On $\mathbb{R}-$complex Finsler spaces

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Abstract. In the present paper we submit for study a new class of Finsler spaces. Through restricting the homogeneity condition from the definition of a complex Finsler metric to real scalars, $\lambda \in \mathbb{R}$, is obtained a wider class of complex spaces, called by us the $\mathbb{R}-$complex Finsler spaces. Two subclasses are taken in consideration: the Hermitian and the non-Hermitian $\mathbb{R}-$complex Finsler spaces. In an $\mathbb{R}-$complex Finsler space we determine a nonlinear connection from the variational problem similar as in the complex Lagrange geometry ([11]) for the Hermitian case and similar as in the real Lagrange geometry ([5, 7, 9]) for the non-Hermitian case. There are studied the $N-$ complex linear connections in each of two classes.

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1 Preliminaries

The notion of complex Finsler space appears for the first time in a paper written by Rizza in 1963, [12], as a generalization of the similar notion from the real case, requering the homogeneity of the fundamental function with respect to complex scalars λ. The first example comes from the complex hyperbolic geometry and was given by S. Kobayashi in 1975, [8]. The Kobayashi metric has given an impulse to the study of complex Finsler geometry.

Geometry means, first of all, distance. The significance of the curve arc length in complex Finsler geometry is pointed out by Abate and Patrizio in 1994, [1]. This distance refers to curves depending on a real parameter and the invariance of the integral to the change of parameters is ensured only for real parameters.

Bearing in mind this fact, in this paper we will extend the definition of a complex Finsler space by reducing the scalars to $\lambda \in \mathbb{R}$ in the homogeneity condition and there is obtained a new class of complex spaces, called by us the $\mathbb{R}$- complex Finsler spaces.

In this section we keep the general settings from [11] and subsequently we recall only some needed notions.

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Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$, $\left(z^k\right)$ be local complex coordinates and $T'M$ the holomorphic tangent bundle which has a natural structure of complex manifold, $\dim_{\mathbb{C}} T'M = 2n$ and the induced coordinates on $u \in T'M$ are denoted by $u = (z^k, \eta^k)$. The changes of local coordinates in u are given by the rules:

$$z'^k = z'^k(z) \ ; \ \eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j. \tag{1.1}$$

The natural frame $\left\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\right\}$ of $T'_u(T'M)$ transforms with the Jacobi matrix of (1.1) changes.

A complex nonlinear connection, briefly (c.n.c.), is a supplementary distribution $H(T'M)$ to the vertical distribution $V(T'M)$ in $T'(T'M)$. The vertical distribution is spanned by $\left\{\frac{\partial}{\partial \eta^k}\right\}$ and an adapted frame in $H(T'M)$ is denoted by $\left\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\right\}$, where N_k^j are the coefficients of the (c.n.c.) and they have a certain rule of change at (1.1) so that $\frac{\delta}{\delta z^k}$ transform like vectors on the base manifold M ($d-$tensors in [11] terminology). Next, we use the abbreviations: $\partial_k := \frac{\partial}{\partial z^k}$, $\delta_k := \frac{\delta}{\delta z^k}$, $\dot{\partial}_k := \frac{\partial}{\partial \eta^k}$ and $\partial_{\bar{k}}$, $\dot{\partial}_{\bar{k}}$, $\delta_{\bar{k}}$ for their conjugates. The dual adapted basis of $\left\{\delta_k, \dot{\partial}_k\right\}$ are $\left\{dz^k, \delta\eta^k = d\eta^k + N_j^k dz^j\right\}$ and $\left\{d\bar{z}^k, \delta\bar{\eta}^k\right\}$ are their conjugates. A notion attached to a (c.n.c.) is that of complex spray on $T'M$, given by a set of coefficients $G^i(z,\eta)$ which transform at (1.1) changes as follows:

$$2G'^i = 2G^k \frac{\partial z'^i}{\partial z^k} - \frac{\partial^2 z'^i}{\partial z^j \partial z^k}\eta^j \eta^k. \tag{1.2}$$

Any complex spray determines a (c.n.c.) by the rule $N_k^h = \frac{\partial G^h}{\partial \eta^k}$. Conversely, any (c.n.c.) N_k^h determines a complex spray by setting $\tilde{G}^h = \frac{1}{2}N_k^h \eta^k$.

