Totally singular Lagrangians and affine Hamiltonians

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Abstract. A Lagrangian or an affine Hamiltonian is called totally singular if it is defined by affine functions in highest velocities or momenta respectively. A natural duality relation between these Lagrangians and affine Hamiltonians is considered. The energy of a second order affine Hamiltonian is related with a dual corresponding Lagrangian of order one. Relations between the curves that are solutions of Euler and Hamilton equations of dual objects are also studied using semi-sprays. In order to generate examples of second order, a natural lifting procedure is considered.

M.S.C. 2000: 53C80, 70H03, 70H05, 70H50.
Key words: totally singular Lagrangians and Hamiltonians, Euler and Hamilton equations, semi-spray.

1 Introduction

In the case of hyperregular Lagrangians and Hamiltonians, a duality between Lagrangians and Hamiltonians is usually given by a Legendre transformation. Using constraints or symplectic technics, one can handle dualities between large classes of singular Lagrangians and Hamiltonians. We are concerned in this paper with Lagrangians and Hamiltonians that have null vertical Hessians, i.e. the ,,most singular" Lagrangians and Hamiltonians; they are affine in velocities and in momenta respectively and are called here *totally singular* ones. The Lagrangian case is remarked in [2], where it is related to a classification of singular Lagrangians. Marsden and Ratiu are concerned in [6] with a special case, called here the regular one, when the vector field given by the Euler equation is uniquely determined. A classification of dynamic solutions using constraints is considered in [3].

We define in the paper allowed totally singular Lagrangians, i.e. Lagrangians affine in velocities together with a vector field that is a solution of its Euler equation (in [2] one say that the Lagrangian allows a global dynamics). We consider a duality of a such Lagrangian with a corresponding allowed totally singular Hamiltonian that is affine in momenta, so that the Legendre map sends lifts of the integral curves of the Euler equation to integral curves of the Hamilton equation (Proposition 2.1). Every

Balkan Journal of Geometry and Its Applications, Vol.14, No.1, 2009, pp. 60-71.

allowed Lagrangian has a dual allowed Hamiltonian, but the converse is true only locally (Proposition 2.2). Certain examples of allowed Lagrangians and Hamiltonians are given.

In the second section we consider allowed totally singular Lagrangians and affine Hamiltonians of second order, having null vertical Hessians of 2-velocities and momenta respectively. A duality is considered also in this case. An allowed Lagrangian has a dual allowed Hamiltonian, but for the converse situation, Theorem 3.4 asserts that, assuming some conditions, an allowed Hamiltonian of second order has a allowed Lagrangian of second order and both can be related to ordinary dual allowed Lagrangians and Hamiltonians on TM. A study of some classes of allowed Lagrangians of second order, including some considered in [6], is performed, obtaining differential equations that have as solutions curves that verify the Euler-Lagrange equations. In order to have consistent examples of allowed Lagrangians and Hamiltonians of second order, a lifting procedure is given from the first to the second order, in the regular case. In this way, certain examples considered in the first section, or examples considered in [6] or [2] can be lifted to allowed Lagrangians and Hamiltonians of second order.

The paper is addressed mainly to a reader interested in geometric mechanics. We avoid abstract and general constructions, or to mention and to extend some related problems (such as affine bundles, some aspects of analytical mechanics or Lie algebroids). We use coordinates on the second order tangent spaces as in [7, 10]. Concerning some new examples we give, we have in mind not to bring over, but to bring in attention. Further developments can be done to foliations (as in [5]), to a complex setting (as in [11]) or to other singular cases (as in [14]).

2 Totally singular Lagrangians and Hamiltonians of order one

A Lagrangian $L : TM \to I\!R$ is *totally singular* if it is affine in velocities, or, equivalently it has a null vertical Hessian. This class of Lagrangians is considered in [6], [1], [2], [3] etc. A totally singular Lagrangian L has the form $L(x, y) = \alpha(y) + \beta$, where $\alpha \in \mathcal{X}^*(M)$ and $\beta \in \mathcal{F}(M)$. Using local coordinates,

$$L(x^i, y^i) = \alpha_i(x^j)y^i + \beta(x^j). \tag{2.1}$$

The Euler equation of L, $\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) = \frac{\partial L}{\partial x^i}$, becomes

$$\left(\frac{\partial \alpha_i}{\partial x^j} - \frac{\partial \alpha_j}{\partial x^i}\right)\varphi^j = \frac{\partial \beta}{\partial x^i}, \tag{2.2}$$

where $\frac{dx^i}{dt} = y^i \overset{not.}{=} \varphi^i$, or $i_\varphi d\alpha = d\beta$.We are interested in the case when $\varphi = \varphi^i \frac{\partial}{\partial x^i} \in \mathcal{X}(M)$.

We say that a (local) vector field φ on M is *allowed* for L if it fulfills the condition (2.2). In the case when φ is a global vector field, one say in [2] that L *allows a global dynamics*. Briefly we say in that follows that the Lagrangian L is *allowed* if an allowed vector field φ exists and it is considered implicitly associated with L if anything else is assumed. A dual totally singular Hamiltonian H (i.e. affine in momenta) is associated here with an allowed L in order to give a natural correlation between the solutions of the Lagrange equation of L and the Hamilton equation of H.

Some important particular cases are briefly described as follows. The *regular* case is when $d\alpha$ is a symplectic form and β is arbitrary. The allowed vector field φ is in this case the Hamiltonian vector field X_β. A trivial case is when α is closed, when $\beta = const.$ and every $\varphi \in \mathcal{X}(M)$ is allowed for L. Another case is when $d\alpha$ has the rank $\dim M - 1$ and $\beta = const.$ The vector subbundle $\ker d\alpha \subset TM$ has a one dimensional fibers, then L becomes allowed considering any section φ in $\ker d\alpha$. The case when the equation (2.2) has multiple solutions is studied in [3], where a classification of such Lagrangians related to their dynamics on TM is performed.

