Minimal lightlike hypersurfaces in $\mathbb{R}^4_2$ with integrable screen distribution

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Abstract. We give the necessary and sufficient condition for a lightlike hypersurface in $\mathbb{R}^4_2$ with integrable screen distribution to be minimal. Using the condition we can get many minimal lightlike hypersurfaces in $\mathbb{R}^4_2$ which are not totally geodesic.

M.S.C. 2000: 53C42, 53C50.
Key words: lightlike hypersurface, minimal, screen distribution, semi-Euclidean space.

1 Introduction

Let M be a submanifold in a semi-Riemannian manifold $(\bar{M}, \bar{g})$. If the induced metric $g = \bar{g}|_M$ is non-degenerate, then (M, g) becomes a semi-Riemannian manifold and it can be studied as a semi-Riemannian submanifold. When g is degenerate, (M, g) is called a lightlike submanifold, and many different situations appear (cf. [2]). In this case, the tangent bundle TM and the normal bundle $TM^\perp$ have a non-trivial intersection, which is called the radical distribution and denoted by $\mathrm{Rad}(TM)$. Then we may choose a (non-unique) semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, which is called the screen distribution and denoted by $S(TM)$.

In particular, in the case of lightlike hypersurfaces, the normal bundle $TM^\perp$ coincides with the radical distribution $\mathrm{Rad}(TM)$, and there exists a canonical transversal vector bundle $\mathrm{tr}(TM)$ corresponding to the screen distribution $S(TM)$ which is called the lightlike transversal vector bundle.

Recently, Bejan and Duggal [1] introduced the notion of minimal lightlike submanifolds. In the proof of Theorem 3.2 of [1], they implicitly show that a lightlike hypersurface M in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with integrable screen distribution $S(TM)$ is minimal if and only if the radical distribution $\mathrm{Rad}(TM)$ contains the mean curvature vector field of any leaf of $S(TM)$. But this statement is a general one, and not easy to use to give some examples of minimal lightlike hypersurfaces.

In this paper, we discuss minimal lightlike hypersurfaces in the 4-dimensional semi-Euclidean space $\mathbb{R}^4_2$ of index 2. We give the necessary and sufficient condition

Balkan Journal of Geometry and Its Applications, Vol.14, No.1, 2009, pp. 84-90.

for a lightlike hypersurface in $\mathbb{R}^4_2$ with integrable screen distribution to be minimal, as follows:

Theorem 1.1. *Let* $(M, g, S(TM))$ *be a lightlike hypersurface in* $\mathbb{R}^4_2$ *with integrable screen distribution* $S(TM)$*. Then* M *is minimal if and only if the eigenvalues of the shape operator of any leaf of* $S(TM)$ *in the direction of the lightlike transversal vector bundle are both zero.*

The necessary and sufficient condition in Theorem 1.1 seems stronger than that from [1, Th.3.2], but they are equivalent in our case, as we will see in Section 3. And in Section 4, using the discussion in the proof of Theorem 1.1, we give a class of minimal lightlike hypersurfaces in $\mathbb{R}^4_2$ which are not totally geodesic.

2 Preliminaries

In this section, following [2] and [1], we recall some basic facts on lightlike hypersurfaces.

Let $\bar{M}$ be a semi-Riemannian manifold with metric $\bar{g}$ and Levi-Civita connection $\bar{\nabla}$. Let M be a lightlike hypersurface in $\bar{M}$, that is, the induce metric $g = \bar{g}|_M$ is degenerate. In the case of lightlike hypersurfaces, the normal bundle $TM^\perp$ coincides with the radical distribution $\mathrm{Rad}(TM)$, defined by

$$\mathrm{Rad}(T_xM) = \{\xi \in T_xM | g(\xi, X) = 0, \quad X \in T_xM\},$$

where $\dim(\mathrm{Rad}(T_xM)) = 1$. There exists a screen distribution $S(TM)$ which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, that is,

$$TM = S(TM) \perp \mathrm{Rad}(TM) = S(TM) \perp TM^\perp.$$

We note that if $\bar{M}$ is of index q, then $S(TM)$ is of index $q-1$. If $S(TM)$ is integrable, then M is locally a product $L \times d$, where d is a null geodesic in $\bar{M}$ as an integral curve of $\mathrm{Rad}(TM)$ and L is a semi-Riemannian submanifold in $\bar{M}$ as a leaf of $S(TM)$.

