + S \wedge S = 0, \quad S \wedge S \textit{ is of type (1,1)}, \quad d^D S = 0 \quad \textit{and} \quad d^D S_{J^\epsilon} = 0. \tag{3.5}$$

4 Pluriharmonic maps into $GL(2r, \mathbb{R})/Sp(\mathbb{R}^{2r})$

In this section we present the notion of ϵpluriharmonic maps and some properties of ϵpluriharmonic maps into the target space $S = S(2r) := GL(2r, \mathbb{R})/Sp(\mathbb{R}^{2r})$.

The following notion was introduced in [1] for holomorphic and in [14] for para-holomorphic vector bundles.

Definition 2. Let (M, J^ϵ) be an ϵcomplex manifold. A connection D on TM is called adapted if it satisfies

$$D_{J^\epsilon Y} X = J^\epsilon D_Y X \tag{4.1}$$

for all vector fields which satisfy $\mathcal{L}_X J^\epsilon = 0$ (i.e. for which $X + \epsilon \hat{i} J^\epsilon X$ is ϵholomorphic).

Definition 3. Let (M, J^ϵ) be an ϵcomplex manifold and (N, h) a pseudo-Riemannian manifold with Levi-Civita connection ∇^h, D an adapted connection on M and ∇ the connection on $T^*M \otimes f^*TN$ which is induced by D and ∇^h and consider $\alpha = \nabla df \in \Gamma(T^*M \otimes T^*M \otimes f^*TN)$. Then f is ϵpluriharmonic if and only if α is of type $(1,1)$, i.e.

$$\alpha(X, Y) - \epsilon\alpha(J^\epsilon X, J^\epsilon Y) = 0$$

for all $X, Y \in TM$.

Remark 2.

1. Note, that an equivalent definition of ϵpluriharmonicity is to say, that f is ϵpluriharmonic if and only if f restricted to every ϵcomplex curve is harmonic. For a short discussion the reader is referred to [3, 17, 19].
2. One knows, that every ϵcomplex manifold (M, J^ϵ) can be endowed with a torsion-free ϵcomplex connection D (cf. [12] in the complex and [19] Theorem 1 for the para-complex case), i.e. D is torsion-free and satisfies $DJ^\epsilon = 0$. Such a connection is adapted. In the rest of the paper, we assume, that the connection D on (M, J^ϵ) is also torsion-free.

The harmonic analogue of the following proposition is well-known.

Proposition 3. *Let (M, J^ϵ) be an ϵcomplex manifold, X, Y be pseudo-Riemannian manifolds and $\Psi : X \to Y$ a totally geodesic immersion. Then a map $f : M \to X$ is ϵpluriharmonic if and only if $\Psi \circ f : M \to Y$ is ϵpluriharmonic.*

The ϵpluriharmonic maps obtained by our construction are exactly the admissible ϵpluriharmonic maps in the sense of the following general definition:

Definition 4. Let (M, J^ϵ) be an ϵcomplex manifold and G/K be a locally symmetric space with associated Cartan-decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. A map $f : (M, J^\epsilon) \to G/K$ is called admissible if the ϵcomplex linear extension of its differential df maps $T^{1,0}M$ to an Abelian subspace of $\mathfrak{p}^{\mathbb{C}_\epsilon}$.

Let ω_0 be the standard symplectic form of $\mathbb{R}^{2r}$, i.e. $\omega_0 = \sum_{i=1}^r e_i \wedge e_{i+r}$ where $(e_i)_{i=1}^{2r}$ is the dual of the standard basis of $\mathbb{R}^{2r}$. Then we define

$$Sym(\omega_0) := \{A \in GL(2r, \mathbb{R}) \,|\, \omega_0(A\cdot, \cdot) = \omega_0(\cdot, A\cdot)\}. \tag{4.2}$$

The adjoint of $g \in GL(2r, \mathbb{R})$ with respect to ω_0 will be denoted by $g^\dagger$. Hence $Sym(\omega_0)$ are the elements $A \in GL(2r, \mathbb{R})$ which satisfy $A^\dagger = A$.
Every element $A \in Sym(\omega_0)$ defines a symplectic form ω_A on $\mathbb{R}^{2r}$ by $\omega_A(\cdot, \cdot) = \omega_0(A\cdot, \cdot)$. To this interpretation corresponds an action

$$GL(2r, \mathbb{R}) \times Sym(\omega_0) \to Sym(\omega_0), \quad (g, A) \mapsto (g^{-1})^\dagger A g^{-1}.$$

This action is used to identify $S(2r)$ and $Sym(\omega_0)$ by a map Ψ in the following proposition.

Proposition 4. *Let Ψ be the canonical map $\Psi : S(2r) \tilde{\to} Sym(\omega_0) \subset GL(2r, \mathbb{R})$ where $GL(2r, \mathbb{R})$ carries the pseudo-Riemannian metric induced by the Ad-invariant trace-form. Then Ψ is a totally geodesic immersion and a map f from an ϵcomplex manifold (M, J^ϵ) to $S(2r)$ is ϵpluriharmonic, if and only if the map $\Psi \circ f : M \to Sym(\omega_0) \subset GL(2r, \mathbb{R})$ is ϵpluriharmonic.*

Proof. The proof is done by relating the map Ψ to the well-known Cartan-immersion. Additional information can be found in [10, 7, 9, 12].

