Geometry of electroelastic materials subject to biasing fields

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Abstract. This paper presents the basic elements of geometrical modeling in the dynamics of electroelastic materials subject to electromechanical initial fields. We derive here the field and the constitutive equations, as well as the boundary conditions, related to the behavior of incremental fields superposed on large static initial deformation and electric fields.

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1 Introduction

The problems related to the mechanics of materials, focusing on the electroelastic materials subject to incremental fields superposed on initial mechanical and electric fields, have attracted considerable attention last period, due their complexity and to multiple applications (see papers [2, 4, 20, 21, 22]). The basic equations of the theory of piezoelectric bodies subject to infinitesimal deformations and fields superposed on large initial mechanical and electric fields were described by Eringen and Maugin in monograph [3].

We present here the fundamental equations of incremental fields superposed on large static deformation and electric fields. Following the paper [1], we derive the balance equations, constitutive equations and boundary conditions for this problem, using the updated Lagrangean description. We analyze the important special case of homogeneous initial state and non-polarizable environment. A detailed analysis may be found in the monographic chapter [17]. Application of these results are analyzed in papers [7]-[19].

2 The quasi-electrostatic approximation of balance equations

We assume the material to be an hyperelastic dielectric, which is nonmagnetizable and conducts neither heat, nor electricity. We shall use the quasi-electrostatic approx-

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imation of the equations of balance. Let B_R be the *reference configuration*, in which at time $t = 0$ the body is undeformed and free of all fields and B_t the *present (current) configuration*. Let V_R and V_t the geometric domains associated with B_R and B_t, endowed with the usual structure of differentiable manifolds. The *material position* (reported to B_R) and the *spatial position* (reported to B_t) of an arbitrary particle X of the body will be denoted by $\mathbf{X}$, resp. $\mathbf{x}$. The first case is known as *Lagrangean description*, while the second as *Eulerian description* of the deformation.

Let ρ_R and ρ be the mass densities of the body, referred to the configurations B_R and B_t. If J is the determinant of the deformation gradient $\mathbf{F}$ from B_R to B_t, then we obtain the *mass conservation law* (in Lagrangean description) in the form:

$$\rho J = \rho_R. \tag{2.1}$$

Similarly, let q_R and q be the volumetric electric charge densities, reported to B_R and B_t. Then, we determine the *charge conservation law* (in Lagrangean description) as:

$$qJ = q_R. \tag{2.2}$$

Let $\mathbf{T}$ be electromechanical stress tensor of Cauchy type and $\mathbf{S}$ its symmetric part. We suppose that the latter tensor is derived from Helmholtz free energy as in non-linear elasticity, but now depending on electromechanical variables, as follows. Let $\mathbf{E}$, $\mathbf{P}$ and $\mathbf{D}$ be the electric field, the electric polarization and the electric displacement vectors, respectively. Then, we have the following relations (see [3]):

$$\mathbf{T} = \mathbf{S} - \mathbf{P} \otimes \mathbf{E}, \quad \mathbf{D} = \mathbf{E} + \mathbf{P}. \tag{2.3}$$

The balance equations in the quasi-electrostatic approximation have the form (see [3]):

$$\rho\dot{\mathbf{v}} = \mathrm{div}_{\mathbf{x}}\mathbf{T} + \rho\mathbf{f} + q\mathbf{E} + (\mathbf{P} \cdot \nabla_{\mathbf{x}})\mathbf{E}, \quad \mathrm{div}_{\mathbf{x}}\mathbf{D} = q, \quad \mathrm{rot}_{\mathbf{x}}\mathbf{E} = 0. \tag{2.4}$$

Here $\mathbf{v}$ is the velocity field vector, $\mathbf{f}$ represents the mechanical body force and q is the volumetric charge density. A superposed dot is used to denote the material time derivative. In equations (2.4) we used the Eulerian description, the fields involved depending on spatial coordinate $\mathbf{x}$ and on time t. Here one finds the electrostatic form of Coulomb and Faraday laws. The first differential relation is derived from the momentum balance.