A Hamiltonian $H : T^*M \to I\!\!R$ is *totally singular* (or affine in momenta) iff H has a null vertical Hessian. The general form of H is

$$H(x,p) = p(\varphi) + \gamma, \tag{2.3}$$

where $\varphi \in \mathcal{X}(M)$ is a vector field and $\gamma \in \mathcal{F}(M)$ is a real function; using coordinates $H(x^i, p_i) = p_i \varphi^i(x^j) + \gamma(x^j)$. The Hamilton equations of H are

$$\begin{cases} \frac{dx^i}{dt} = \varphi^i, \\ \frac{dp_i}{dt} = -\frac{\partial \varphi^j}{\partial x^i} p_j - \frac{\partial \gamma}{\partial x^i}. \end{cases} \tag{2.4}$$

It follows that a integral curve $\gamma = (x^i, p_i) : I = (a,b) \to T^*M$ (of the Hamilton equations) projects to a curve $\pi_1 \circ \gamma = x^i : I \to M$, that is an integral curve of the vector field $\varphi = \varphi^i \frac{\partial}{\partial x^i} \in \mathcal{X}(M)$.

All the Hamiltonians considered in this section are totally singular. Notice that every curve γ in M that has the local form $t \to (\gamma^i(t))$ lifts to a curve $\gamma^{(1)}$ in TM that has the local form $t \to (\gamma^i(t), \frac{d\gamma}{dt}(t))$. We say that $\gamma^{(1)}$ is the *tangent lift* of γ. If φ is an allowed vector field for L, we denote by $\Gamma_{L,\varphi}$ the set of its integral curves (in M); these curves are solutions of the Euler equation of L. If $H : T^*M \to I\!\!R$ is a Hamiltonian, we denote by Γ_H the set of curves in T^*M that are solutions of Hamilton equation of H and by $X_H \in \mathcal{X}(T^*M)$ the Hamiltonian vector field of H. In the sequel $\pi : TM \to M$ and $\pi' : T^*M$ are the canonical projections. We say that a Hamiltonian H having the form (2.3) and an allowed Lagrangian L having the form (2.1) are in *duality* if the vector field φ that corresponds to H is allowed for L (i.e. $i_\varphi d\alpha = d\beta$) and $L(x,\varphi) + H(x,\alpha) - \alpha(\varphi) = const.$ Using coordinates, the two conditions are equivalent to (2.2) and $\beta + \gamma + \alpha_j \varphi^j = const.$ respectively. Since the Lie derivative has the form $\mathcal{L}_\varphi = i_\varphi d + d i_\varphi$, it follows that the condition $i_\varphi d\alpha = d\beta$ is equivalent to the condition $\mathcal{L}_\varphi \alpha = d(\alpha(\varphi) + \beta)$. Thus H and L are in duality iff the conditions $i_\varphi d\alpha = d\beta$ and $\mathcal{L}_\varphi \alpha = -d\gamma$ hold. The following characterization is straightforward using the definitions and the above remarks.

Proposition 2.1. *The allowed totally singular Lagrangian L and the totally singular Hamiltonian H are in duality iff the following two conditions hold:*

1. *The projection of the Hamiltonian vector field X_H, $\varphi = \pi'_* X_H$, is an allowed vector field for L.*
2. *The Legendre map $\mathcal{L} : TM \to T^*M$ of L sends every tangent lift of a curve $\gamma \in \Gamma_{L,\varphi}$ to a curve $\mathcal{L} \circ \gamma^{(1)} \in \Gamma_H$.*

Proposition 2.2. *1. If L is an allowed totally singular Lagrangian, then it has a dual H. If H and H' are two duals that correspond to a same allowed vector field, then $H - H' = const$.*

2. Let H be a totally singular Hamiltonian having the form (2.3). If $\varphi \neq 0$, then, locally, there is a totally singular Lagrangian L that is a dual with H and has φ as an allowed vector field.

Notice that the essential relation is

$$\frac{\partial \alpha_i}{\partial x^j}\varphi^j + \alpha_j \frac{\partial \varphi^j}{\partial x^i} + \frac{\partial \gamma}{\partial x^i} = 0, \tag{2.5}$$

or $\mathcal{L}_\varphi \alpha + d\gamma = 0$. It is a system of partial differential equations that (α_i) must satisfy. The final argument in the proof is obtained using the following Lemma.

Lemma 2.3. *If $\varphi \neq 0$, then the system (2.5) allows always α as a local solution.*

We consider below some examples. Other ones can be found in [2, 3].

Example 1. Let $L : TM \to I\!R$ be a regular Lagrangian and $f \in \mathcal{F}(TM)$. Then $L_0 : TTM \to I\!R$, $L_0(x^i, y^i, X^i, Y^i) = X^i \frac{\partial L}{\partial y^i}(x^j, y^j) + f(x^j, y^j)$ is a regular totally singular Lagrangian. The Euler equations of L_0, gives a vector field $G(f) = -g^{ij}\frac{\partial f}{\partial y^j}\frac{\partial}{\partial x^i} + g^{ij}\left(\left(\frac{\partial^2 L}{\partial x^p \partial y^j} - \frac{\partial^2 L}{\partial x^j \partial y^p}\right) g^{pq}\frac{\partial f}{\partial y^q} + \frac{\partial f}{\partial x^j}\right)\frac{\partial}{\partial y^i} \in \mathcal{X}(TM)$. If $f \in \pi^*\mathcal{F}(M)$, then $G(f)$ is a vertical vector field. In the particular case when L comes from a Riemannian metric on M, then $G(f)$ is the vertical lift of the usual gradient of f. The dual Hamiltonian is $H_0 : T^*TM \to I\!R$, $H_0(x^i, y^i, p_{(0)i}, p_{(1)i}) = X^i p_{(0)i} + Y^i p_{(1)i} + \gamma(x^j, y^j)$, where $X^i = -g^{ij}\frac{\partial f}{\partial y^j}$ and $Y^i = g^{ij}\left(\left(\frac{\partial^2 L}{\partial x^j \partial y^k} - \frac{\partial^2 L}{\partial x^k \partial y^j}\right) X^k + \frac{\partial f}{\partial x^j}\right)$, thus $\gamma = g^{ij}\frac{\partial f}{\partial y^j}\frac{\partial L}{\partial y^i} - f$. In particular, when $L = \frac{1}{2}F$, F is 2-homogeneous and $f(x^i, y^j) = -\frac{1}{2}F(x^i, y^j) + h(x^i)$, then $X^i = y^i$ and the solutions of the Euler-Lagrange equations are the integral curves of a (first order) semi-spray $(x^i, y^i) \xrightarrow{S} (x^i, y^i, y^i, S^i(x^i, y^i))$. For example, it is the case when $L = \frac{1}{2}g_{ij}(x^k)y^i y^j$ comes from a (pseudo) Riemannian metric g.