From [2, p.79], we know that for a screen distribution $S(TM)$, there exists a unique vector bundle $\mathrm{tr}(TM)$ of rank 1 such that, for any non-zero local section ξ of $TM^\perp$ on U there is a unique section N of $\mathrm{tr}(TM)|_U$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, W) = 0$$

for all $W \in \Gamma(S(TM)|_U)$. This vector bundle $\mathrm{tr}(TM)$ is called the lightlike transversal vector bundle with respect to $S(TM)$, and we have the decomposition

$$T\bar{M}|_M = TM \oplus \mathrm{tr}(TM).$$

From now on, ξ denotes a non-zero local section of $\mathrm{Rad}(TM)$. According to the above decomposition, we have the Gauss formula

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N,$$

where $X, Y \in \Gamma(TM)$. Then ∇ is a torsion-free linear connnection on M, and B is a symmetric $C^\infty(M)$-bilinear form on $\Gamma(TM)$. This form B is called the local second fundamental form of M, which is independent of the choice of $S(TM)$. When $B = 0$, M is called totally geodesic.

Following the Definition 2 of [1] in the case of lightlike hypersurfaces, M is called minimal if $\mathrm{trace}(B) = 0$, where the trace is written with respect to g restricted to $S(TM)$. This condition is independent of the choice of $S(TM)$ and ξ.

3 Proof of Theorem 1.1

Proof of Theorem 1.1. Let $(M, g, S(TM))$ be a lightlike hypersurface in $\mathbb{R}^4_2$ with integrable screen distribution $S(TM)$. Then M is locally a product $L \times d$, where d is an open set of a lightlike line in $\mathbb{R}^4_2$ as an integral curve of $\mathrm{Rad}(TM)$ and L is a Lorentzian surface in $\mathbb{R}^4_2$ as a leaf of $S(TM)$.

Let L be an arbitrary leaf of $S(TM)$, and $f : L \to \mathbb{R}^4_2$ be the inclusion map. Along L, we choose a local frame field $\{e_1, e_2\}$ so that $\{f_*e_1, f_*e_2\}$ is orthonormal with signature $(+,-)$, and a local normal orthonormal frame field $\{e_3, e_4\}$ with signature $(+,-)$. Then we may assume that the inclusion map $F : M \to \mathbb{R}^4_2$ is given by

$$F(p,t) = f(p) + t(e_3(p) + e_4(p)), \quad p \in L, \quad t \in (-\varepsilon, \varepsilon).$$

We shall use the following ranges of indices:

$$1 \le A, B, ... \le 4, \quad 1 \le i, j, ... \le 2, \quad 3 \le \alpha, \beta, ... \le 4.$$

Let ω_A^B be the connection forms which satisfy

$$d(f_*e_i) = \sum_{j=1}^{2} \omega_i^j f_*e_j + \sum_{\alpha=3}^{4} \omega_i^\alpha e_\alpha, \quad de_\alpha = \sum_{i=1}^{2} \omega_\alpha^i f_*e_i + \sum_{\beta=3}^{4} \omega_\alpha^\beta e_\beta.$$