A covariant derivative law D on $T'M$ is said to be an $N-$complex linear connection, $N-(c.l.c.)$, if its coefficients satisfy:

$D_{\delta_k}\delta_j = L^i_{jk}\delta_i$; $D_{\delta_k}\dot{\partial}_j = L^i_{jk}\dot{\partial}_i$; $D_{\dot{\partial}_k}\delta_j = C^i_{jk}\delta_i$; $D_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i$;

$D_{\delta_{\bar{k}}}\delta_j = L^i_{j\bar{k}}\delta_i$; $D_{\delta_{\bar{k}}}\dot{\partial}_j = L^i_{j\bar{k}}\dot{\partial}_i$; $D_{\dot{\partial}_{\bar{k}}}\delta_j = C^i_{j\bar{k}}\delta_i$; $D_{\dot{\partial}_{\bar{k}}}\dot{\partial}_j = C^i_{j\bar{k}}\dot{\partial}_i$

and their conjugates in account of $\overline{D_X Y} = D_{\bar{X}}\bar{Y}$. If D is of $(1,0)$ type, then $L^i_{j\bar{k}} = C^i_{j\bar{k}} = L^{\bar{i}}_{\bar{j}k} = C^{\bar{i}}_{\bar{j}k} = 0$.

For more details concerning the geometry of $T'M$ bundle see [11].

In this note, as a rule, the straightforward computations are omitted but the reader will have at hand all elements to check the assertions which involve them.

2 On $\mathbb{R}-$complex Finsler metrics

We recall that the homogeneity of the fundamental function of a complex Finsler space is with respect to complex scalars and the metric tensor of the space is one Hermitian. Now we slightly change the definition of a complex Finsler space.

Definition 2.1. An $\mathbb{R}-$complex Finsler space is a pair (M, F), where F is a continuous function $F : T'M \longrightarrow \mathbb{R}_+$ satisfying:

i) $L := F^2$ is smooth on $\widetilde{T'M}$ (except the 0 section);
ii) $F(z,\eta) \geq 0$, the equality holds if and only if $\eta = 0$;
iii) $F(z,\lambda\eta,\bar{z},\lambda\bar{\eta}) = |\lambda| F(z,\eta,\bar{z},\bar{\eta}), \forall \lambda \in \mathbb{R}$;

The assertion *iii*) says that L is $(2,0)$ homogeneous with respect to the real scalar λ, i.e. $L(z,\lambda\eta,\bar{z},\lambda\bar{\eta}) = \lambda^2 L(z,\eta,\bar{z},\bar{\eta}), \forall\lambda \in \mathbb{R}$.

Let us set the following metric tensors:

$$g_{ij} := \frac{\partial^2 L}{\partial\eta^i\partial\eta^j}\ ;\ g_{i\bar{j}} := \frac{\partial^2 L}{\partial\eta^i\partial\bar{\eta}^j};\ g_{\bar{i}\bar{j}} := \frac{\partial^2 L}{\partial\bar{\eta}^i\partial\bar{\eta}^j} \tag{2.1}$$

Proposition 2.2. *In an* $\mathbb{R}-$*complex Finsler space the following conditions hold:*
i) $\frac{\partial L}{\partial\eta^i}\eta^i + \frac{\partial L}{\partial\bar{\eta}^i}\bar{\eta}^i = 2L$;
ii) $g_{ij}\eta^i + g_{\bar{j}i}\bar{\eta}^i = \frac{\partial L}{\partial\eta^j}$;
iii) $2L = g_{ij}\eta^i\eta^j + g_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j + 2g_{i\bar{j}}\eta^i\bar{\eta}^j$;
iv) $\frac{\partial g_{ik}}{\partial\eta^j}\eta^j + \frac{\partial g_{ik}}{\partial\bar{\eta}^j}\bar{\eta}^j = 0$; $\frac{\partial g_{i\bar{k}}}{\partial\eta^j}\eta^j + \frac{\partial g_{i\bar{k}}}{\partial\bar{\eta}^j}\bar{\eta}^j = 0$.

