Compact Einstein Kaehler submanifolds of a complex projective space

Yoshio Matsuyama

Abstract. In the present paper we show that if M_n is a compact irreducible (resp. reducible) Einstein Kaehler submanifold of a complex projective space, then M is parallel and totally geodesic in $CP_{n+p}(c)$, $CP_n(\frac{c}{r}), p = {}_{n+r}\mathrm{C}_r - 1 - n$, the complex quadric $Q_n(c)$ in the totally geodesic submanifold $CP_{n+1}(c)$ of $CP_{n+p}(c)$, $SU(\frac{n}{2}+2)/SU(\frac{n}{2}) \times U(2), n > 6, p = \frac{n(n-6)}{8}$; $SU(10)/U(5), n = 10, p = 5$ or $E_6/Spin(10) \times S^1, n = 16, p = 10$ (resp. $P_{n_1}(c) \times P_{n_1}(c), n = 2n_1$).

M.S.C. 2000: 53C40, 53B25
Key words: complex projective space, Einstein Kaehler submanifold, second fundamental form, parallel submanifold.

1 Introduction

Let $P_{n+p}(c)$ (resp. D_{n+p}) be an $(n+p)$-dimensional complex projective space with the Fubini-Study metric (resp. Bergman metric) of constant holomorphic sectional curvature c. Let C_{n+p} be an $(n+p)$-dimensional complex Euclidean space. Let $M_{n+p}(c)$ be an $(n+p)$-dimensional complex space form with constant holomorphic sectional curvature c. We remark that if $c > 0$ (resp. $c < 0, c = 0$), then $M_{n+p}(c) = P_{n+p}(C)$ (resp. D_{n+p}, C_{n+p}). Let M_n be an n-dimensional complex Kaehler submanifold of $M_{n+p}(c)$. There are a number of conjecture for Kaehler submanifolds in $P_{n+p}(c)$ suggested by K. Ogiue ([5]); some have been resolved under a suitable topological restriction (e.g. M_n is complete) (cf. [1], [5], [6] and [7]). In this direction, one of the open problems so far is as follows.

Conjecture (K. Ogiue). Let M_n be an n-dimensional Kaehler submanifold immersed in $M_{n+p}(c), c > 0$. If M is irreducible (or Einstein) and if the second fundamental form is parallel, is M one of the following ? $M_n(c), M_n(\frac{c}{2})$ or locally the complex quadric $Q_n(c)$.

In the case that M_n is an Einstein Kaehler submanifold of the codimension two immersed in $P_{n+2}(c)$, it was proved in [1] and [6] that such a submanifold M_n is totally geodesic in $P_{n+2}(c)$ or the complex quadric $Q_n(c)$ in the totally geodesic hypersurface

Balkan Journal of Geometry and Its Applications, Vol.14, No.1, 2009, pp. 40-45.

$P_{n+1}(c)$ of $P_{n+2}(c)$. Moreover, if M_n is an Einstein Kaehler submanifolds immersed in a complex linear or hyperbolic space, then M_n is totally geodesic ([7]).

In the present paper we would like to consider that M_n is compact and Einstein, so that the above conjucture is resolved partially. The main result is the following:

Theorem *Let M_n be an n-dimensional compact irreducible (resp. reducible) Einstein Kaehler submanifold immersed in $P_{n+p}(c)$. Then M_n is parallel and totally geodesic in $P_{n+p}(c)$, $CP_n(\frac{c}{r}), p = {}_{n+r}\mathrm{C}_r - 1 - n$, the complex quadric $Q_n(c)$ in the totally geodesic submanifold $CP_{n+1}(c)$ of $CP_{n+p}(c)$, $SU(\frac{n}{2}+2)/SU(\frac{n}{2}) \times U(2), n > 6, p = \frac{n(n-6)}{8}$; $SU(10)/U(5), n = 10, p = 5$ or $E_6/Spin(10) \times S^1, n = 16, p = 10$ (resp. $P_{n_1}(c) \times P_{n_1}(c), n = 2n_1$).*

2 Preliminaries

Let M_n be an n-dimensional compact Riemannian manifold. We denote by UM the unit tangent bundle over M and by UM_x its fibre over $x \in M$. If dx, dv and dv_x denote the canonical measures on M, UM and UM_x, respectively, then for any continuous function $f : UM \to R$, we have:

$$\int_{UM} f dv = \int_M \{\int_{UM_x} f dv_x\} dx.$$

If T is a k-covariant tensor on M and ∇T is its covariant derivative, then we have:

$$\int_{UM} \{\sum_{i=1}^{n} (\nabla T)(e_i, e_i, v, \cdots, v)\} dv = 0,$$

where $e_1, \cdots, e_n$ is an orthonormal basis of $T_x M, x \in M$.