Example 2. The vector field given by Euler equations for the rotational dynamics of a rigid body about its center of mass, in absence of external forces (see [6, section 1.2 and chapter 15]), can be associated with a totally singular Lagrangian, as follows. Consider $(\Pi_1, \Pi_2, \Pi_3) \in I\!R^3$, $I_i \in I\!R^*_+$, $i = \overline{1,3}$, and the 1-form $F = I_1\Pi_1(\Pi_2^2 + \Pi_3^2)d\Pi_1 + I_2\Pi_2(\Pi_3^2 + \Pi_1^2)d\Pi_2 + I_3\Pi_3(\Pi_1^2 + \Pi_2^2)d\Pi_3$. If $I_1 \neq I_2 \neq I_3 \neq I_1$, then dF has rank 2 on an open domain, dense in $I\!R^3$. The totally singular Lagrangian $L(\Pi_i, \dot{\Pi}_i) = I_1\Pi_1\dot{\Pi}_1(\Pi_2^2 + \Pi_3^2) + I_2\Pi_2\dot{\Pi}_2(\Pi_3^2 + \Pi_1^2) + I_3\Pi_3\dot{\Pi}_3(\Pi_1^2 + \Pi_2^2)$ is allowed according to $\varphi = \Pi_2\Pi_3\,(I_2 - I_3)\frac{\partial}{\partial \Pi_1} + \Pi_3\Pi_1\,(I_3 - I_1)\frac{\partial}{\partial \Pi_2} + \Pi_1\Pi_2\,(I_1 - I_2)\frac{\partial}{\partial \Pi_1} \in \ker F$.

Example 3. Let $\Omega = d\omega$ be a symplectic form on M, $X_f \in \mathcal{X}(M)$ be the Hamiltonian vector field of an $f \in \mathcal{F}(M)$. Then the totally singular Lagrangian $L(x^i, y^i) = y^i\omega_i + f$ is allowed and the solutions of the Euler equation of this Lagrangian are the integral curves of X_f. A particular case is contained in Example 1.

Example 4. Let $X \in \mathcal{X}(D)$, $f \in \mathcal{F}(D)$, where $D \subset I\!R^m$ is open , $m \geq 2$, $X_x \neq 0$, $df_x \neq 0$, $(\forall)x \in D$, and $X(f) = 0$ on D. Let us say that f is a *critical function* for X.

It can be easily proved that every point $x_0 \in D$ has a neighborhood $D_0 \subset D$ such that there are local coordinates $(z^i)_{i=\overline{1,m}}$ on D_0 such that $X = \frac{\partial}{\partial z^2}$ and $f(z^i) = z^1$. Let us suppose that $D = D_0$ and define $\omega \in \mathcal{X}^*(D)$, $\omega = z^1 dz^2 + \tilde{\omega}$, where $\tilde{\omega} \in \mathcal{X}^*(D)$ has the form $\pi^*\omega_1$, $\pi : D \to D_1$, $(z^i) \xrightarrow{\pi} (z^3, \ldots, z^m)$, $D_1 \subset I\!R^{m-2}$ is open and $\omega_1 \in \mathcal{X}^*(D_1)$. We have obviously $i_X d\omega = df$, thus the totally singular Lagrangian $L(x^i,\ y^i) = y^i\omega_i + f$ is allowed. When $m = 2n$ we can consider $\omega = \sum\limits_{\alpha=1}^{n} z^{2\alpha-1} dz^{2\alpha}$; in this case $d\omega$ is symplectic and the totally singular Lagrangian $L(z^i, w^i) = \sum\limits_{\alpha=1}^{n} z^{2\alpha-1} w^{2\alpha} + z^1$ is regular, thus it is allowed with the corresponding vector field X.

Example 5. The totally singular Lagrangian $L(x^1, x^2, y^1, y^2) = e^{x^1} y^2$ on $I\!R^2$ is regular.

Example 6. The Hamiltonian $H(x^1, x^2, p_1, p_2) = e^{x^1} p_2$ on $I\!R^2$ is totally singular. The corresponding vector field is $\varphi = e^{x^1} \frac{\partial}{\partial y^2}$. If we look for a differential form $\alpha = \alpha_1(x^1, x^2)dx^1 + \alpha_2(x^1, x^2)dx^2$ such that the Lie derivative $L_\varphi \alpha = 0$, we obtain that $\alpha_1 = -x^2 f(x^1) + c$ and $\alpha_2 = f(x^1)$. Then $d\alpha = (f(x^1) + f'(x^1))dx^1 \wedge dx^2$. The condition that α be non-degenerated is $f + f' \neq 0$.