Furthermore, the jump conditions and the electromechanical surface stress vector $\mathbf{t_n}$, defined on the boundary ∂V_t, are given by:

$$\mathbf{n} \cdot [\mathbf{D}] = w, \quad \mathbf{n} \times [\mathbf{E}] = \mathbf{0}, \quad \mathbf{t_n} = \mathbf{Tn} = (\mathbf{S} - \mathbf{P} \otimes \mathbf{E})\mathbf{n} \quad \text{on}\,\partial V_t. \tag{2.5}$$

Here w represents the surface charge density, $\mathbf{n}$ the exterior normal unit vector to the boundary and $[\phi] = \phi^+ - \phi^-$ is the jump of the field ϕ across the boundary. From now on, we shall denote simply by ϕ the inside limit value ϕ^-.

The previous field equations and boundary conditions may be expressed in Lagrangean description using electromechanical stress tensors of Piola-Kirchhoff type. They are related to the reference configuration B_R and are defined by the relations:

$$\mathbf{\Theta} = J\mathbf{F}^{-1}\mathbf{S}, \quad \mathbf{\Pi} = J\mathbf{F}^{-1}\mathbf{S}\mathbf{F}^{-T}, \quad \mathbf{\Theta} = \mathbf{\Pi}\mathbf{F}^{T}. \tag{2.6}$$

Here $\mathbf{\Theta}$ and $\mathbf{\Pi}$ are the nominal electromechanical stress tensors of Piola-Kirchhoff type related to the symmetric part $\mathbf{S}$ of the electromechanical stress tensor $\mathbf{T}$ of Cauchy type. It is obvious that $\mathbf{\Pi}$ is a symmetric tensor.

Similarly, we introduce the lagragean version of vectors $\mathbf{P}$, $\mathbf{E}$ and $\mathbf{D}$, as follows:

$$\boldsymbol{\mathcal{P}} = J\mathbf{F}^{-1}\mathbf{P}, \quad \boldsymbol{\mathcal{E}} = \mathbf{F}^{T}\mathbf{E}, \quad \boldsymbol{\mathcal{D}} = J\mathbf{F}^{-1}\mathbf{D} = J\mathbf{F}^{-1}\mathbf{E} + \boldsymbol{\mathcal{P}}. \tag{2.7}$$

Finally, using the first relation (2.3) we obtain the associated material version of tensor $\mathbf{T}$:

$$\boldsymbol{\mathcal{T}} = J\mathbf{F}^{-1}\mathbf{T} = \mathbf{\Theta} - \boldsymbol{\mathcal{P}} \otimes \mathbf{E}. \tag{2.8}$$

Consequently, we derive from the momentum balance the Lagrangean form of the balance equations:

$$\rho_R \ddot{\mathbf{u}} = \operatorname{div}_{\mathbf{X}} \boldsymbol{\mathcal{T}} + \rho_R \mathbf{f} + q_R \mathbf{E} + (\boldsymbol{\mathcal{P}} \cdot \nabla_{\mathbf{X}})\mathbf{E}, \quad \operatorname{div}_{\mathbf{X}} \boldsymbol{\mathcal{D}} = q_R, \quad \operatorname{rot}_{\mathbf{X}} \boldsymbol{\mathcal{E}} = \mathbf{0}. \tag{2.9}$$

Here $\mathbf{u}$ is the displacement vector from B_R to B_t. The differential operators are associated with the reference configuration, while the various fields involved depend on the material coordinate $\mathbf{X}$ and on time t.

Furthermore, the jump conditions and the electromechanical surface stress vector $\mathbf{t}_{\mathbf{N}}$, reported to the material configuration, are given by:

$$\mathbf{N} \cdot [\boldsymbol{\mathcal{D}}] = w_R, \quad \mathbf{N} \times [\boldsymbol{\mathcal{E}}] = \mathbf{0}, \quad \mathbf{t}_{\mathbf{N}} = \boldsymbol{\mathcal{T}}\mathbf{N} = (\mathbf{\Theta} - \boldsymbol{\mathcal{P}} \otimes \mathbf{E})\mathbf{N} \quad \text{on } \partial V_R. \tag{2.10}$$

Here $\mathbf{N}$ is the exterior normal unit vector to the boundary ∂V_R and w_R is surface charge density per material surface area.