We note that, in our situation, $\omega_A^B = -\omega_B^A$ if $|A - B|$ is even, and $\omega_A^B = \omega_B^A$ if $|A - B|$ is odd. Then

$$de_3 = \omega_3^1 f_*e_1 + \omega_3^2 f_*e_2 + \omega_3^4 e_4 = -\omega_1^3 f_*e_1 + \omega_2^3 f_*e_2 + \omega_4^3 e_4,$$

$$de_4 = \omega_4^1 f_*e_1 + \omega_4^2 f_*e_2 + \omega_4^3 e_3 = \omega_1^4 f_*e_1 - \omega_2^4 f_*e_2 + \omega_4^3 e_3.$$

Let h_{ij}^α denote the components of the second fundamental form h of L, so that

$$\omega_i^\alpha = \sum_{j=1}^{2} h_{ij}^\alpha \omega^j,$$

where $\{\omega^1, \omega^2\}$ is the coframe field dual to $\{e_1, e_2\}$.

Set

$$\tilde{e}_i(p,t) = (e_i(p), 0) \in T_{(p,t)}M = T_pL \times T_td.$$

Then $\{\tilde{e}_1, \tilde{e}_2, \partial_t\}$ is a natural frame field on $M = L \times d$, and we obtain

$$F_*\tilde{e}_1 = (1 - tA_{11})f_*e_1 + tA_{12}f_*e_2 + t\omega^3_4(e_1)(e_3 + e_4),$$

$$F_*\tilde{e}_2 = -tA_{12}f_*e_1 + (1 + tA_{22})f_*e_2 + t\omega^3_4(e_2)(e_3 + e_4),$$

$$F_*\partial_t = e_3 + e_4 =: \xi,$$

where we set

$$A_{ij} = h^3_{ij} - h^4_{ij}.$$

As the induced metric g is given by

$$g(X, Y) = \langle F_*X, F_*Y\rangle, \quad X, Y \in \Gamma(TM),$$

we have the components of g by

$$g(\tilde{e}_1, \tilde{e}_1) = 1 - 2tA_{11} + t^2(A^2_{11} - A^2_{12}),$$

$$g(\tilde{e}_2, \tilde{e}_2) = -1 - 2tA_{22} + t^2(A^2_{12} - A^2_{22}),$$

$$g(\tilde{e}_1, \tilde{e}_2) = -2tA_{12} + t^2A_{12}(A_{11} - A_{22}),$$

$$g(\tilde{e}_1, \partial_t) = g(\tilde{e}_2, \partial_t) = g(\partial_t, \partial_t) = 0.$$

Thus, for sufficiently small t, M is a lightlike hypersurface, and ξ is a non-zero local section of the radical distribution $\mathrm{Rad}(TM)$.

We choose the screen distribution $S(TM)$ so that it is spanned by $\{F_*\tilde{e}_1, F_*\tilde{e}_2\}$. Then, for each t, the map $F_t : L \to \mathbb{R}^4_2$ defined by $F_t(p) = F(p, t)$ becomes an inclusion map of a leaf of $S(TM)$. Let $\mathrm{tr}(TM)$ be the lightlike transversal vector bundle corresponding to $S(TM)$, and N be the local section of $\mathrm{tr}(TM)$ which corresponds to ξ as in Section 2. Then A_{ij} are the components of the second fundamental form h of L in the direction N.

With respect to the local second fundamental form B, we may obtain

$$B(\tilde{e}_1, \tilde{e}_1) = \langle D_{\tilde{e}_1}F_*\tilde{e}_1,\ \xi\rangle = A_{11} - t(A^2_{11} - A^2_{12}),$$

$$B(\tilde{e}_1, \tilde{e}_2) = \langle D_{\tilde{e}_2}F_*\tilde{e}_1,\ \xi\rangle = A_{12} - tA_{12}(A_{11} - A_{22}),$$

$$B(\tilde{e}_2, \tilde{e}_2) = \langle D_{\tilde{e}_2}F_*\tilde{e}_2,\ \xi\rangle = A_{22} - t(A^2_{12} - A^2_{22}),$$

where D is the induced connection from the flat connection on $\mathbb{R}^4_2$. By the definition, M is minimal if and only if $\mathrm{trace}(B) = 0$ where the trace is written with respect to g restricted to $S(TM)$, which is now equivalent to that

$$g(\tilde{e}_2, \tilde{e}_2)B(\tilde{e}_1, \tilde{e}_1) - 2g(\tilde{e}_1, \tilde{e}_2)B(\tilde{e}_1, \tilde{e}_2) + g(\tilde{e}_1, \tilde{e}_1)B(\tilde{e}_2, \tilde{e}_2) = 0. \tag{3.1}$$