Now, we suppose that M_n is an n-dimensional compact Kaehler submanifold of complex dimension n, immersed in the complex projective space $P_{n+p}(c)$. We denote by J and $<,>$ the complex structure and the Fubini-Study metric. Let ∇ and h be the Riemannian connection and the second fundamental form of the immersion, respectively. A and $\nabla^{\perp}$ are the Weingarten endomorphism and the normal connection. The first and the second covariant derivatives of the normal valued tensor h are given by

$$(\nabla h)(X, Y, Z) = \nabla^{\perp}_X (h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

and

$$\begin{aligned}(\nabla^2 h)(X, Y, Z, W) &= \nabla^{\perp}_X ((\nabla h)(Y, Z, W)) - (\nabla h)(\nabla_X Y, Z, W) \\ &- (\nabla h)(Y, \nabla_X Z, W) - (\nabla h)(Y, Z, \nabla_X W),\end{aligned}$$

respectively, for any vector fields X, Y, Z and W tangent to M.

Let R and $R^{\perp}$ denote the curvature tensor associated with ∇ and $\nabla^{\perp}$, respectively. Then h and ∇h are symmetric and for $\nabla^2 h$ we have the Ricci-identity

$$\begin{aligned}&(\nabla^2 h)(X, Y, Z, W) - (\nabla^2 h)(Y, X, Z, W) \\ &= R^{\perp}(X, Y) h(Z, W) - h(R(X, Y) Z, W) - h(Z, R(X, Y) W).\end{aligned} \tag{2.1}$$

We also consider the relations

$$h(JX,Y)=Jh(X,Y) \text{ and } A_{J\xi}=JA_\xi=-A_\xi J,$$

where ξ is a normal vector to M_n.
Now, let $v\in UM_x, x\in M$. If $e_2,\dots,e_{2n}$ are orthonormal vectors in UM_x orthogonal to v, then we can consider $\{e_2,\dots,e_{2n}\}$ as an orthonormal basis of $T_v(UM_x)$. We remark that $\{v=e_1,e_2,\dots,e_{2n}\}$ is an orthonormal basis of T_xM. We denote the Laplacian of $UM_x\cong S^{2n-1}$ by Δ.
If S and ρ is the Ricci tensor of M and the scalar curvature of M, respectively, and since M is a complex Kaehler submanifold in in $P_{n+p}(c)$, then from the Gauss equation we have

$$S(v,w) = \frac{n+1}{2}c<v,w>-\sum_{i=1}^{2n}<A_{h(v,e_i)}e_i,w>, \tag{2.2}$$

$$\rho = n(n+1)c-|h|^2. \tag{2.3}$$

Define a function f_1 on $UM_x, x\in M$, by

$$f_1(v)=|h(v,v)|^2,$$

$$f_2(v)=\sum_{i=1}^{2n}<A_{h(v,e_i)}e_i,v>,$$

Using the minimality of M we can prove that

$$(\Delta f_1)(v) = -4(2n+2)f_1(v)^2+8\sum_{i=1}^{2n}<A_{h(v,e_i)}e_i,v> \tag{2.4}$$

$$(\Delta f_2)(v) = -4nf_2(v)+2|h|^2 \tag{2.5}$$

Since

$$\frac{1}{2}\sum_{i=1}^{2n}(\nabla^2 f_1)(e_i,e_i,v) = \sum_{i=1}^{2n}<(\nabla^2 h)(e_i,e_i,v,v),h(v,v)> +\sum_{i=1}^{2n}<(\nabla h)(e_i,v,v),(\nabla h)(e_i,v,v)>,$$

we have the following (See [2] and [3]):
Lemma *Let M be an n-dimensional complex Kaehler submanifold of $P_{n+p}(c)$. Then for $v\in UM_x$ we have*

$$\text{(2.6)}\qquad \frac{1}{2}\sum_{i=1}^{2n}(\nabla^2 f_1)(e_i,e_i,v) = \sum_{i=1}^{2n}|(\nabla h)(e_i,v,v)|^2+\frac{n+2}{2}c|h(v,v)|^2 +2\sum_{i=1}^{2n}< A_{h(v,v)}e_i, A_{h(e_i,v)}v > -2\sum_{i=1}^{2n}< A_{h(v,e_i)}e_i, A_{h(v,v)}v > -\sum_{i=1}^{2n}< A_{h(v,v)}e_i, A_{h(v,v)}e_i > .$$

3 Proof of Theorem

Since M is Einstein, we have

$$\text{(3.1)}\qquad S=\frac{n+1}{2}cI-\sum_{i=1}^{2n}A_{h(v,e_i)}e_i=\frac{\rho}{2n}I,$$

where I denotes the identity transformation. From (2.2) and (2.3) the equation (3.1) yields

$$\text{(3.2)}\qquad \sum_{i=1}^{2n}A_{h(v,e_i)}e_i=\frac{|h|^2}{2n}v.$$

We see the following equation holds for $v\in UM_x, x\in M$.