1. First we study the identification $S(2r) \tilde{\to} Sym(\omega_0)$.
   $GL(2r, \mathbb{R})$ operates on $Sym(\omega_0)$ via

   $$GL(2r, \mathbb{R}) \times Sym(\omega_0) \to Sym(\omega_0), \;\; (g, B) \mapsto g \cdot B := (g^{-1})^\dagger B g^{-1}.$$

   The stabiliser of the $\mathbb{1}_{2r}$ is $Sp(\mathbb{R}^{2r})$ and the action is seen to be transitive by choosing a symplectic basis. Using the orbit-stabiliser theorem we get by identifying orbits and rest-classes a diffeomorphism

   $$\Psi : S(2r) \tilde{\to} Sym(\omega_0), \;\; g\, Sp(\mathbb{R}^{2r}) \mapsto g \cdot \mathbb{1}_{2r} = (g^{-1})^\dagger \mathbb{1}_{2r} g^{-1} = (g^{-1})^\dagger g^{-1}.$$

2. We recall some results about symmetric spaces (see: [7, 13]). Let G be a Lie-group and $\sigma : G \to G$ a group-homomorphism with $\sigma^2 = Id_G$. Let K denote

the subgroup $K = G^\sigma = \{g \in G \,|\, \sigma(g) = g\}$. The Lie-algebra $\mathfrak{g}$ of G decomposes in $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ with $d\sigma_{Id_G}(\mathfrak{h}) = \mathfrak{h}$, $d\sigma_{Id_G}(\mathfrak{p}) = -\mathfrak{p}$. One has the following information: The map $\phi : G/K \to G$ with $\phi : [gK] \mapsto g\sigma(g^{-1})$ defines a totally geodesic immersion called the Cartan-immersion.
We want to utilise this in the case $G = GL(2r, \mathbb{R})$ and $K = Sp(\mathbb{R}^{2r})$. In this spirit we define $\sigma : GL(2r, \mathbb{R}) \to GL(2r, \mathbb{R}),\ g \mapsto (g^{-1})^\dagger$. The map σ is obviously a homomorphism and an involution with $GL(2r, \mathbb{R})^\sigma = Sp(\mathbb{R}^{2r})$.
By a direct calculation one gets $d\sigma_{Id_G} = -h^\dagger$ and hence

$$\mathfrak{h} = \{h \in \mathfrak{gl}_{2r}(\mathbb{R}) \,|\, h^\dagger = -h\} = \mathfrak{sp}(\mathbb{R}^{2r}),\ \ \mathfrak{p} = \{h \in \mathfrak{gl}_{2r}(\mathbb{R}) \,|\, h^\dagger = h\} =: \mathrm{sym}(\omega_0).$$

Thus we end up with $\phi : S(2r) \to GL(2r, \mathbb{R}),\ \ g \mapsto g\sigma(g^{-1}) = gg^\dagger = \Psi \circ \Lambda(g)$. Here Λ is the map induced by $\Lambda : G \to G, h \mapsto (h^{-1})^\dagger$. This map is an isometry of the invariant metric. Hence Ψ is a totally geodesic immersion. Using proposition 3 the proof is finished. □

Remark 3. Above we have identified $S(2r)$ with $Sym(\omega_0)$ via Ψ.
Let us choose $o = eSp(\mathbb{R}^{2r})$ as base point and suppose that Ψ is chosen to map o to $\mathbb{1}_{2r}$. By construction Ψ is $\mathrm{GL}(2r, \mathbb{R})$-equivariant. We identify the tangent-space $T_\omega \mathrm{Sym}(\omega_0)$ at $\omega \in \mathrm{Sym}(\omega_0)$ with the (ambient) vector space of ω_0-symmetric matrices in $\mathfrak{gl}_{2r}(\mathbb{R})$

$$T_\omega \mathrm{Sym}(\omega_0) = \mathrm{sym}(\omega_0). \tag{4.3}$$

For $\tilde\omega \in S(2r)$ such that $\Psi(\tilde\omega) = \omega$, the tangent space $T_{\tilde\omega} S(2r)$ is canonically identified with the vector space of ω-symmetric matrices:

$$T_{\tilde\omega} S(2r) = \mathrm{sym}(\omega) := \{A \in \mathfrak{gl}_{2r}(\mathbb{R}) | A^\dagger \omega = \omega A\}\,. \tag{4.4}$$

Note that $\mathrm{sym}(\mathbb{1}_{2r}) = \mathrm{sym}(\omega_0)$.

Proposition 5. *The differential of $\varphi := \Psi^{-1}$ at $\omega \in \mathrm{Sym}(\omega_0)$ is given by*

$$\mathrm{sym}(\omega_0) \ni X \mapsto -\frac{1}{2}\omega^{-1} X \in \omega^{-1}\mathrm{sym}(\omega_0) = \mathrm{sym}(\omega)\,. \tag{4.5}$$

Using this proposition we relate now the differentials

$$df_x : T_x M \to \mathrm{sym}(\omega_0) \tag{4.6}$$

of a map $f : M \to \mathrm{Sym}(\omega_0)$ at $x \in M$ and

$$d\tilde f_x : T_x M \to \mathrm{sym}(f(x)) \tag{4.7}$$

of a map $\tilde f = \varphi \circ f : M \to S(2r)$: $d\tilde f_x = d\varphi\, df_x = -\frac{1}{2} f(x)^{-1} df_x$.

We interpret the one-form $A = -2d\tilde f = f^{-1}df$ with values in $\mathfrak{gl}_{2r}(\mathbb{R})$ as connection form on the vector bundle $E = M \times \mathbb{R}^{2r}$. We note, that the definition of A is the pure gauge, i.e. A is gauge-equivalent to $A' = 0$. Since for $A' = 0$ one has $A = f^{-1}A'f + f^{-1}df = f^{-1}df$, the curvature vanishes. This yields the next proposition:

Proposition 6. *Let $f : M \to GL(2r,\mathbb{R})$ be a C^∞-mapping and $A := f^{-1}df$: $TM \to \mathfrak{gl}_{2r}(\mathbb{R})$. Then the curvature of A vanishes, i.e. for $X,Y \in \Gamma(TM)$ it is*

$$Y(A_X) - X(A_Y) + [A_Y, A_X] + A_{[X,Y]} = 0. \tag{4.8}$$

In the next proposition we recall the equations for ϵpluriharmonic maps from an ϵcomplex manifold to $GL(2r,\mathbb{R})$:

Proposition 7. *(cf. [17, 19]) Let (M, J^ϵ) be an ϵcomplex manifold, $f : M \to GL(2r,\mathbb{R})$ a C^∞-map and A defined as in proposition 6.*
The ϵpluriharmonicity of f is equivalent to the equation

$$Y(A_X) + \frac{1}{2}[A_Y, A_X] - \epsilon J^\epsilon Y(A_{J^\epsilon X}) - \epsilon\frac{1}{2}[A_{J^\epsilon Y}, A_{J^\epsilon X}] = 0, \tag{4.9}$$

for all $X,Y \in \Gamma(TM)$.