3 Second order Hamiltonians and Lagrangians

A coordinate construction of T^kM, $k \geq 2$, can be found in [7]-[10], or in [13] most used in this paper. The adapted coordinates on T^2M on the domain of an adapted chart are $(x^i,\ y^i = y^{(1)i},\ y^{(2)i})$; they change according to the rules $x^{i'} = x^{i'}(x^i)$, $y^{(1)i'} = \frac{\partial x^{i'}}{\partial x^i} y^{(1)i}$, $y^{(2)i'} = \frac{1}{2} \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j} y^{(1)i} y^{(1)j} + \frac{\partial x^{i'}}{\partial x^i} y^{(2)i}$. A section $S : TM \to T^2M$ of the affine bundle $T^2M \xrightarrow{\pi_2} TM$ is called a (first order) *semi-spray* on TM. It can be regarded as well as a vector field Γ_0 on the manifold TM , since $T^2M \subset TTM$. The total space of the induced vector bundle $\pi_1^* T^*M$ is also the total space of a fibered manifold $(TM \times_M T^*M, r_2, M)$. It is used in [8, 13] in the study of the dual geometrical objects of second order on M, in particular the second order Hamiltonians on M. In the sequel we denote $\pi_1^* T^*M = T^{2*}M$, considered as a vector bundle over TM. The tensors defined on the fibers of the vertical vector bundle $V_1^2 M \to T^2M$ of the affine bundle $T^2M \xrightarrow{\pi_2} TM$, or on the fibers of an open fibered submanifold, are called *d-tensors* of second order on M. For example, *d-vector fields* of second order are sections of $V_1^2 M \to T^2M$, or, equivalently, of $T^2M \times_M TM = \pi_2^* TM \to T^2M$; a *d-covector field of second order* (or a *second order d-form* on T^2M, for short) is a section of $T^2M \times_M T^*M = \pi_2^* T^*M \to T^2M$; a *bilinear d-form* on T^2M is a section of $T^2M \times_M (T^*M \otimes T^*M) = \pi_2^*(T^*M \otimes T^*M) \to T^2M$; and so on.

A *Lagrangian of second order* on M is a differentiable function $L : T^2M \to I\!R$ or $L : W \to M$, where $W \subset T^2M$ is an open subfibered manifold. For example, in [7]-[10] $W = \widetilde{T^2M}$ is $T^2M \backslash \{0\}$ (where $\{0\}$ is the image of the „null" velocities) and $L : T^2M \to I\!R$ is continuous. If $u \in TM$, then the fiber $T_u^2 M = \pi_2^{-1}(u) \subset T^2M$ is a real affine space, modeled on the real vector space $T_{\pi(u)}M$. The *vectorial dual* of the affine space $T_u^2 M$ is the vector space of affine morphisms $T_u^2 M^\dagger = Aff(T_u^2 M, I\!R)$. Denoting by $T^2M^\dagger = \bigcup_{u\in T^{k-1}M} T^2_uM^\dagger$ and $\pi^\dagger : T^2M^\dagger \to TM$ the canonical projection, then $(T^2M^\dagger, \pi^\dagger, TM)$ is a vector bundle. There is a canonical vector bundle epimorphism over the base TM, $\Pi : T^2M^\dagger \to T^{2*}M$. This projection is also the canonical projection of an affine bundle with the fiber $I\!R$. An *affine Hamiltonian* of second order on M is a section $h : T^{2*}M \to T^2M^\dagger$ of this affine bundle (or of an open fibered submanifold $W \subset T^{2*}M$), i.e. $\Pi \circ h = 1_{T^{2*}M}$ (or $\Pi \circ h = 1_W$ respectively). Thus an affine Hamiltonian is not a real function, but a section in an affine bundle with a one dimensional fiber.

The definition of a affine Hamiltonian was first considered in [13], in order to create a canonical duality Lagrangian - Hamiltonian, via a Legendre map. The Hamiltonians considered in [9] are called in [13] as em vectorial Hamiltonians; they are not in a canonical duality with Lagrangians of higher order (see [13] for more details).

We consider some local coordinates (x^i) on M, (x^i, y^i) on TM, and (x^i, y^i, p_i, T) on $T^{2\dagger}M$. The vector bundle morphism Π is given in local coordinates by $(x^i, y^i, p_i, T) \xrightarrow{\Pi} (x^i, y^i, p_i)$. The coordinates p_i and T change according to the rules $p_{i'} = \dfrac{\partial x^i}{\partial x^{i'}} p_i$ and $T' = T + \frac{1}{2} y^j \frac{\partial y^{i'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} p_i$ respectively. Thus the local function H_0 changes according to the rule

$$H_0'(x^{i'}, y^{i'}, p_{i'}) = H_0(x^i, y^i, p_i) + \frac{1}{2} y^j y^k \frac{\partial^2 x^{i'}}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial x^{i'}} p_i. \tag{3.1}$$

The corresponding map $h : T^{2*}M \to T^{2\dagger}M$ has the local form $h(x^i, y^i, p_i) = (x^i, y^i, p_i, H_0(x^i, y^i, p_i))$. It is easy to see that $\frac{\partial H_0'}{\partial p_{i'}} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial H_0}{\partial p_i} + \frac{1}{2} y^j y^k \frac{\partial^2 x^{i'}}{\partial x^j \partial x^k}$. Thus there is a *Legendre** *map* $\mathcal{H} : T^{2*}M \to T^2M$, given in local coordinates by $(x^i, y^i, p_i) \xrightarrow{\mathcal{H}} (x^i, y^i, \mathcal{H}^i)$, $\mathcal{H}^i(x^i, y^i, p_i) = \frac{\partial H_0}{\partial p_i}(x^i, y^i, p_i)$. It is easy to see that $h^{ij} = \frac{\partial^2 H_0}{\partial p_i \partial p_j}$ is a symmetric bilinear d-form on TM, which we call the *vertical Hessian* of h. For an affine Hamiltonian h of second order and the local domain U, the energy $\mathcal{E}_U$ is given by the local formula $\mathcal{E}_U = p_{(0)i} y^i + 2H_0(x^i, y^i, p_{(1)i})$. It can be proved (see [13]) that the local functions $\mathcal{E}_U$ glue together in a global function $\mathcal{E}_0 : \widetilde{T^*T^1M} \to I\!R$. We say that an affine Hamiltonian is *totally singular* if its vertical Hessian is null. A totally singular affine Hamiltonian h of second order on M has a local form

$$H_0(x^j, y^i, p_i) = p_i S^i(x^j, y^j) + f(x^j, y^j), \tag{3.2}$$

where (S^i) defines an affine section $S : TM \to T^2M$ given locally by $(x^i, y^i) \xrightarrow{S} (x^i, y^i, S^i)$ and $f \in \mathcal{F}(TM)$. It defines also a (first order) semi-spray, denoted by $\Gamma_0 \in \mathcal{X}(TM)$, given locally by the formula

$$\Gamma_0 = y^i \frac{\partial}{\partial x^i} + 2S^i \frac{\partial}{\partial y^i}, \tag{3.3}$$

that we call the *associated* semi-spray of h.