Proof. By differentiating the equality $L(z,\lambda\eta,\bar{z},\lambda\bar{\eta}) = \lambda^2 L(z,\eta,\bar{z},\bar{\eta})$, with respect to $\lambda \in \mathbb{R}$ and then setting $\lambda = 1$, it results *i*). Now, *ii*) results by differentiating *i*) with respect to η^j. Contracting *ii*) with η^j, then adding it with its conjugate and using *i*), we obtain *iii*). *iv*) follows by differentiating *ii*) with respect to η^k and $\bar{\eta}^k$ respectively. □

An immediate consequence of the homogeneity conditions and Proposition 2.1 concerns the following Cartan type complex tensors:

$$C_{ijk} := \frac{\partial g_{ij}}{\partial\eta^k}\ ;\ C_{ij\bar{k}} := \frac{\partial g_{ij}}{\partial\bar{\eta}^k}\ ;\ C_{i\bar{j}\bar{k}} := \frac{\partial g_{i\bar{j}}}{\partial\bar{\eta}^k} \tag{2.2}$$

and their conjugates.

Let us denote with 0 or $\bar{0}$ the contracting of the Cartan tensors with η^k or $\bar{\eta}^k$, respectively. From the above Proposition 2.1 *iv*) and the definitions of metric tensors g_{ij}, $g_{i\bar{j}}$, it follows that

Proposition 2.3. *The Cartan complex tensors are symmetric in the indices of the same type (without bar or with bar) and*

$$C_{ij0} + C_{ij\bar{0}} = 0\ ;\ C_{i0\bar{j}} + C_{i\bar{0}\bar{j}} = 0\ ;\ C_{0ij} + C_{\bar{0}ij} = 0.$$

In complex Finsler geometry a strongly pseudo-convex requirement is assumed, that is the metric tensor $g_{i\bar{j}}$ defines a positive-definite quadratic form, and then the matrix $(g_{i\bar{j}})$ is invertible.

From Proposition 2.1 *iii*) we remark that by restricting the homogeneity of the Finsler function in the definition of a complex Finsler space, the associated Lagrangian function L acquires a more general form than in the complex Finsler geometry. Consequently, a naturally question arises now: which of the (2.1) metric tensors g_{ij} or $g_{i\bar{j}}$ need be invertible? Need both of them be invertible?.

A first remark is that if there exists a set of local charts on M in which $g_{ij} = 0$, then $C_{ijk} = C_{ij\bar{k}} = C_{i\bar{k}j} = 0$, and consequently $g_{i\bar{j}}$ is a function of z alone, that is in terminology from [11] the space is then purely Hermitian. If there exists a set of local charts in which $g_{i\bar{j}} = 0$, then $C_{i\bar{j}k} = C_{ik\bar{j}} = 0$, and consequently g_{ij} are holomorphic functions with respect to η.

Clearly, for a proper Hermitian geometry, the existence of the inverse of the $g_{i\bar{j}}$ tensor is a compulsive requirement. On the other hand, from some physicists' point of view, for which the Hermitian condition is an impediment, it seems more attractive that g_{ij} should be an invertible metric tensor.

Definition 2.4. An $\mathbb{R}-$ complex Hermitian Finsler space is a pair (M, F) where F satisfies the regularity condition:

$$g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j} \tag{2.3}$$

is nondegenerated, i.e. det $\left(g_{i\bar{j}}\right) \neq 0$ in any $u \in \widetilde{T'M}$, and defines a positive definite Levi-form for all $z \in M$.

This ensures the strongly pseudo-convexity of the indicatrix.

Any $\mathbb{R}-$ complex Hermitian Finsler space (M, F) is a particular complex Lagrange one with $L = F^2$, in the sense that we leave the homogeneity condition in a Lagrangian space. Therefore the geometric machinery may be studied following the general framework described in [11].

The fundamental problem in the study of $T'M$ geometry is that of the existence of a complex nonlinear connection, depending only on the Lagrangian function L.