$$\text{(3.3)}\qquad \sum_{i=1}^{2n}< A_{h(Jv,Jv)}e_i, A_{h(e_i,Jv)}Jv >= -\sum_{i=1}^{2n}< A_{h(v,v)}e_i, A_{h(e_i,v)}v > .$$

From (3.3) we have

$$\text{(3.4)}\qquad \frac{1}{4}\sum_{i=1}^{2n}(\nabla^2 f_1)(e_i,e_i,v)+\frac{1}{4}\sum_{i=1}^{2n}(\nabla^2 f_1)(e_i,e_i,Jv) = \sum_{i=1}^{2n}|(\nabla h)(e_i,v,v)|^2+\frac{n+2}{2}c|h(v,v)|^2 -2\sum_{i=1}^{2n}< A_{h(v,e_i)}e_i, A_{h(v,v)}v > -\sum_{i=1}^{2n}< A_{h(v,v)}e_i, A_{h(v,v)}e_i > .$$

Integrating over UM_x and using (3.2) , we have

$$\text{(3.5)}\qquad \int_{UM_x}\sum_{i=1}^{2n}|(\nabla h)(e_i,v,v)|^2dv_x+\frac{n+2}{2}c\int_{UM_x}|h(v,v)|^2dv_x -\frac{1}{n}\int_{UM_x}|h|^2|h(v,v)|^2dv_x-\int_{UM_x}\sum_{i=1}^{2n}< A_{h(v,v)}e_i, A_{h(v,v)}e_i > dv_x=0.$$

On the other hand, we get

$$\int_{UM_x} |h(v,v)|^2 dv_x = \frac{2}{2n(2n+2)} \int_{UM_x} |h|^2 dv_x,$$

$$\int_{UM_x} \sum_{i=1}^{2n} < A_{h(v,v)} e_i, A_{h(v,v)} e_i > dv_x = \frac{2}{2n(2n+2)} \int_{UM_x} \sum (\mathrm{trace} A_{\xi_i} A_{\xi_j})^2 dv_x.$$

Hence we obtain

$$\begin{aligned} 0 &\geq \int_{UM_x} \sum_{i=1}^{2n} |(\nabla h)(e_i, v, v)|^2 dv_x \\ &+ \int_{UM_x} \{ \frac{n+2}{4n(n+1)} c|h|^2 - \frac{2}{4n^2(n+1)} |h|^4 - \frac{1}{4n(n+1)} |h|^4 \} dv_x \\ &= \int_{UM_x} \sum_{i=1}^{2n} |(\nabla h)(e_i, v, v)|^2 dv_x \\ &+ \frac{n+2}{2n(n+1)} \int_{UM_x} (S(v,v) - \frac{n}{2} c)|h|^2 dv_x, \end{aligned}$$

noting that $\sum (\mathrm{trace} A_{\xi_i} A_{\xi_j})^2 \leq \frac{1}{2}|h|^4$. Put

$$S'(v,v) = S(v,v) - \frac{n}{2} c$$

Let α be the 1-form on UM_x given by

$$\alpha_v(e) = S'(e,v)|h|^2$$

with $v \in UM_x$ and $e \in T_v UM_x$, we get

$$\begin{aligned} (\delta\alpha)(v) &= -S'(v,v)|h|^2 \\ &+ \sum_{i=1}^{2n} S'(e_i, e_i)|h|^2 \\ &= (2n-1) S'(v,v)|h|^2 \end{aligned}$$

Integrating this equation over UM_x, we have

$$0 = \int_{UM_x} S'(v,v)|h|^2 dv_x.$$

Hence we obtain

$$0 \geq \int_{UM_x} \sum_{i=1}^{2n} |(\nabla h)(e_i, v, v)|^2 dv_x.$$

Thus M is parallel.

References

[1] Y. Matsuyama, *On a* 2*-dimensional Einstein Kaehler submanifolds of a complex space form*, Proc. Amer. Math. Soc. 95 (1985), 595-603.

[2] Y. Matsuyama, *On totally real submanifolds of a complex projective space*, Nihonkai Math. J. 13 (2002), 153-157.

[3] S. Montiel, A. Ros and F. Urbano, *Curvature pinching and eigenvalue rigidity for minimal submanifolds*, Math. Z. 191 (1986), 537-548.

[4] H. Naitoh, *Totally real parallel submanifolds of* $P^n(C)$, Tokyo J. Math. 4 (1981), 279-306.

[5] K. Ogiue, *Differential geometry of Kaehler submanifolds*, Adv. Math. 13 (1974), 73-114.

[6] K. Tsukada, *Parallel Kaehler submanifolds of Hermitian symmetric spaces*, Math. Z. 190 (1985), 129-150.

[7] M. Umehara, *Kaehler submanifolds of complex space forms*, Tokyo J. Math. 10 (1987), 203-214.

Author's address:

Yoshio Matsuyama
Department of Mathematics,
Chuo University, 1-13-27 Kasuga,
Bunkyo-ku,Tokyo 112-8551, Japan.
E-mail address: matuyama@math.chuo-u.ac.jp