For example, let $\Gamma_0 = y^{(1)i} \frac{\partial}{\partial x^i} + 2S^i \frac{\partial}{\partial y^{(1)i}}$ be the local form of a first order semi-spray. Then the formula $H_0(x^i, p_i) = S^i p_i$ defines a totally singular affine Hamiltonian of second order. In particular, one Γ_0 can be the first order semi-spray defined by a regular Lagrangian $L_0 : TM \to I\!R$ (see [7]).

Given a totally singular affine Hamiltonian h having the local form 3.2, its energy $\mathcal{E} = p_{(0)i}y^{(1)i} + 2p_{(1)i}S^i + 2f$ is a totally singular Hamiltonian $\mathcal{E} : TM \to T^*TM$. Let us look for a totally singular Lagrangian on TM, that is in duality with $\mathcal{E}$. The vector field on TM, that corresponds to $\mathcal{E}$ is $\varphi = \Gamma_0 = y^{(1)i}\frac{\partial}{\partial x^i} + 2S^i\frac{\partial}{\partial y^{(1)i}}$, the first order semi-spray defined by h. We denote $\varphi^{(0)i} = y^{(1)i}$, $\varphi^{(1)i} = 2S^i(x^j, y^{(1)j})$ and $\gamma = f$. The system (2.5), where additionally $\alpha := \bar{\alpha}$, with

$$\bar{\alpha} = \alpha_{(0)i}dx^i + \alpha_{(1)i}dy^{(1)i}, \tag{3.4}$$

becomes: $2\frac{\partial S^j}{\partial x^i}\alpha_{(1)j} + 2\frac{\partial f}{\partial x^i} + \Gamma_0(\alpha_{(0)j}) = 0$, $\alpha_{(0)i} + 2\frac{\partial S^j}{\partial y^{(1)i}}\alpha_{(1)j} + 2\frac{\partial f}{\partial y^{(1)i}} + \Gamma_0(\alpha_{(1)j}) = 0$,

Denoting by $H_0(x^i, y^{(1)j}, p_{(1)j}) = S^i p_{(1)i} + f$, then this system of partial differential equations becomes:

$$\begin{cases} 2\frac{\partial H_0}{\partial x^i}(x^i, y^{(1)j}, \alpha_{(1)j}) + \Gamma_0(\alpha_{(0)j}) = 0, \\ \alpha_{(0)i} + 2\frac{\partial H_0}{\partial y^{(1)i}} + \Gamma_0(\alpha_{(1)j}) = 0, \end{cases} \tag{3.5}$$

Taking $\alpha_{(0)i}$ from the second equation, the first equation becomes

$$\Gamma_0(\alpha_{(1)i}) = 2\frac{\partial H_0}{\partial x^i}(x^i, y^{(1)i}, \alpha_{(1)i}). \tag{3.6}$$

Let us denote by $F_i \in \mathcal{F}(T^{2*}M)$ the right side of this equation. Let $\{z^\alpha\}_{\alpha=\overline{1,3m}}$ be a system of local coordinates on the manifold $T^{2*}M$ such that $\Gamma_0 = \frac{\partial}{\partial z^1}$. Then the local form of the differential equation (3.6) is $\frac{\partial^1 \alpha_{(1)i}}{\partial z^1} = F_i(z^\alpha, \alpha_{(1)i})$. It is obvious that this differential equation has local solutions.

There are canonical projections $T^{2\dagger}M \xrightarrow{\pi} T^{2*}M = TM \times_M T^*M \xrightarrow{p_1} TM$. If $\alpha = (\alpha_i(x^j, y^{(1)j}))$ is a d-form on TM, let $\alpha' : TM \to T^{2*}M = T^1M \times_M T^*M$ be the map defined by $\alpha'(z) = (z, \alpha_z)$. We say that a map $h_\alpha : TM \to T^{2\dagger}M$ is an *α-Hamiltonian* if $\pi \circ h_\alpha = \alpha'$. Using local coordinates, the local form of h_α is $(x^j, y^{(1)j}) \xrightarrow{h} (x^j, y^{(1)j}, \alpha_i(x^j, y^{(1)j}), -h_0(x^j, y^{(1)j}))$ and the local functions h_0 change on the intersection of two coordinate charts according to the rule $2h'_0(x^{j'}, y^{(1)j'}) = 2h_0(x^j, y^{(1)j}) + \Gamma(y^{(1)i'})\alpha_{i'}$. For example, if $\chi : T^{2*}M \to T^{2\dagger}M$ is an affine Hamiltonian and $\alpha : TM \to TM \times_M T^*M$ is a d-form on TM, then $h_\alpha = \chi \circ \alpha'$ is an α-Hamiltonian.

Proposition 3.1. *Let*

$$L(x^j, y^{(1)j}, y^{(2)j}) = 2y^{(2)i}\alpha_i(x^j, y^{(1)j}) - 2h_0(x^j, y^{(1)j}), \tag{3.7}$$

where $\alpha = (\alpha_i)$ is a d-form on TM and h_0 is the local form an α-Hamiltonian h_α. Then $L \in \mathcal{F}(T^2M)$.