It is a cubic identity for t, and is equivalent to that

$$A_{11} = A_{22}, \quad A^2_{11} = A^2_{12}. \tag{3.2}$$

Let us consider

$$A_i{}^j = (h^3)_i{}^j - (h^4)_i{}^j,$$

which are the components of the shape operator of L in the direction N. Noting that

$$A_1{}^1 = A_{11}, \quad A_2{}^1 = A_{21} = A_{12}, \quad A_1{}^2 = -A_{12}, \quad A_2{}^2 = -A_{22},$$

we can see that the condition (3.2) is equivalent to that the trace and the determinant of $(A_i{}^j)$ are both zero, which is also equivalent to that the eigenvalues of $(A_i{}^j)$ are both zero. Thus we have proved the theorem. $\square$

Let M be a lightlike hypersurface in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with integrable $S(TM)$. As noted in the introduction, the proof of [1, Th.3.2] implies that M is minimal if and only if $(*)$ "Rad(TM) contains the mean curvature vector field of any leaf of $S(TM)$". So the necessary and sufficient condition in Theorem 1.1 seems stronger than the above condition $(*)$, but they are equivalent in our case, as we have shown. We note that the condition $(*)$ corresponds to the equation (3.1), which is equivalent to (3.2) for an arbitrary leaf. Namely, for a lightlike hypersurface M in $\mathbb{R}^4_2$ with integrable $S(TM)$, if Rad(TM) contains the mean curvature vector field of any leaf of $S(TM)$, then the leaves must satisfy another condition.

4 A class of minimal lightlike hypersurfaces

In this section, using the discussion in Section 3, we give a class of minimal lightlike hypersurfaces in $\mathbb{R}^4_2$ which are not totally geodesic. In fact, we give a class of Lorentzian surfaces in $\mathbb{R}^4_2$ which satisfy the condition (3.2).

Let $\{x_1, x_2, x_3, x_4\}$ be the standard coordinate system for $\mathbb{R}^4_2$ with metric

$$ds^2 = dx_1^2 - dx_2^2 + dx_3^2 - dx_4^2.$$

Proposition 4.1. *Let $Q_1(z), Q_2(z), Q_3(z)$ and $Q_4(z)$ be smooth functions. Set*

$$f(u,v) = \begin{pmatrix} Q_1(u+v) + Q_2(u-v) \\ Q_1(u+v) - Q_2(u-v) \\ Q_3(u+v) + Q_4(u-v) \\ Q_3(u+v) - Q_4(u-v) \end{pmatrix},$$

and assume that

$$Q_1'(u+v)Q_2'(u-v) + Q_3'(u+v)Q_4'(u-v) > 0.$$

Then f gives a Lorentzian surface in $\mathbb{R}^4_2$ which satisfies the condition (3.2).

Proof. First we have

$$f_u = \begin{pmatrix} Q_1'(u+v) + Q_2'(u-v) \\ Q_1'(u+v) - Q_2'(u-v) \\ Q_3'(u+v) + Q_4'(u-v) \\ Q_3'(u+v) - Q_4'(u-v) \end{pmatrix}, \quad f_v = \begin{pmatrix} Q_1'(u+v) - Q_2'(u-v) \\ Q_1'(u+v) + Q_2'(u-v) \\ Q_3'(u+v) - Q_4'(u-v) \\ Q_3'(u+v) + Q_4'(u-v) \end{pmatrix},$$

and

$$\langle f_u, f_u\rangle = 4\{Q_1'(u+v)Q_2'(u-v) + Q_3'(u+v)Q_4'(u-v)\} =: E > 0,$$
$$\langle f_u, f_v\rangle = 0, \qquad \langle f_v, f_v\rangle = -E.$$

So f gives a Lorentzian surface in $\mathbb{R}^4_2$.