The previous balance equations are supplemented by the constitutive equations (see [3]):

$$\mathbf{\Pi} = \frac{\partial \mathcal{H}}{\partial \mathbf{G}}, \quad \boldsymbol{\mathcal{P}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{\mathcal{E}}} \tag{2.11}$$

where

$$\rho_R \psi = \mathcal{H}(\mathbf{G}, \boldsymbol{\mathcal{E}}) \tag{2.12}$$

is the electromechanical Helmholtz free energy, and

$$\mathbf{G} = \frac{1}{2}(\mathbf{C} - \mathbf{1}) = \frac{1}{2}(\mathbf{F}^{T}\mathbf{F} - \mathbf{1}) \tag{2.13}$$

is Green strain tensor.

3 Small deformation and electric fields superposed on large static deformation and electric fields

In this part we describe the behaviour of incremental electromechanical fields superposed on large initial electromechanical fields. In [3] one obtains these equations using the field equations corresponding to the spatial (eulerian) description of the deformation. The perturbed electromechanical surface forces, as well as the surface and volumetric charges, are taken in the perturbed current configuration. In [1], these quantities are referred to the initially deformed static configuration, which is supposed as being known (i.e. the *updated Lagrangean description* of the deformation). We shall follow the second approach.

To describe this situation we use three different configurations : the *reference configuration* B_R in which at time $t = 0$ the body is undeformed and free of all fields; the *initial configuration* $\overset{\circ}{B}$ in which the body is deformed statically and carries the initial fields; the *present (current) configuration* B_t obtained from $\overset{\circ}{B}$ by applying time dependent incremental deformations and fields. Let V_R, $\overset{\circ}{V}$ and V_t the geometric domains associated with B_R, $\overset{\circ}{B}$ and B_t, endowed with the usual structure of differentiable manifolds.

In what follows, all the fields related to the initial configuration $\overset{\circ}{B}$ will be denoted by a superposed "$\circ$". The static deformation from B_R to $\overset{\circ}{B}$ is described by the relation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, the associated deformation gradient is $\overset{\circ}{\mathbf{F}}=\overset{\circ}{\mathbf{F}}(\mathbf{X})$ and $\overset{\circ}{J}= \det \overset{\circ}{\mathbf{F}}(\mathbf{X})$. Thus, we obtain the mass conservation law, resp. the charge conservation law, in the form:

$$\overset{\circ}{\rho}\overset{\circ}{J}= \rho_R, \quad \overset{\circ}{q}\overset{\circ}{J}= q_R. \tag{3.1}$$

Furthermore, due to the relations (2.3) we derive that:

$$\overset{\circ}{\mathbf{T}}=\overset{\circ}{\mathbf{S}} - \overset{\circ}{\mathbf{P}} \otimes \overset{\circ}{\mathbf{E}}, \quad \overset{\circ}{\mathbf{D}}=\overset{\circ}{\mathbf{E}} + \overset{\circ}{\mathbf{P}}\,. \tag{3.2}$$

According to (2.4), the field equations for these static fields are:

$$\mathrm{div}_{\mathbf{x}}\, \overset{\circ}{\mathbf{T}} + \overset{\circ}{\rho}\overset{\circ}{\mathbf{f}} + \overset{\circ}{q}\overset{\circ}{\mathbf{E}} +(\overset{\circ}{\mathbf{P}}\cdot\nabla_{\mathbf{x}})\,\overset{\circ}{\mathbf{E}}= \mathbf{0}, \quad \mathrm{div}_{\mathbf{x}}\, \overset{\circ}{\mathbf{D}}=\overset{\circ}{q}, \quad \mathrm{rot}_{\mathbf{x}}\, \overset{\circ}{\mathbf{E}}= 0. \tag{3.3}$$

Here all the electromechanical fields depend on the variable $\mathbf{x}$ and on time t.