With a similar argument as in proposition 4 we have shown in [18, 22]:

Proposition 8. *Let (M, J^ϵ) be an ϵcomplex manifold. A map*

$$\phi : M \to GL(r,\mathbb{C}_\epsilon)/U^\epsilon(p,q),$$

where the target-space is carrying the (pseudo-)metric induced by the trace-form on $GL(r,\mathbb{C}_\epsilon)$, is ϵpluriharmonic if and only if

$$\psi = \Psi^\epsilon \circ \phi : M \to GL(r,\mathbb{C}_\epsilon)/U^\epsilon(p,q) \tilde{\to} Herm^\epsilon_{p,q}(\mathbb{C}^r_\epsilon) \subset GL(r,\mathbb{C}_\epsilon)$$

is ϵpluriharmonic.

To be complete we mention the related symmetric decomposition:

$$\mathfrak{h} = \{A \in \mathfrak{gl}_r(\mathbb{C}_\epsilon)\,|\,A^h = -A\} = \mathfrak{u}^\epsilon(p,q),\ \ \mathfrak{p} = \{A \in \mathfrak{gl}_r(\mathbb{C}_\epsilon)\,|\,A^h = A\} =: \mathrm{herm}^\epsilon_{p,q}(\mathbb{C}^r_\epsilon).$$

5 tt^*-geometry and pluriharmonic maps

In this section we are going to state and prove the main results. Like in section 4 one regards the mapping $A = f^{-1}df$ as a map $A : TM \to \mathfrak{gl}_{2r}(\mathbb{R})$.

Theorem 1. *Let (M, J^ϵ) be a simply connected ϵcomplex manifold. Let (E, D, S, ω) be a symplectic ϵtt^*-bundle where E has rank $2r$ and M dimension n.*
The matrix representation $f : M \to Sym(\omega_0)$ of ω in a D^θ-flat frame of E induces an admissible ϵpluriharmonic map $\tilde{f} : M \xrightarrow{f} Sym(\omega_0) \tilde{\to} S(2r)$, where $S(2r)$ carries the (pseudo-Riemannian) metric induced by the trace-form on $GL(2r,\mathbb{R})$. Let s' be another D^θ-flat frame. Then $s' = s \cdot U$ for a constant matrix and the ϵpluriharmonic map associated to s' is $f' = U^\dagger f U$.

Proof. Thanks to remark 1.1) we can restrict to the case $\theta = \pi$ for $\epsilon = -1$ and $\theta = 0$ for $\epsilon = 1$.

Let $s := (s_1, \dots, s_{2r})$ be a D^θ-flat frame of E (i.e. $Ds = -\epsilon Ss$), f the matrix $\omega(s_k, s_l)$ and further S^s the matrix-valued one-form representing the tensor S in the frame s. For $X \in \Gamma(TM)$ we get:

$$\begin{aligned} X(f) &= X\omega(s,s) = \omega(D_X s, s) + \omega(s, D_X s) \\ &= -\epsilon[\omega(S_X s, s) + \omega(s, S_X s)] \\ &= -2\epsilon\omega(S_X s, s) = -2\epsilon f \cdot S^s_X. \end{aligned}$$

It follows $A_X = -2\epsilon S^s_X$. We now prove the ϵpluriharmonicity using

$$d^D S(X,Y) = D_X(S_Y) - D_Y(S_X) - S_{[X,Y]} = 0, \tag{5.1}$$

$$d^D S_{J^\epsilon}(X,Y) = D_X(S_{J^\epsilon Y}) - D_Y(S_{J^\epsilon X}) - S_{J^\epsilon [X,Y]} = 0. \tag{5.2}$$

The equation (5.2) implies

$$\begin{aligned} 0 = d^D S_{J^\epsilon}(J^\epsilon X, Y) &= D_{J^\epsilon X}(S_{J^\epsilon Y}) - \underbrace{\epsilon D_Y(S_X)}_{\overset{(5.1)}{=} \epsilon(D_X(S_Y) - S_{[X,Y]})} - S_{J^\epsilon[J^\epsilon X, Y]} \\ &= D_{J^\epsilon X}(S_{J^\epsilon Y}) - \epsilon D_X(S_Y) + \epsilon S_{[X,Y]} - S_{J^\epsilon[J^\epsilon X, Y]}. \end{aligned}$$

In local ϵholomorphic coordinate fields X, Y on M we get in the frame s

$$J^\epsilon X(S^s_{J^\epsilon Y}) - \epsilon X(S^s_Y) + [S^s_X, S^s_Y] - \epsilon[S^s_{J^\epsilon X}, S^s_{J^\epsilon Y}] = 0.$$

Now $A = -2\epsilon S^s$ gives equation (4.9) and proves the ϵpluriharmonicity of f.
Using $A_X = -2\epsilon S^s_X = -2d\tilde{f}(X)$, we find the property of the differential, as $S \wedge S$ is of type (1,1) by the ϵtt^*-equations, see proposition 2. The last statement is obvious.
□

Theorem 2. *Let (M, J^ϵ) be a simply connected ϵcomplex manifold and put $E = M \times \mathbb{R}^{2r}$. Then an ϵpluriharmonic map $\tilde{f} : M \to S(2r)$ gives rise to an ϵpluriharmonic map $f : M \xrightarrow{\tilde{f}} S(2r) \tilde{\to} Sym(\omega_0) \overset{i}{\hookrightarrow} GL(2r, \mathbb{R})$.*
If the map $\tilde{f}$ is admissible, then the map f induces a symplectic ϵtt^*-bundle $(E, D = \partial - \epsilon S, S = \epsilon d\tilde{f}, \omega = \omega_0(f\cdot, \cdot))$ on M where ∂ is the canonical flat connection on E.