A second order Lagrangian $L : T^2M \to I\!R$ gives rise to an integral action $I(\gamma) = \int_0^1 L\left(x^i, \frac{dx^i}{dt}, \frac{1}{2}\frac{d^2x^i}{dt^2}\right)dt$ on curves $\gamma : [0,1] \to M$, $t \xrightarrow{\gamma} (x^i(t))$, $\gamma^{(1)}(0) = x_0^{(1)}$, $\gamma^{(1)}(1) = x_1^{(1)}$. The critical curves of this action, obtained by a variational condition, verify the well-known Lagrange equation of second order (see, for example, [7]):

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^{(1)i}} + \frac{1}{2}\frac{d^2}{dt^2}\frac{\partial L}{\partial y^{(2)i}} = 0. \tag{3.8}$$

The dual counterpart of this action is the *integral action* of an affine Hamiltonian h on a curve $\gamma : [0,1] \to T^*M$, using the formula:

$$I(\gamma) = \int_0^1 \left[p_i \frac{d^2x^i}{dt^2} - 2H_0\left(x^i, \frac{dx^i}{dt}, p_i\right)\right] dt, \tag{3.9}$$

where γ has the local form $t \xrightarrow{\gamma} (x^i(t), p_i(t))$. The critical condition (or Fermat condition in the case of an extremum) can be used for the integral action; one obtains the Hamilton equation in the condensed form:

$$\begin{cases} \frac{1}{2}\frac{d^2p_i}{dt^2} - \frac{\partial H_0}{\partial x^i} + \frac{d}{dt}\frac{\partial H_0}{\partial y^{(1)i}} = 0, \\ \frac{1}{2}\frac{d^2x^i}{dt^2} - \frac{\partial H_0}{\partial p_i} = 0. \end{cases} \tag{3.10}$$

(See [13] for more details.)

Let h and L be totally singular of second order, having the local forms (3.2) and (3.7) respectively. Then a d-form on TM, given locally by $\alpha = (\alpha_i(x^j, y^{(1)j}))$, corresponds to L and a first order semi-spray Γ_0, given by formula (3.3), corresponds to h. We say that L is in *duality* with h if the formula (3.6) holds, with $\alpha_{(1)i} = \alpha_i$ and $h_\alpha = h \circ \alpha'$ (i.e. the α-Hamiltonian h_α corresponds to h and α). Using relation (3.7), one can prove the following fact.

Lemma 3.2. *If L is in duality with h, then $\frac{\partial H_0}{\partial x^i}(x^i,\, y^{(1)i}, \alpha_i) = -\frac{\partial L}{\partial x^i}(x^i,\, y^{(1)i},\, S^i)$ and $\frac{\partial H_0}{\partial y^{(a)i}}(x^i,\, y^{(1)i},\, \alpha_i) = -\frac{\partial L}{\partial y^{(a)i}}(x^i,\, y^{(1)i},\, S^i)$.*

We say also that a totally singular Lagrangian L of second order is *allowed* if there is a first order semi-spray Γ_0 and a d-form $\alpha = (\alpha_i)$ such that the following formula holds $\Gamma_0(\alpha_i) = -2\frac{\partial L}{\partial x^i}(x^i, y^{(1)i}, \alpha_i) + 2\Gamma_0\left(\frac{\partial L}{\partial y^{(1)i}}\right)$. Notice that for a second order Lagrangian, this condition is analogous to the condition (2.2), obtained in the first order case.

It is easy to see that a totally singular Lagrangian of second order is allowed if it is in duality with a totally singular Hamiltonian of second order. Thus a local dual of a totally singular Hamiltonian of second order is allowed. One can easily check the validity of the following result.

Proposition 3.3. *Let $\alpha = (\alpha_i(x^j, y^{(1)j}))$ be a d-form on TM and $h : TM \to T^{2\dagger}M$ be an* α-Hamiltonian *such that there is a* 1-*form $\bar{\alpha} \in \mathcal{X}^*(TM)$ such that α is the top component of $\bar{\alpha}$, i.e. $\bar{\alpha} = \alpha_{(0)i}dx^i + \alpha_{(1)i}dy^{(1)i}$, with $\alpha_{(1)i} = \alpha_i$. Then the formula*

$$L(x^j, y^{(1)j}, Y^{(0)i}, Y^{(1)i}) = (Y^{(0)i} - y^{(1)i})\alpha_{(0)i} + Y^{(1)i}\alpha_{(1)i} - h \tag{3.11}$$

defines a totally singular Lagrangian $L : TTM \to \mathbb{R}$.

The restriction of L to T^2M, has the form $L_0(x^j, y^{(1)j}, y^{(2)j}) = y^{(2)i}\alpha_i - h$. Thus if a totally singular Lagrangian L_0 on T^2M has the property that $\alpha = (\alpha_i)$ is the top component of a 1-form α' on TM, then L_0 is the restriction to T^2M of a totally singular Lagrangian L on TM (since $T^2M \subset TTM$).

Let h be a totally singular Hamiltonian of second order on M, i.e. $H_0(x^j,\, y^{(1)j},\, p_i) = p_iS^i(x^j,\, y^{(1)j}) + f(x^j,\, y^{(1)j})$. We can consider a local 1-form $\bar{\alpha} = \alpha_{(0)i}dx^i + \alpha_{(1)i}dy^{(1)i}$ that is a solution of the system (3.5). Considering the d-form α on TM, defined by its top component, we can construct a totally singular Lagrangian of second order on M.

Theorem 3.4. *Let h be a totally singular affine Hamiltonian of second order. If the system (3.6) has a d-form $\alpha = (\alpha_{(1)i})$ on TM as a global solution, then there is a totally singular allowed Lagrangian, $L : TTM \to I\!R$ (on TM), such that:*

1. *The energy $\mathcal{E}$ of h is a dual Hamiltonian of L.*
2. *The restriction of L to $T^2M \subset TTM$ is an allowed totally singular Lagrangian $L_1 : T^2M \to I\!R$ (of second order on M).*
3. *The pairs (h, L_1) and $(\mathcal{E}, L)$ are each dual pairs.*

Notice that Theorem 3.4 can be adapted in the case when the d-form $\alpha = (\alpha_{(1)i})$ has a solution on an open fibered submanifold of $T^{2*}M \to TM$.

Every curve γ in M that has the local form $t \to (\gamma^i(t))$ lifts to a curve $\gamma^{(2)}$ in T^2M that has the local form $t \to (\gamma^i(t), \frac{d\gamma^i}{dt}(t), \frac{1}{2}\frac{d^2\gamma^i}{dt^2}(t))$, called the 2-*tangent* lift of γ.