In a complex Finsler space a special derivative law is usually considered, namely the Chern-Finsler connection. It is one metrical with respect to the lift of the metric tensor $g_{i\bar{j}}$ on $T'M$, is of $(1,0)-$ type and its coefficients are given with respect to a (c.n.c.) obtained by constraining the derivative law to an extra condition ([1]). In [11] we proved that the Chern-Finsler (c.n.c.) actually is derived from a complex spray.

Similar reasonings lead us to a complex nonlinear connection in an $\mathbb{R}-$ complex Hermitian Finsler space and then to obtaining a derivative law.

Let us consider $c(t)$ a curve on complex manifold M and $(z^k(t), \eta^k(t) = \frac{dz^k}{dt})$ its extension on $T'M$. The Euler-Lagrange equations with respect to a complex Lagrangian L are

$$\frac{\partial L}{\partial z^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \eta^i}\right) = 0, \tag{2.4}$$

where L is considered along the curve c on $T'M$.

If we develop the calculus in (2.4) taking into account that L depends on the parameter $t \in \mathbb{R}$ by means of $z^k(t), \eta^k(t)$ and their conjugates, we will get

$$\left(g_{ij}\frac{d^2 z^j}{dt^2} + \frac{\partial^2 L}{\partial z^j \partial \eta^i}\eta^j - \frac{\partial L}{\partial z^i}\right) + \left(g_{i\bar{j}}\frac{d^2 \bar{z}^j}{dt^2} + \frac{\partial^2 L}{\partial \bar{z}^j \partial \eta^i}\bar{\eta}^j\right) = 0. \tag{2.5}$$

Now, following the same arguments form [13] concerning the complex geodesic curves, we impose that both brackets in (2.5) vanish:

$$g_{ij}\frac{d^2 z^j}{dt^2} + \frac{\partial^2 L}{\partial z^j \partial \eta^i}\eta^j - \frac{\partial L}{\partial z^i} = 0 \tag{2.6}$$

and

$$g_{i\bar{j}}\frac{d^2\bar{z}^j}{dt^2}+\frac{\partial^2 L}{\partial\bar{z}^j\partial\eta^i}\bar{\eta}^j=0. \tag{2.7}$$

From the conjugate of (2.7), taking into account Proposition 2.1 *ii*), by direct calculation there is obtained

$$\frac{d^2z^h}{dt^2}+2G^h\left(z^h\left(t\right),\eta^h\left(t\right)\right)=0, \tag{2.8}$$

with

$$G^h(z,\eta)=\frac{1}{2}g^{\bar{\imath}h}\left(\frac{\partial g_{\bar{r}\bar{\imath}}}{\partial z^j}\bar{\eta}^r+\frac{\partial g_{s\bar{\imath}}}{\partial z^j}\eta^s\right)\eta^j. \tag{2.9}$$

Proposition 2.5. *The functions G^h from (2.9) are the coefficients of a complex spray on $T'M$.*

The proof consists directly checking that G^h obey the (1.2) change rule of a complex spray. Then, by using the fact that any complex spray determines a (c.n.c.), it results that a complex nonlinear connection, depending only on the complex Hermitian function L, is given by $\overset{c}{N}{}^h_k=\frac{\partial G^h}{\partial\eta^k}$, where G^h are given in (2.9), and called the *canonical* (c.n.c.). On the other hand, if we pay more attention to (2.9), this spray comes from a (c.n.c.). Thus we may conclude

Theorem 2.6. *A complex nonlinear connection for the $\mathbb{R}-$ complex Hermitian Finsler space (M,F), called the Chern-Finsler (c.n.c.), is given by*

$$\overset{CF}{N}{}^h_j=g^{\bar{\imath}h}\left(\frac{\partial g_{\bar{r}\bar{\imath}}}{\partial z^j}\bar{\eta}^r+\frac{\partial g_{s\bar{\imath}}}{\partial z^j}\eta^s\right). \tag{2.10}$$

A similarly computation as in [1, 11] gives that the adapted frame of $\overset{CF}{N}{}^h_j$ (c.n.c.) satisfies $[\delta_i,\delta_j]=0$.