Remark 4. We observe, that for ϵRiemannian surfaces $M = \Sigma$ every ϵpluriharmonic map is admissible, since $T^{1,0}\Sigma$ is one-dimensional.

Proof.
Let $\tilde{f} : M \to S(2r)$ be an ϵpluriharmonic map. Then due to proposition 4 we know, that $f : M \tilde{\to} Sym(\omega_0) \overset{i}{\hookrightarrow} GL(2r, \mathbb{R})$ is ϵpluriharmonic.
Since $E = M \times \mathbb{R}^{2r}$, we want to regard sections of E as 2r-tuples of $C^\infty(M, \mathbb{R})$-functions.
As in section 4 we consider the one-form $A = -2d\tilde{f} = f^{-1}df$ with values in $\mathfrak{gl}_{2r}(\mathbb{R})$ as a connection on E. The curvature of this connection vanishes (proposition 6). First, the constraints on ω are fulfilled:

Lemma 1. *The connection D is compatible with the symplectic form ω and S is symmetric with respect to ω.*

Proof. This is a direct computation with $X \in \Gamma(TM)$ and $v, w \in \Gamma(E)$:

$$\begin{aligned} X\omega(v,w) &= X\omega_0(fv,w) = \omega_0(X(f)v,w) + \omega_0(f(Xv),w) + \omega_0(fv,Xw) \\ &= \frac{1}{2}\omega_0(X(f)v,w) + \frac{1}{2}\omega_0(v,X(f)w) + \omega_0(f(Xv),w) + \omega_0(fv,Xw) \\ &= \frac{1}{2}\omega_0(f\cdot f^{-1}(Xf)v,w) + \frac{1}{2}\omega_0(v,f\cdot f^{-1}(Xf)w) \\ &\quad +\omega_0(fXv,w) + \omega_0(fv,Xw) \\ &= \omega(Xv - \epsilon S_X v, w) + \omega(v, Xw - \epsilon S_X w) = \omega(D_X v, w) + \omega(v, D_X w). \end{aligned}$$

S is ω-symmetric, since for $x \in M$ $d\tilde{f}_x$ takes by definition values in $sym(f(x))$. □
To finish the proof, we have to check the ϵtt^*-equations. The second ϵtt^*-equation

$$-\epsilon[S_X, S_Y] = [S_{J^\epsilon X}, S_{J^\epsilon Y}] \tag{5.3}$$

for S follows from the assumption that the image of $T^{1,0}M$ under $(d\tilde{f})^{\mathbb{C}_\epsilon}$ is Abelian. In fact, this is equivalent to $[d\tilde{f}(J^\epsilon X), d\tilde{f}(J^\epsilon Y)] = -\epsilon[d\tilde{f}(X), d\tilde{f}(Y)]$ $\forall X, Y \in TM$.

$$\begin{aligned} d^D S(X,Y) &= [D_X, S_Y] - [D_Y, S_X] - S_{[X,Y]} \\ &= \partial_X(S_Y) - \partial_Y(S_X) - 2\epsilon[S_X, S_Y] - S_{[X,Y]} = 0 \end{aligned}$$

is equivalent to the vanishing of the curvature of $A = -2\epsilon S$ interpreted as a connection on E (see proposition 6).
Finally one has for ϵholomorphic coordinate fields $X, Y \in \Gamma(TM)$:

$$\begin{aligned} d^D S_{J^\epsilon}(J^\epsilon X, Y) &= [D_{J^\epsilon X}, S_{J^\epsilon Y}] - \epsilon[D_Y, S_X] \\ &= [\partial_{J^\epsilon X} - \epsilon S_{J^\epsilon X}, S_{J^\epsilon Y}] - \epsilon[\partial_Y - \epsilon S_Y, S_X] \\ &= \partial_{J^\epsilon X}(S_{J^\epsilon Y}) - \epsilon\partial_Y(S_X) - \epsilon[S_{J^\epsilon X}, S_{J^\epsilon Y}] - [S_X, S_Y] \\ &\overset{(5.3)}{=} -\frac{1}{2}\epsilon\left(\partial_{J^\epsilon X}(A_{J^\epsilon Y}) - \epsilon\partial_Y(A_X)\right) \\ &\overset{(4.8)}{=} -\frac{1}{2}\epsilon\left(\partial_{J^\epsilon X}(A_{J^\epsilon Y}) - \epsilon\partial_X(A_Y) - \epsilon[A_X, A_Y]\right) \\ &\overset{(5.3)}{=} -\frac{1}{2}\epsilon\{\partial_{J^\epsilon X}(A_{J^\epsilon Y}) - \epsilon\partial_X(A_Y) \\ &\quad -\frac{1}{2}\epsilon[A_X, A_Y] + \frac{1}{2}[A_{J^\epsilon X}, A_{J^\epsilon Y}]\} \\ &\overset{(4.9)}{=} 0. \end{aligned}$$

This shows the vanishing of the tensor $d^D S_{J^\epsilon}$. It remains to show the curvature equation for D. We observe, that $D + \epsilon S = \partial - \epsilon S + \epsilon S = \partial$ and that the connection ∂ is flat, to find $0 = R^{D+\epsilon S}_{X,Y} = R^D_{X,Y} + \epsilon d^D S(X,Y) + [S_X, S_Y] \overset{d^D S = 0}{=} R^D_{X,Y} + [S_X, S_Y]$. □

In the situation of theorem 2 the two constructions are inverse.

Proposition 9.