Set
$$e_1 = \frac{1}{\sqrt{E}}\frac{\partial}{\partial u}, \qquad e_2 = \frac{1}{\sqrt{E}}\frac{\partial}{\partial v}.$$

Then $\{e_1, e_2\}$ is an orthonormal frame field with signature $(+,-)$. Set

$$e_3 = \frac{1}{\sqrt{E}}\begin{pmatrix} -Q_3'(u+v) - Q_4'(u-v) \\ Q_3'(u+v) - Q_4'(u-v) \\ Q_1'(u+v) + Q_2'(u-v) \\ -Q_1'(u+v) + Q_2'(u-v) \end{pmatrix}, \quad e_4 = \frac{1}{\sqrt{E}}\begin{pmatrix} -Q_3'(u+v) + Q_4'(u-v) \\ Q_3'(u+v) + Q_4'(u-v) \\ Q_1'(u+v) - Q_2'(u-v) \\ -Q_1'(u+v) - Q_2'(u-v) \end{pmatrix}.$$

Then $\{e_3, e_4\}$ is a normal orthonormal frame field with signature $(+,-)$.

Let h^α_{ij} denote the components of the second fundamental form of f with respect to these frames. Then we can get

$$\begin{aligned} h^3_{11} &= \frac{1}{E}\langle f_{uu}, e_3\rangle \\ &= 2E^{-3/2}\{Q_1'(u+v)Q_3''(u+v) - Q_1''(u+v)Q_3'(u+v) + Q_2'(u-v)Q_4''(u-v) - Q_2''(u-v)Q_4'(u-v)\} \\ &= h^3_{22} = h^4_{12}, \end{aligned}$$

and

$$\begin{aligned} h^3_{12} &= h^4_{11} = h^4_{22} \\ &= 2E^{-3/2}\{Q_1'(u+v)Q_3''(u+v) - Q_1''(u+v)Q_3'(u+v) - Q_2'(u-v)Q_4''(u-v) + Q_2''(u-v)Q_4'(u-v)\}. \end{aligned}$$

Thus the condition (3.2) is satisfied, and we have proved the proposition. □

Remark 1 This construction is inspired by the previous paper [5] and the structure of complex curves in $\mathbb{R}^4 = C^2$.

By the discussion in the proof of Theorem 1.1, we get the following result

Theorem 4.2. *Let f, e_3, e_4 be as in Proposition 4.1. Then the map*

$$F(u, v, t) = f(u, v) + t(e_3 + e_4)$$

gives a minimal lightlike hypersurface in $\mathbb{R}^4_2$, which is not totally geodesic if

$$Q_2'(u-v)Q_4''(u-v) - Q_2''(u-v)Q_4'(u-v) \neq 0.$$

By Proposition 4.1 and Theorem 4.2, we can get many minimal lightlike hypersurfaces in $\mathbb{R}^4_2$ which are not totally geodesic. For example, when

$$Q_1(z) = Q_2(z) = z, \qquad Q_3(z) = Q_4(z) = e^z,$$

we have

$$E = 4\{Q_1'(u+v)Q_2'(u-v) + Q_3'(u+v)Q_4'(u-v)\} = 4(1 + e^{2u}) > 0,$$

and

$$Q_2'(u-v)Q_4''(u-v) - Q_2''(u-v)Q_4'(u-v) = e^{u-v} \neq 0.$$

We point out, that related information to this subject can be found in [3] and [4].

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