Proposition 3.5. *Let $t \overset{\gamma_1}{\to} (\gamma_1^i(t), \gamma_1^{(1)i}(t))$ be an integral curve of the first order semi-spray Γ_0 that corresponds to h and L be a dual Lagrangian. Then:*

1. *The curve γ_1 is the tangent lift of a curve $t \overset{\gamma}{\to} (\gamma^i(t))$, i.e. $\gamma_1 = \gamma^{(1)}$.*
2. *The curve $t \overset{\gamma_2}{\to} (\gamma^i, \omega_i)$ in T^*M, where $\omega_i(t) = \alpha_i(\gamma^i(t), \frac{d\gamma}{dt}(t))$, is a solution of Hamilton equation of h.*
3. *The curve γ given by 1. is a solution of Euler equation of L.*

We remark in that follows that not all the solutions of the Lagrange equation of a totally singular Lagrangian of second order come from the integral curves of a first order semi-spray, but some come from the integral curves of a second order semi-spray.

Let $L : T^2M \to I\!R$ be a totally singular Lagrangian, i.e. $L(x^i, y^{(1)i}, y^{(2)i}) = 2y^{(2)i}\alpha_i(x^j, y^{(1)j}) - 2\beta(x^i, y^{(1)i})$. The Lagrange equation of L is $\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^{(1)i}} + \frac{1}{2!}\frac{d^2}{dt^2}\frac{\partial L}{\partial y^{(2)i}} = 0$.

Let us consider a curve on T^2M that is a solution of the Lagrange equation, that has the form $t \to (x^i(t), y^{(1)i}(t) = \frac{dx^i}{dt}, y^{(2)i}(t) = \frac{1}{2}\frac{dy^{(1)i}}{dt})$. By a straightforward computation one obtain

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^{(1)i}} + \frac{1}{2}\frac{d^2}{dt^2}\frac{\partial L}{\partial y^{(2)i}} =$$
$$2\frac{dy^{(2)j}}{dt}\left(\frac{\partial \alpha_i}{\partial y^{(1)j}} - \frac{\partial \alpha_j}{\partial y^{(1)i}}\right) + 2y^{(2)j}\left(\frac{\partial \alpha_j}{\partial x^i} - \frac{\partial^2 \alpha_j}{\partial x^k \partial y^{(1)i}}y^{(1)k} - \right.$$
$$2\frac{\partial^2 \alpha_j}{\partial y^{(1)k}\partial y^{(1)i}}y^{(2)k} + 2\frac{\partial^2 \beta}{\partial y^{(1)j}\partial y^{(1)i}} + 2\frac{\partial^2 \alpha_i}{\partial y^{(1)j}\partial x^k}y^{(1)k} +$$
$$\left. 2\frac{\partial^2 \alpha_i}{\partial y^{(1)k}\partial y^{(1)j}}y^{(2)k} + \frac{\partial \alpha_i}{\partial x^j}\right) - 2\frac{\partial \beta}{\partial x^i} + 2\frac{\partial^2 \beta}{\partial x^k \partial y^{(1)i}}y^{(1)k} + \frac{\partial^2 \alpha_i}{\partial x^k \partial x^j}y^{(1)k}y^{(1)j}.$$

First let us suppose that $\frac{\partial \alpha_i}{\partial y^{(1)j}} - \frac{\partial \alpha_j}{\partial y^{(1)i}} = 0$, i.e. locally there is a function α such that $\alpha_i = \frac{\partial \alpha}{\partial y^{(1)i}}$. In this case, since $\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^{(1)i}} + \frac{1}{2}\frac{d^2}{dt^2}\frac{\partial L}{\partial y^{(2)i}} =$

$$2y^{(2)j}\left(\frac{\partial \alpha_j}{\partial x^i} + \frac{\partial \alpha_i}{\partial x^j} + \frac{\partial^2 \alpha_i}{\partial y^{(1)j}\partial x^k}y^{(1)k} + \frac{\partial^2 \beta}{\partial y^{(1)j}\partial y^{(1)i}}\right) -$$
$$2\frac{\partial \beta}{\partial x^i} + 2\frac{\partial^2 \beta}{\partial x^k \partial y^{(1)i}}y^{(1)k} + \frac{\partial^2 \alpha_i}{\partial x^k \partial x^j}y^{(1)k}y^{(1)j}.$$

It follows that along a curve that is a lift of a solution of the Euler equation, $\{y^{(2)j}\}$ is a solution of an algebraic system of equations.

Let us denote by $g_{ij} = \frac{\partial\alpha_j}{\partial x^i} + \frac{\partial\alpha_i}{\partial x^j} + \frac{\partial^2\alpha_i}{\partial y^{(1)j}\partial x^k}y^{(1)k} + \frac{\partial^2\beta}{\partial y^{(1)j}\partial y^{(1)i}}$, by $f_i = 2\frac{\partial\beta}{\partial x^i} - 2\frac{\partial^2\beta}{\partial x^k\partial y^{(1)i}}y^{(1)k} - \frac{\partial^2\alpha_i}{\partial x^k\partial x^j}y^{(1)k}y^{(1)j}$ and let $U \subset TM$ be their domain of definition.

Proposition 3.6. *Let us suppose that there is a local vector field $S = y^i\frac{\partial}{\partial x^i} + S^i(x^j, y^{(1)j})\frac{\partial}{\partial y^{(1)i}} : U \to TTM$ such that $g_{ij}S^j = f_j$. Then the image $U' = S(U)$ is included in T^2M and the image of an integral curve of S has the form $\gamma^{(2)}$, where γ is a solution of the Euler equation of the Lagrangian L.*

Corollary 3.7. *Let us suppose that the local matrix (g_{ij}) is invertible and $(g^{ij}) = (g_{ij})^{-1}$. Then, considering the local vector field $S = y^i\frac{\partial}{\partial x^i} + g^{jk}f_k\frac{\partial}{\partial y^{(1)i}} : U \to TTM$, the image $U' = S(U)$ is included in T^2M and the image of an integral curve of S has the form $\gamma^{(2)}$, where γ is a solution of the Euler equation of the Lagrangian L.*

Notice that the above result is related with [4, Sect. 5].