Now, having this (2.10) (c.n.c.), our aim is to obtain a N-(c.l.c.) of Chern-Finsler type. Let us consider the $\overset{CF}{N}-$lift of the fundamental tensor $g_{i\bar{j}}$ to the complexified bundle $T_C(T'M)$,

$$\mathcal{G}=g_{i\bar{j}}dz^i\otimes d\bar{z}^j+g_{i\bar{j}}\delta\eta^i\otimes\delta\bar{\eta}^j. \tag{2.11}$$

A N-(c.l.c.) D is said to be metrical if $D\mathcal{G}=0$, that is $\left(D_X\mathcal{G}\right)(Y,Z)=X\left(\mathcal{G}\left(Y,Z\right)\right)-\mathcal{G}\left(D_XY,Z\right)-\mathcal{G}\left(Y,D_XZ\right)=0$ for any $X,Y,Z\in\Gamma T_C(T'M)$. Recalling that the notations for the coefficients of a $N-(c.l.c.)$ were settled at the end of the preview section, and taking in turn X, Y,Z the adapted frame of Chern-Finsler (c.n.c.) $\{\delta_i,\dot{\partial}_i,\delta_{\bar{\imath}},\dot{\partial}_{\bar{\imath}}\}$, we easily check that

Theorem 2.7. *In a $\mathbb{R}-$ complex Hermitian Finsler space, a $\overset{CF}{N}-$complex linear connection, which is metrical of $(1,0)-$type, is given by:*

$$L^i_{jk}=g^{\bar{m}i}\delta_j(g_{i\bar{m}})\ ;\ C^i_{jk}=g^{\bar{m}i}\dot{\partial}_j(g_{i\bar{m}})\ ;\ L^i_{j\bar{k}}=C^i_{j\bar{k}}=0. \tag{2.12}$$

The (2.12) $\overset{CF}{N}$ $-(c.l.c.)$ will be called the *Chern-Finsler connection* of the $\mathbb{R}-$ complex Hermitian Finsler space.

Moreover, as in [1] we prove that $L^i_{jk} = \frac{\partial N^i_k}{\partial \eta^j}$. Therefore, the Chern-Finsler connection of $\mathbb{R}-$ complex Hermitian Finsler space has some nonzero torsion and curvature coefficients like the Chern-Finsler (c.n.c.) of a complex Finsler space, namely:

$$\begin{aligned} T^i_{jk} &: = L^i_{jk} - L^i_{kj} \ ; \ Q^i_{jk} := C^i_{jk} \ ; \ \Theta^i_{j\bar{k}} := \delta_{\bar{k}} \overset{CF}{N^i_j} \ ; \ \Theta^i_{j\bar{k}} := \partial_{\bar{k}} \overset{CF}{N^i_j}; \\ R^i_{j\bar{h}k} &: = -\delta_{\bar{h}} L^i_{jk} - \delta_{\bar{h}} (\overset{CF}{N^l_k}) C^i_{jl} \ ; \ P^i_{j\bar{h}k} := -\delta_{\bar{h}} C^i_{jk} = \Xi^i_{k\bar{h}j}; \\ Q^i_{j\bar{h}k} &: = \dot{\partial}_{\bar{h}} L^i_{jk} + \dot{\partial}_{\bar{h}} (\overset{CF}{N^l_k}) C^i_{jl} \ ; \ S^i_{j\bar{h}k} := -\dot{\partial}_{\bar{h}} C^i_{jk} = S^i_{k\bar{h}j}. \end{aligned} \tag{2.13}$$

3 Non-Hermitian metric structure on $T'M$

In this section, we make a similar approach when the tensor g_{ij} is nondegenerate. Note that in the $\mathbb{R}-$ complex Hermitian Finsler case the condition (2.6) is one algebraic, to whom we do not assign any geometrical meaning. Actually, in complex Finsler geometry this ones is just the weakly Kähler condition with respect to Chern-Finsler connection.