1. *Given a symplectic ϵtt^*-bundle (E, D, S, ω) on an ϵcomplex manifold (M, J^ϵ). Let $\tilde{f}$ be the associated admissible ϵpluriharmonic map constructed to a D^θ-flat frame s in theorem 1. Then the symplectic ϵtt^*-bundle $(M \times \mathbb{R}^r, \tilde{S}, \tilde{\omega})$ associated to $\tilde{f}$ of theorem 2 is the representation of (E, D, S, ω) in the frame s.*

2. *Given an admissible ϵpluriharmonic map $\tilde{f} : (M, J^\epsilon) \to S(2r)$, then one obtains via theorem 2 a symplectic ϵtt^*-bundle $(M \times \mathbb{R}^{2r}, D, S, \omega)$. The ϵpluriharmonic map associated to this symplectic ϵtt^*-bundle is conjugated to the map $\tilde{f}$ by a constant matrix.*

Proof. Using again remark 1.1) we can set $\theta = \pi$ in the complex and $\theta = 0$ in the para-complex case.

1. The map f is obviously ω in the frame s and in the computations of theorem 1 one gets $A = -2d\tilde{f} = f^{-1}df = -2\epsilon S^s$. From $0 = D^\theta s = Ds + \epsilon Ss$ we obtain that the connection D in the frame s is just $\partial - \epsilon S^s = \partial + \frac{A}{2}$.
2. We have to find a D^θ-flat frame s. It is $D^\theta = \partial - \epsilon S + \epsilon S = \partial$. Hence we can take s as the standard-basis of $\mathbb{R}^{2r}$ and we get f. Every other basis gives a conjugated result. □

6 Harmonic bundle solutions

In this section we use the notation $H^\epsilon(p,q) := GL(r, \mathbb{C}_\epsilon)/U^\epsilon(p,q)$. As motivation for considering ϵharmonic bundles we prove :

Proposition 10. *The canonical inclusion*

$$i : GL(r, \mathbb{C}_\epsilon)/U^\epsilon(p,q) \hookrightarrow GL(2r, \mathbb{R})/Sp(\mathbb{R}^{2r})$$

is totally geodesic. Let further (M, J^ϵ) be an ϵcomplex manifold, then a map $\alpha : M \to GL(r, \mathbb{C}_\epsilon)/U^\epsilon(p,q)$ is ϵpluriharmonic if and only if $i \circ \alpha : M \to GL(2r, \mathbb{R})/Sp(\mathbb{R}^{2r})$ is ϵpluriharmonic.

Proof. Looking at the inclusion of the symmetric decompositions

$$\mathfrak{gl}_r(\mathbb{C}_\epsilon) = \mathrm{herm}_{p,q}(\mathbb{C}^r_\epsilon) \oplus \mathfrak{u}^\epsilon(p,q) \subset \mathfrak{gl}_{2r}\mathbb{R} = \mathrm{sym}(\omega_0) \oplus \mathfrak{o}(k,l),$$

we see, that $\mathrm{herm}^\epsilon_{p,q}(\mathbb{C}^r_\epsilon) \subset \mathrm{sym}(\omega_0)$ is a Lie-triple system, i.e.

$$[\mathrm{herm}^\epsilon_{p,q}(\mathbb{C}^r_\epsilon), [\mathrm{herm}^\epsilon_{p,q}(\mathbb{C}^r_\epsilon), \mathrm{herm}^\epsilon_{p,q}(\mathbb{C}^r_\epsilon)]] \subset \mathrm{herm}^\epsilon_{p,q}(\mathbb{C}^r_\epsilon)$$

and that therefore the inclusion $GL(r, \mathbb{C}_\epsilon)/U^\epsilon(p,q) \overset{i}{\hookrightarrow} GL(2r, \mathbb{R})/Sp(\mathbb{R}^{2r})$ is totally geodesic. The second statement follows from proposition 3. □

In [18, 22] we related ϵpluriharmonic maps from an ϵcomplex manifold (M, J^ϵ) into $H^\epsilon(p,q) = GL(r, \mathbb{C}_\epsilon)/U^\epsilon(p,q)$ with $r = p+q$ to ϵharmonic bundles over (M, J^ϵ). First we recall the definition of an ϵharmonic bundle:

Definition 5. An ϵharmonic bundle $(E \to M, D, C, \bar{C}, h)$ consists of the following data:
An ϵcomplex vector bundle E over an ϵcomplex manifold (M, J^ϵ), an ϵhermitian metric h, a metric connection D with respect to h and two C^∞-linear maps $C : \Gamma(E) \to \Gamma(\Lambda^{1,0}T^*M \otimes E)$ and $\bar{C} : \Gamma(E) \to \Gamma(\Lambda^{0,1}T^*M \otimes E)$, such that the connection

$$D^{(\lambda)} = D + \lambda C + \bar{\lambda}\bar{C}$$

is flat for all $\lambda \in \mathbb{S}^1_\epsilon$ and $h(C_Z a, b) = h(a, \bar{C}_{\bar{Z}} b)$ with $a, b \in \Gamma(E)$ and $Z \in \Gamma(T^{1,0}M)$.

Remark 5. In the complex case with positive definite metric h this definition is equivalent to the definition of a harmonic bundle in Simpson [23]. Equivalent structures in the complex case with metrics of arbitrary signature have been also regarded in Hertling's paper [11].

The relation of ϵharmonic bundles to ϵpluriharmonic maps is stated in the following theorem.

Theorem 3. *(cf. [18, 22])*

(i) Let $(E \to M, D, C, \bar{C}, h)$ be an ϵharmonic bundle over the simply connected ϵcomplex manifold (M, J^ϵ). Then the representation of h in a $D^{(\lambda)}$-flat frame defines an ϵpluriharmonic map $\phi_h : M \to Herm^\epsilon_{p,q}(\mathbb{C}^r_\epsilon)$. The map ϕ_h induces an admissible ϵpluriharmonic map $\tilde{\phi}_h = \Psi^\epsilon \circ \phi_h : M \to H^\epsilon(p,q)$ (cf. proposition 8 for Ψ^ϵ) .