An other interesting situation arises when the d-tensor $\bar{\alpha}_{ij} = \frac{\partial\alpha_j}{\partial y^{(1)i}} - \frac{\partial\alpha_i}{\partial y^{(1)j}}$ is non-degenerated. In this case we look for a second order semi-spray, i.e. a vector field $X \in \mathcal{X}(T^2M)$, $X : T^2M \to T^3M \subset TT^2M$, having the local form $X = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + 3f^i(x^i, y^{(1)i}, y^{(2)i})\frac{\partial}{\partial y^{(2)i}} = \Gamma + 3f^i\frac{\partial}{\partial y^{(2)i}}$, such that the Euler equation holds along its integral curves. Let us consider an integral curve of X. Then along the curve we have $\frac{d}{dt} = X$ and

$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^{(1)i}} + \frac{1}{2}\frac{d^2}{dt^2}\frac{\partial L}{\partial y^{(2)i}} = 2y^{(2)j}\frac{\partial\alpha_j}{\partial x^i} - 2\frac{\partial\beta}{\partial x^i} - 6f^j\frac{\partial\alpha_j}{\partial y^{(1)i}} -$
$2y^{(2)j}\Gamma\left(\frac{\partial\alpha_j}{\partial y^{(1)i}}\right) + 2\Gamma\left(\frac{\partial\beta}{\partial y^{(1)i}}\right) + \frac{1}{2}\Gamma^2(\alpha_i) + 6f^j\frac{\partial\alpha_i}{\partial y^{(1)j}} =$
$\frac{\partial L}{\partial x^i} - \Gamma(\frac{\partial L}{\partial y^{(1)i}}) + \frac{1}{2}\Gamma^2(\alpha_i) - 6f^j(\frac{\partial\alpha_j}{\partial y^{(1)i}} - \frac{\partial\alpha_i}{\partial y^{(1)j}})$. If we suppose that the d-tensor $\bar{\alpha}_{ij} = \frac{\partial\alpha_j}{\partial y^{(1)i}} - \frac{\partial\alpha_i}{\partial y^{(1)j}}$ is non-degenerated and we denote by $(\bar{\alpha}^{ij}) = (\bar{\alpha}_{ij})^{-1}$, then $f^j(x^i, y^{(1)i}, y^{(2)i}) = \frac{1}{6}\bar{\alpha}^{ij}\left(\frac{\partial L}{\partial x^i} - \Gamma\left(\frac{\partial L}{\partial y^{(1)i}}\right) + \Gamma^2(\alpha_i)\right) =$
$\frac{1}{6}\bar{\alpha}^{ij}\left(\frac{\partial L}{\partial x^i} - \Gamma\left(\frac{\partial L}{\partial y^{(1)i}}\right) + \frac{1}{2}\Gamma^2\left(\frac{\partial L}{\partial y^{(2)i}}\right)\right)$.

In order to have consistent examples of totally singular Lagrangians and affine Hamiltonians of second order, we give an algorithm for lifting an allowed non-degenerated non-singular Lagrangian of first order, to an allowed non-singular Lagrangian of second order, also non-degenerated.

We recall that if $\bar{\alpha} \in \mathcal{X}^*(TM)$ has the local expression $\bar{\alpha} = \alpha_{(0)i}dx^i + \alpha_{(1)i}dy^{(1)i}$, then the d-form α defined by $(\alpha_{(1)i})$ is called its *top component*. We say that the d-form α on TM is *non-degenerated* if the matrix $\left(\alpha_{ij} = \frac{\partial\alpha_i}{\partial y^{(1)j}}\right)_{i,j=\overline{1,m}}$ is non-degenerate in every point of TM of coordinates $(x^j, y^{(1)j})$. We denote $(\alpha^{ij}) = (\alpha_{ij})^{-1}$. Notice that the condition does not depend on coordinates. We say that the d-form α on TM is *s-non-degenerated* (the initial s comes from skew-symmetric) if the matrix

$$\left(\tilde{\alpha}_{ij} = \frac{\partial\alpha_i}{\partial y^{(1)j}} - \frac{\partial\alpha_j}{\partial y^{(1)i}}\right)_{i,j=\overline{1,m}} \tag{3.12}$$

is non-degenerate in every point of TM of coordinates $(x^j, y^{(1)j})$. We denote $(\tilde{\alpha}^{ij}) = (\tilde{\alpha}_{ij})^{-1}$. Notice that the condition does not depend on coordinates.

Let $L : TM \to I\!R$, $L(x^i, y^{(1)i}) = \alpha_i(x^j)y^{(1)i} + \beta(x^j)$, be the general form of a totally singular Lagrangian, where $\alpha \in \mathcal{X}^*(M)$, $\beta \in \mathcal{F}(M)$. We denote by $\alpha_{ij} = (d\alpha)_{ij} = \frac{\partial\alpha_i}{\partial x^j} - \frac{\partial\alpha_j}{\partial x^i}$, $\bar{\omega}_i = y^{(1)j}\alpha_{ji}$ and $\bar{\theta}_i = \frac{1}{2}y^{(1)j}\frac{\partial\bar{\omega}_j}{\partial x^i}$.

Lemma 3.8. *The couple* $(\bar{\omega}_i, \bar{\theta}_i)$ *defines a covector field* $\omega : TM \to T^*TM$.

Let us suppose that $d\alpha$ is non-degenerate, i.e. L is regular (in our sense). Using formula

$$2S^i = \alpha^{ij} \left(\Gamma \left(\alpha_j \right) - \beta_j \right), \tag{3.13}$$

we obtain a section $S : TM \to T^2M$. Then $\bar{L}(x^i, y^{(1)i}, y^{(2)i}) = (y^{(2)i} - S^i(x^j, y^{(1)j}))\bar{\omega}_i$ $+\beta(x^j)$ is a Lagrangian of order 2 on M that has a null Hessian, and it is s-non-degenerated, since $\frac{\partial \bar{\omega}_i}{\partial y^{(1)j}} - \frac{\partial \bar{\omega}_j}{\partial y^{(1)i}} = 2\alpha_{ji}$.

We say that the totally singular Lagrangian $\bar{L}$, of order 2, is the *lift* of the Lagrangian L; it is easy to see that $\bar{L}$ is s-non-degenerated.

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