Definition 3.1. An $\mathbb{R}-$ complex non-Hermitian Finsler space is the pair (M, F) where F satisfies the regularity condition: $g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}$ is nondegenerated, i.e. $\det(g_{ij}) \neq 0$ at any point $u \in \widetilde{T'M}$, and defines a positive definite quadratic form for all $z \in M$.

In this circumstances the indicatrix is one strongly convex like in the real case.

Now in the non-Hermitian case we can use (2.6) with another purpose. From (2.6) and Proposition 2.1 *iii)* like above, we find by direct calculation that

$$\frac{d^2 z^h}{dt^2} + 2G^h\left(z^h(t), \eta^h(t)\right) = 0, \tag{3.1}$$

with

$$\begin{aligned} G^h(z,\eta) &= \frac{1}{4} g^{ih} \left(\frac{\partial g_{ri}}{\partial z^j} + \frac{\partial g_{ji}}{\partial z^r} - \frac{\partial g_{rj}}{\partial z^i} \right) \eta^r \eta^j - \frac{1}{2} g^{ih} \frac{\partial g_{\bar{l}\bar{s}}}{\partial z^i} \bar{\eta}^l \bar{\eta}^s \\ &\quad + g^{ih} \left(\frac{\partial g_{i\bar{s}}}{\partial z^j} - \frac{\partial g_{j\bar{s}}}{\partial z^i} \right) \eta^j \bar{\eta}^s \end{aligned} \tag{3.2}$$

and a straightforward computation proves that the coefficients G^h obey the (1.2) rule, thus:

Proposition 3.2. *The functions G^h from (3.2) are the coefficients of a complex spray on $T'M$.*

Following the general theory of a spray it results that $\overset{c}{N^h_k} = \frac{\partial G^h}{\partial \eta^k}$ is a complex nonlinear connection, depending only on the complex Lagrangian L. With respect to this connection, we can consider its adapted coframe $\left\{dz^i, \delta\eta^i, d\bar{z}^i, \delta\bar{\eta}^i\right\}$ and the $\overset{c}{N}-$ lift (or Sasaki-type lift) of g_{ij} fundamental tensor to a metric structure G on $T_C(T'M)$ defined by

$$G = g_{ij}dz^i \otimes dz^j + g_{ij}\delta\eta^i \otimes \delta\eta^j + g_{\bar{\imath}\bar{\jmath}}d\bar{z}^i \otimes d\bar{z}^j + g_{\bar{\imath}\bar{\jmath}}\delta\bar{\eta}^i \otimes \delta\bar{\eta}^j. \tag{3.3}$$

A $N-$(c.l.c.) D is a metric connection with respect to the non-Hermitian metric structure G if $DG = 0$.

If a $N-$(c.l.c.) D is metrical with respect to the non-Hermitian Sasaki-type lift G, then the local coefficients of D connection satisfy the following system of derivations:

$$g_{ij\shortmid k} = 0,\ g_{ij|k} = 0,\ g_{ij\shortmid \bar{k}} = 0,\ g_{ij|\bar{k}} = 0.$$

Theorem 3.3. *There exists a unique metric $N-$(c.l.c.) D with respect to the metrical structure (3.3) with $hT(h,h) = 0,\ vT(v,v) = 0$ torsions and satisfying the condition*

$$G\left(D_{\bar{X}}Y, Z\right) = G\left(D_{\bar{X}}Z, Y\right), \forall X, Y, Z \in T'\left(T'M\right). \tag{3.4}$$

It has the following local coefficients:

$$\begin{aligned} L^i_{jk} &= \frac{1}{2}g^{im}\left\{\delta_j(g_{km}) + \delta_k(g_{jm}) - \delta_m(g_{jk})\right\}\ ;\ C^i_{jk} = \frac{1}{2}g^{im}\dot{\partial}_j(g_{km}), \\ L^i_{j\bar{k}} &= \frac{1}{2}g^{im}\delta_{\bar{k}}(g_{jm})\ ;\ C^i_{j\bar{k}} = \frac{1}{2}g^{im}\dot{\partial}_{\bar{k}}(g_{jm}). \end{aligned} \tag{3.5}$$