*(ii) Let (M, J^ϵ) be a simply connected ϵcomplex manifold and $E = M \times \mathbb{C}^r_\epsilon$. Given an admissible ϵpluriharmonic map $\tilde{\phi}_h : M \to H^\epsilon(p,q)$, then $(E, D = \partial - \epsilon C - \epsilon\bar{C}, C = \epsilon(d\tilde{\phi}_h)^{1,0}, h = (\phi_h \cdot, \cdot)_{\mathbb{C}^r_\epsilon})$ defines an ϵharmonic bundle, where ∂ is the ϵcomplex linear extension on $TM^{\mathbb{C}_\epsilon}$ of the flat connection on $E = M \times \mathbb{C}^r_\epsilon$.
In the complex case of signature $(r,0)$ and $(0,r)$ every pluriharmonic map $\tilde{\phi}_h$ is admissible.*

The last theorem and proposition 10 yield ϵpluriharmonic maps to $GL(2r,\mathbb{R})/Sp(\mathbb{R}^{2r})$. We are going to identify the related symplectic ϵtt^*-bundles. Therefore we construct symplectic ϵtt^*-bundles from ϵharmonic bundles, via the next proposition.

Proposition 11. *Let $(E \to M, D, C, \bar{C}, h)$ be an ϵharmonic bundle over the ϵcomplex manifold (M, J^ϵ), then $(E, D, S, \Omega = Im\, h)$ with $S_X := C_Z + \bar{C}_{\bar{Z}}$ for $X = Z + \bar{Z} \in TM$ and $Z \in T^{1,0}M$ is a symplectic ϵtt^*-bundle.*

Proof. For $\lambda = \cos_\epsilon(\alpha) + \hat{i}\sin_\epsilon(\alpha) \in \mathbb{S}^1_\epsilon$ we compute $D^{(\lambda)}$:

$$\begin{aligned} D^{(\lambda)}_X &= D_X + \lambda C_Z + \bar{\lambda}\bar{C}_{\bar{Z}} = D_X + \cos_\epsilon(\alpha)(C_Z + \bar{C}_{\bar{Z}}) + \sin_\epsilon(\alpha)(\hat{i}C_Z - \hat{i}\bar{C}_{\bar{Z}}) \\ &= D_X + \cos_\epsilon(\alpha)S_X + \sin_\epsilon(\alpha)(C_{J^\epsilon Z} + \bar{C}_{J^\epsilon \bar{Z}}) \\ &= D_X + \cos_\epsilon(\alpha)S_X + \sin_\epsilon(\alpha)S_{J^\epsilon X} = D^\alpha_X. \end{aligned}$$

Hence we see

$$D^\alpha = D^{(\lambda)} \tag{6.1}$$

and D^α is flat if and only if $D^{(\lambda)}$ is flat.
Further we claim, that S is Ω-symmetric. With $X = Z + \bar{Z}$ for $Z \in T^{1,0}M$ one finds

$$h(S_X \cdot, \cdot) = h(C_Z + \bar{C}_{\bar{Z}} \cdot, \cdot) = h(\cdot, C_Z + \bar{C}_{\bar{Z}} \cdot) = h(\cdot, S_X \cdot).$$

This yields the symmetry of S with respect to $\Omega = \text{Im}h$.
Finally we show $D\Omega = 0$:

$$\begin{aligned}
2\hat{i}X\Omega(e,f) &= X.(h(e,f)-h(f,e)) = (Z+\bar{Z}).(h(e,f)-h(f,e)) \\
&= h(D_Z e,f)+h(e,D_{\bar{Z}}f)+h(D_{\bar{Z}}e,f)+h(e,D_Z f) \\
&- [h(D_Z f,e)+h(f,D_{\bar{Z}}e)+h(D_{\bar{Z}}f,e)+h(f,D_Z e)] \\
&= h((D_Z+D_{\bar{Z}})e,f)+h(e,(D_{\bar{Z}}+D_Z)f) \\
&- h((D_Z+D_{\bar{Z}})f,e)-h(f,(D_{\bar{Z}}+D_Z)e) \\
&= h(D_X e,f)-h(f,D_X e)+h(e,D_X f)-h(D_X f,e) \\
&= 2\hat{i}(\Omega(D_X e,f)+\Omega(e,D_X f)).
\end{aligned}$$

This proves, that $(E, D, S, \Omega = \operatorname{Im} h)$ is a symplectic ϵtt^*-bundle. □
From theorem 1 one obtains the next corollary.

Corollary 1. *Let $(E \to M, D, C, \bar{C}, h)$ be an ϵharmonic bundle over the simply connected ϵcomplex manifold (M, J^ϵ), then the representation of $\Omega = Im\, h$ in a $D^{(\lambda)}$-flat frame defines an ϵpluriharmonic map $\Phi_\Omega : M \to GL(\mathbb{R}^{2r})/Sp(\mathbb{R}^{2r})$.*

Proof. This follows from the identity (6.1), i.e. $D^{(\lambda)}_X = D^\alpha_X$ for $\lambda = \cos_\epsilon(\alpha)+\hat{i}\sin_\epsilon(\alpha) \in \mathbb{S}^1_\epsilon$ and from proposition 11 and theorem 1. □

Our aim is to understand the relations between the ϵpluriharmonic maps in theorem 3 and corollary 1. Therefore we need to have a closer look at the map $h \mapsto \operatorname{Im} h$. First, we identify $\mathbb{C}^r_\epsilon$ with $\mathbb{R}^r \oplus \hat{i}\,\mathbb{R}^r = \mathbb{R}^{2r}$. In this model the multiplication with $\hat{i}$ coincides with the automorphism $j^\epsilon = \begin{pmatrix} 0 & \epsilon \mathbb{1}_r \\ \mathbb{1}_r & 0 \end{pmatrix}$ and $GL(r,\mathbb{C}_\epsilon)$ (respectively $\mathfrak{gl}_r(\mathbb{C}_\epsilon)$) are the elements in $GL(2r,\mathbb{R})$ (respectively $\mathfrak{gl}_{2r}(\mathbb{R})$), which commute with j^ϵ.
An endomorphism $C \in \operatorname{End}(\mathbb{C}^r_\epsilon)$ decomposes in its real-part A and its imaginary part B, i.e. $C = A+\hat{i}\,B$ with $A, B \in \operatorname{End}(\mathbb{R}^r)$. In the above model C is given by the matrix

$$\iota(C) = \begin{pmatrix} A & \epsilon B \\ B & A \end{pmatrix}.$$

The ϵcomplex conjugated $\bar{C} = A-\hat{i}B$, the transpose $C^t = A^t+\hat{i}B^t$ and the ϵhermitian conjugated C^h of C correspond to

$$\iota(\bar{C}) = \begin{pmatrix} A & -\epsilon B \\ -B & A \end{pmatrix},\ \iota(C^t) = \begin{pmatrix} A^t & \epsilon B^t \\ B^t & A^t \end{pmatrix},\ \iota(C^h) = \iota(\bar{C}^t) = \begin{pmatrix} A^t & -\epsilon B^t \\ -B^t & A^t \end{pmatrix}.$$