Consequently, the jump conditions and the electromechanical surface stress vector $\overset{\circ}{\mathbf{t}}_{\mathbf{n}}$ at the boundary $\partial \overset{\circ}{V}$ are:

$$\overset{\circ}{\mathbf{n}}\cdot[\overset{\circ}{\mathbf{D}}] =\overset{\circ}{w}, \quad \overset{\circ}{\mathbf{n}}\times[\overset{\circ}{\mathbf{E}}] = \mathbf{0}, \quad \overset{\circ}{\mathbf{t}}_{\mathbf{n}}=\overset{\circ}{\mathbf{T}}\overset{\circ}{\mathbf{n}}= (\overset{\circ}{\mathbf{S}} - \overset{\circ}{\mathbf{P}} \otimes \overset{\circ}{\mathbf{E}})\,\overset{\circ}{\mathbf{n}} \quad \text{on}\, \partial \overset{\circ}{V}\,. \tag{3.4}$$

Here $\overset{\circ}{\mathbf{n}}$ is the exterior normal unit vector to the boundary $\partial \overset{\circ}{V}$.

The constitutive equations (2.11), giving the electromechanical stress tensor and the electric polarization in the statical configuration $\overset{\circ}{B}$, become:

$$\overset{\circ}{\boldsymbol{\Pi}}= \frac{\partial \mathcal{H}}{\partial \mathbf{G}}(\overset{\circ}{\mathbf{G}}, \overset{\circ}{\boldsymbol{\mathcal{E}}}), \quad \overset{\circ}{\boldsymbol{\mathcal{P}}}= -\frac{\partial \mathcal{H}}{\partial \boldsymbol{\mathcal{E}}}(\overset{\circ}{\mathbf{G}}, \overset{\circ}{\boldsymbol{\mathcal{E}}}), \tag{3.5}$$

where

$$\overset{\circ}{\boldsymbol{\Pi}}=\overset{\circ}{J}\overset{\circ}{\mathbf{F}}^{-1}\overset{\circ}{\mathbf{S}}\overset{\circ}{\mathbf{F}}^{-T}, \quad \overset{\circ}{\boldsymbol{\mathcal{P}}}=\overset{\circ}{J}\overset{\circ}{\mathbf{F}}^{-1}\overset{\circ}{\mathbf{P}} \tag{3.6}$$

and

$$\overset{\circ}{\mathbf{G}} = \frac{1}{2}(\overset{\circ}{\mathbf{F}}{}^{T}\,\overset{\circ}{\mathbf{F}} - \mathbf{1}), \qquad \overset{\circ}{\boldsymbol{\mathcal{E}}} = \overset{\circ}{\mathbf{F}}{}^{T}\,\overset{\circ}{\mathbf{E}}\,. \tag{3.7}$$

Now, we assume that time-dependent incremental deformations and fields are applied to the body in the initial configuration $\overset{\circ}{B}$, determining their description into the current configuration B_t. Here, all the fields referred to $\overset{\circ}{B}$ as reference configuration, will be denoted by a subscript "o". Let $\mathbf{u}(\mathbf{x}, t)$ the small displacement from $\overset{\circ}{B}$ to B_t and let $\mathbf{F}_o = \mathbf{F}_o(\mathbf{x}, t)$ the gradient of deformation from $\overset{\circ}{B}$ to B_t, $\overset{\circ}{B}$ being taken as reference configuration. We define the gradient of the displacement $\mathbf{u}(\mathbf{x}, t)$ by $\mathbf{H}_o(\mathbf{x}, t)$, and $J_o(\mathbf{x}, t)$ as the determinant of $\mathbf{F}_o(\mathbf{x}, t)$. All the fields involved are regarded as functions of $\mathbf{x}$ and t, when reported to $\overset{\circ}{B}$. For simplicity, we shall suppress the argument $\mathbf{x}$ in the following notations. Thus, we obtain:

$$\mathbf{F}_o(t) = \mathbf{1} + \mathbf{H}_o(t), \quad \mathbf{F}(t) = \mathbf{F}_o(t)\,\overset{\circ}{\mathbf{F}}, \quad J(t) = J_o(t)\,\overset{\circ}{J}\,. \tag{3.8}$$

Here $\mathbf{F}(t)$ is the deformation gradient from B_R to B_t and $J(t)$ is its determinant.