Proof. The conditions $hT(h,h) = 0,\ vT(v,v) = 0$ suggest to take D like the Levi-Civita connection of G:

$$\begin{aligned} 2G\left(D_XY, Z\right) &= X\left(G\left(Y,Z\right)\right) + Y\left(G\left(Z,X\right)\right) - Z\left(G\left(X,Y\right)\right) \\ &\quad + G\left(\left[X,Y\right],Z\right) + G\left(\left[Z,X\right],Y\right) - G\left(\left[Y,Z\right],X\right), \end{aligned} \tag{3.6}$$

which is $h-$ and $v-$ metrical. If we take $X = \delta_k,\ Y = \delta_j,\ Z = \delta_i$ in (3.6) and using (3.2.6) p. 43, [11], we find by a straightforward computation $(3.5)_1$. Taking within in (3.6): $X = \dot{\partial}_{\bar{k}}, Y = \delta_i, Z = \delta_j$ and using (3.4) we obtain $(3.5)_2$. Now for $X = \dot{\partial}_{\bar{k}}, Y = \delta_j, Z = \delta_i$, and taking into account (3.4) we find $(3.5)_4$ and finally for $X = \delta_{\bar{k}}, Y = \delta_j, Z = \delta_i$ in (3.6) we have $(3.5)_3$. □

Other nonzero torsion components of $N-$(c.l.c.) D from above Theorem 3.1 are:

$$\begin{aligned} \tau^i_{j\bar{k}} &= L^i_{j\bar{k}};\ R^i_{jk} = \delta_j N^i_k - \delta_k N^i_j;\ \Theta^i_{j\bar{k}} = \delta_{\bar{k}}N^i_j\ ;\ \Upsilon^i_{j\bar{k}} = C^i_{j\bar{k}}; \\ Q^i_{jk} &= C^i_{jk};\ \rho^i_{j\bar{k}} = \dot{\partial}_{\bar{k}}N^i_j;\ \chi^i_{j\bar{k}} = C^i_{j\bar{k}};\ P^i_{jk} = \dot{\partial}_k N^i_j - L^i_{kj};\ \Sigma^i_{j\bar{k}} = L^i_{j\bar{k}}. \end{aligned} \tag{3.7}$$

We note that if D is of $(1,0)-$type then g_{ij} is holomorphic with respect to z and η.

The connection $1-$form of the $N-$(c.l.c.) given in Theorem 3.1 has the following simplified expression

$$\omega^i_j = L^i_{jk}dz^k + L^i_{j\bar{k}}d\bar{z}^k + C^i_{jk}\delta\eta^k + C^i_{j\bar{k}}\delta\bar{\eta}^k.$$

The calculus for the curvature components of the $N-$(c.l.c.)D is based on the local components of the curvature given in (3.2.10), p.45, [11] and on the local coefficients of D given by (3.5). It is one trivial and we leave this computation as an exercise for the reader.

We end this paper with a class of examples which prove that our study is not trivial, per contra it illustrates the interest for such study. Consider

$$\alpha^2 = Re\{a_{ij}(z)\eta^i\eta^j + a_{i\bar{j}}\eta^i\bar{\eta}^j\} \text{ and } \beta = Re\{b_i(z)\eta^i\},$$

where $a_{ij}(z)$ define a positive-definite Riemannian metric or $a_{i\bar{j}}(z)$ a Hermitian metric on the complex manifold M and $b_i(z)$ is a $1-$form. All $\mathbb{R}-$homogenous functions F of α^2 and β define $\mathbb{R}-$ complex Finsler metrics. For instance $F = \alpha + \beta$ defines $\mathbb{R}-$Randers metrics, or $F = \frac{\alpha^2}{\beta}$ defines $\mathbb{R}-$Kropina metrics. Certainly, demanding that $a_{i\bar{j}}$ or a_{ij} should be nondegenerate we obtain the $\mathbb{R}-$ complex (α, β) spaces of Hermitian or non-Hermitian type. A detailed study of this class of $\mathbb{R}-$ complex Finsler spaces will be done in a forthcoming paper. Related results can be found in [10, 4, 6].

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