We observe, that $\iota(\bar{C}^t) = I^\epsilon \iota(C)^T I^\epsilon$ where $\cdot^T$ is the transpose in $\operatorname{End}(\mathbb{R}^{2r})$ and

$$I^\epsilon = \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & -\epsilon\mathbb{1}_r \end{pmatrix}.$$

The ϵhermitian matrices $Herm^\epsilon_{p,q}(\mathbb{C}^r_\epsilon)$ (of signature (p,q) for $\epsilon = -1$, i.e. in the complex case) coincide with the subset of symmetric matrices $H \in \operatorname{Sym}_{k,l}(\mathbb{R}^{2r})$, which commute with j^ϵ, i.e. $[H, j^\epsilon] = 0$, where the pair (k,l) is

$$(k,l) = \begin{cases} (2p,2q) \text{ for } \epsilon = -1, \\ (r,r) \text{ for } \epsilon = 1. \end{cases}$$

Likewise, $T_{I_{k,l}}Herm^{\epsilon}_{p,q}(\mathbb{C}^r_\epsilon)$ consists of symmetric matrices $h \in \mathrm{sym}_{k,l}(\mathbb{R}^{2r})$, which commute with j^ϵ, i.e. the ϵhermitian matrices in $\mathrm{sym}_{k,l}(\mathbb{R}^{2r})$ which we have denoted by $\mathrm{herm}^{\epsilon}_{p,q}(\mathbb{C}^r_\epsilon)$.
An ϵhermitian sesquilinear scalar product h (of signature (p,q) for $\epsilon = -1$) corresponds to an ϵhermitian matrix $H \in Herm^{\epsilon}_{p,q}(\mathbb{C}^r_\epsilon)$ (of hermitian signature (p,q) for $\epsilon = -1$) defined by $h(\cdot,\cdot) = (H\cdot,\cdot)_{\mathbb{C}^r_\epsilon}$. The condition $C^h = \bar{C}^t = C$, i.e. C is ϵhermitian, means in our model, that C has the form

$$\iota(C) = \begin{pmatrix} A & \epsilon B \\ B & A \end{pmatrix}$$

with $A = A^t$ and $B = -B^t$.
Using this information we find the explicit representation of the map which corresponds to taking the imaginary part $\mathrm{Im}\, h$ of h. This is the map $\Im$ satisfying $\mathrm{Im}\, h = (\Im(H)\cdot,\cdot)_{\mathbb{R}^{2r}}$, where $(\cdot,\cdot)_{\mathbb{R}^{2r}}$ is the Euclidean standard scalar product on $\mathbb{R}^{2r}$. With $z, w \in \mathbb{C}^r_\epsilon$ we define

$$\beta(z,w) := \mathrm{Im}\,(z,w)_{\mathbb{C}^r_\epsilon} = \frac{1}{2\hat{i}}\,(z\cdot\bar{w} - \bar{z}\cdot w)$$

and find $\mathrm{Im}\, h(z,w) = \mathrm{Im}\,(Hz,w)_{\mathbb{C}^r_\epsilon} = \frac{1}{2\hat{i}}\left[(Hz)\cdot\bar{w} - (\overline{Hz})\cdot w\right] = \beta(Hz,w)$. Further we remark that $\beta(\cdot,\cdot) = \mathrm{Im}\,(\cdot,\cdot)_{\mathbb{C}^r_\epsilon} = (I^\epsilon j^\epsilon\cdot,\cdot)_{\mathbb{R}^{2r}}$, where $(\cdot,\cdot)_{\mathbb{R}^{2r}}$ is the Euclidean standard scalar product on $\mathbb{R}^{2r}$.
This yields $\mathrm{Im}\, h(z,w) = (I^\epsilon j^\epsilon \iota(H)z,w)_{\mathbb{R}^{2r}} = -\epsilon(j^\epsilon I^\epsilon \iota(H)z,w)_{\mathbb{R}^{2r}}$ and for $H = A + \hat{i}B$ with $A, B \in \mathrm{End}(\mathbb{R}^r)$ one obtains

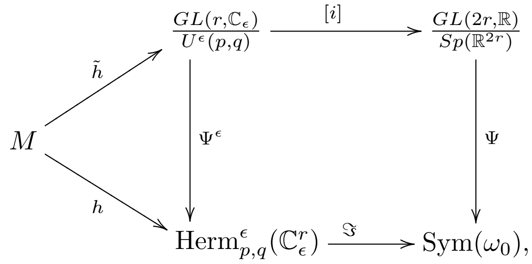
$$\Im(H) = I^\epsilon j^\epsilon \begin{pmatrix} A & \epsilon B \\ B & A \end{pmatrix} = \epsilon \begin{pmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{pmatrix} \begin{pmatrix} A & \epsilon B \\ B & A \end{pmatrix} = \epsilon \begin{pmatrix} B & A \\ -A & -\epsilon B \end{pmatrix}.$$

This map is easily seen to have maximal rank and to be equivariant with respect to the following $GL(r,\mathbb{C}_\epsilon)$-action on $\mathrm{Herm}^{\epsilon}_{p,q}(\mathbb{C}^r_\epsilon)$:

$$GL(r,\mathbb{C}_\epsilon) \times \mathrm{Herm}^{\epsilon}_{p,q}(\mathbb{C}^r_\epsilon) \to \mathrm{Herm}^{\epsilon}_{p,q}(\mathbb{C}^r_\epsilon), \quad (g,H) \mapsto (g^{-1})^h H g^{-1}$$

and the $GL(2r,\mathbb{R})$-action on $\mathrm{Sym}(\omega_0)$ which was considered in section 4. Summarising we have the commutative diagram in which all maps apart $\Im$ of the square were shown to be totally geodesic:

(6.2)



where $[i]$ is induced by the inclusion $i : GL(r,\mathbb{C}_\epsilon) \hookrightarrow GL(2r,\mathbb{R})$. Hence $\Im$ is totally geodesic. Utilising this diagram we show the next proposition.

Proposition 12. *A map $h : M \to Herm^{\epsilon}_{p,q}(\mathbb{C}^r_{\epsilon})$ is ϵpluriharmonic, if and only if $\Omega = Im\, h : M \to Sym(\omega_0)$ is ϵpluriharmonic.*

Proof. As discussed above, the map $\Im$ is a totally geodesic immersion and therefore we are in the situation of proposition 3. □

From this proposition it follows:

Proposition 13. *Let $(E \to M, D, C, \bar{C}, h)$ be an ϵharmonic bundle over the ϵcomplex manifold (M, J^{ϵ}), $(E, D, S, \Omega = Im\, h)$ the symplectic ϵtt^*-bundle constructed in proposition 11 and $\tilde{\Phi}_{\Omega} : M \to GL(\mathbb{R}^{2r})/Sp(\mathbb{R}^{2r})$ the ϵpluriharmonic map given in corollary 1. Then $\tilde{\Phi}_{\Omega} = [i] \circ \tilde{\Phi}_h$ and these ϵpluriharmonic maps are admissible.*

Proof. This follows using the definition of (E, D, S, Ω) (cf. proposition 11) from corollary 1 and proposition 12. For the second part one observes, that the differential of $[i]$ is a homomorphism of Lie-algebras. □

This describes the ϵpluriharmonic maps coming from symplectic ϵtt^*-bundles induced by ϵharmonic bundles. Conversely, this gives an Ansatz to construct ϵharmonic bundles from ϵpluriharmonic maps to $GL(r, \mathbb{C}_{\epsilon})/U^{\epsilon}(p, q)$. For metric ϵtt^*-bundles we have gone this way in [18, 22] to obtain theorem 3.

References

[1] N. Abe and S. Kurosu, *A decomposition of a holomorphic vector-bundle with connection and its applications to complex affine immersions*, Res. in Math. 44 (2003), 3–24.

[2] V. Cortés, C. Mayer, T. Mohaupt and F. Saueressig, *Special Geometry of Euclidean Supersymmetry I: Vectormultiplets*, J. High Energy Phys. JHEP03 (2004) 028, hep-th/0312001 2004.

[3] V. Cortés and L. Schäfer, *Topological-antitopological fusion equations, pluriharmonic maps and special Kähler manifolds*, Progress in Mathematics 234, Birkhäuser 2005, 59–74.

[4] V. Cortés and L. Schäfer, *Flat nearly Kähler manifolds,* Annals of Global Analysis and Geometry 32 (2007), 379-389.

[5] V. Cortés and L. Schäfer, manuscript in preparation.

[6] S. Cecotti and C. Vafa, *Topological-Antitopological fusion,* Nuclear Physics B 367 (1991), 351–461.

[7] J. Cheeger and D. G. Ebin, *Comparsion Theorems in Riemannian Geometry,* North-Holland (1975).

[8] B. Dubrovin, *Geometry and integrability of topological-antitopological fusion,* Commun. Math. Phys 152 (1992) 539–564.

[9] S. Gallot, D. Hulin and J. Lafontaine, *Riemannian Geometry,* Springer (1993).

[10] S. Helgason, *Differential Geometry and Symmetric Spaces,* Academic Press (1962).

[11] C. Hertling, *tt*-Geometry, Frobenius manifolds, their connections, and the construction for singularities*, J. Reine Angew. Math. 555 (2003), 77–161.

[12] S. Kobayashi and K. Nomizu, *Foundations of differential geometry volume I/II,* Interscience Publishers (1964/1969).

[13] Loos O., Symmetric spaces, W.A. Benjamin, Inc. (1969).

[14] M.-A. Lawn and L. Schäfer, *Decompositions of para-complex vector-bundles and affine para-complex immersions,* Results Math. 48 (2005), 246–274.

[15] D. Mc Duff and D. Salamon, *Introduction to symplectic topology*, Oxford University Press (1995).

[16] J. H. Sampson, *Applications of harmonic maps to Kähler geometry,* Contemporary Mathematics Volume 49, (1986), 125–134.

[17] L. Schäfer, *tt*-geometry and pluriharmonic maps*, Ann. Global Anal. Geom. 28, no. 3 (2005), 285–300.

[18] L. Schäfer, *Harmonic bundles, topological-antitoplological fusion and the related puriharmonic maps*, J. Geom. Phys. 56 (2006), no. 5, 830–842.

[19] L. Schäfer, *tt*-bundles in para-complex geometry, special para-Kähler manifolds and para-pluriharmonic maps*, Diff. Geom. Appl. 24 (2006), 60–89.

[20] L. Schäfer, *tt*-bundles on the tangent bundle of an almost complex manifold*, Journal of Geometry and Physics 57 (2007), 999-1014.

[21] L. Schäfer, *Para-tt*-bundles on the tangent bundle of an almost para-complex manifold*, Annals of Global Analysis and Geometry 32 (2007), 125-145.

[22] L. Schäfer, *Harmonic bundle solutions of topological-antitoplological fusion in para-complex geometry*, to appear in Diff. Geom. Appl.

[23] C.T. Simpson: *Higgs-bundles and local systems,* Institut des hautes Etudes scientifiques - Publication mathematiques, No. 75 (1992), 5–95.

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