Consequently, to obtain the field equations referred to the configuration $\overset{\circ}{B}$, we introduce the following Piola-Kirchhoff type fields:

$$\begin{gathered} \boldsymbol{\Theta}_o(t) = J_o\mathbf{F}_o^{-1}(t)\mathbf{S}(t) = \overset{\circ}{J}{}^{-1}\,\overset{\circ}{\mathbf{F}}\,\boldsymbol{\Pi}(t)\,\overset{\circ}{\mathbf{F}}{}^{T}\,\mathbf{F}_o^{T}(t), \\ \boldsymbol{\Pi}_o(t) = J_o\mathbf{F}_o^{-1}\mathbf{S}\mathbf{F}_o^{-T}, \quad \boldsymbol{\Theta}_o = \boldsymbol{\Pi}_o\mathbf{F}_o^{T}, \end{gathered} \tag{3.9}$$

$$\boldsymbol{\mathcal{P}}_o(t) = J_o(t)\mathbf{F}_o^{-1}(t)\mathbf{P}(t) = \overset{\circ}{J}{}^{-1}\,\overset{\circ}{\mathbf{F}}\,\boldsymbol{\mathcal{P}}(t), \quad \boldsymbol{\mathcal{E}}_o(t) = \mathbf{F}_o^{T}(t)\mathbf{E}(t), \tag{3.10}$$

$$\boldsymbol{\mathcal{D}}_o(t) = J_o(t)\mathbf{F}_o^{-1}(t)\mathbf{D}(t) = J_o(t)\mathbf{F}_o^{-1}(t)\mathbf{E}(t) + \boldsymbol{\mathcal{P}}_o(t), \tag{3.11}$$

$$\boldsymbol{\mathcal{T}}_o(t) = J_o(t)\mathbf{F}_o^{-1}(t)\mathbf{T}(t) = \boldsymbol{\Theta}_o(t) - \boldsymbol{\mathcal{P}}_o(t) \otimes \mathbf{E}(t). \tag{3.12}$$

Furthermore,

$$\mathbf{H}_o(0) = \mathbf{0}, \quad \mathbf{F}_o(0) = \mathbf{1}, \quad J_o(0) = 1. \tag{3.13}$$

Therefore,

$$\boldsymbol{\Theta}_o(0) = \overset{\circ}{\mathbf{S}}, \quad \boldsymbol{\mathcal{P}}_o(0) = \overset{\circ}{\mathbf{P}}, \quad \boldsymbol{\mathcal{E}}_o(0) = \overset{\circ}{\mathbf{E}}, \quad \boldsymbol{\mathcal{D}}_o(0) = \overset{\circ}{\mathbf{D}}, \quad \boldsymbol{\mathcal{T}}_o(0) = \overset{\circ}{\mathbf{T}}\,. \tag{3.14}$$

In conclusion, we obtain the balance equations in updated Lagrangean description:

$$\rho(t)J_o(t) = \overset{\circ}{\rho}, \quad q(t)J_o(t) = q_o, \tag{3.15}$$

$$\begin{gathered} \overset{\circ}{\rho}\,\ddot{\mathbf{u}}(t) = \mathrm{div}_{\mathbf{X}}\boldsymbol{\mathcal{T}}_o(t) + \overset{\circ}{\rho}\,\mathbf{f}(t) + q_o(t)\mathbf{E}(t) + (\boldsymbol{\mathcal{P}}_o(t)\cdot\nabla_{\mathbf{X}})\mathbf{E}(t), \\ \mathrm{div}_{\mathbf{X}}\boldsymbol{\mathcal{D}}_o(t) = q_o(t), \quad \mathrm{rot}_{\mathbf{X}}\boldsymbol{\mathcal{E}}_o(t) = \mathbf{0}. \end{gathered} \tag{3.16}$$

Here $q_o(t)$ is the current volumetric electric charge density per unit material volume in the configuration $\overset{\circ}{B}$.

We find that the jump conditions and the electromechanical surface stress vector of Piola-Kirchhoff type are given by:

$$\overset{\circ}{\mathbf{n}} \cdot [\boldsymbol{\mathcal{D}}_o(t)] = w_o(t), \quad \overset{\circ}{\mathbf{n}} \times [\boldsymbol{\mathcal{E}}_o(t)] = \mathbf{0}, \tag{3.17}$$
$$\mathbf{t}_{o\mathbf{n}}(t) = \boldsymbol{\mathcal{T}}_o(t) \overset{\circ}{\mathbf{n}} = (\boldsymbol{\Theta}_o(t) - \boldsymbol{\mathcal{P}}_o(t) \otimes \mathbf{E}(t)) \overset{\circ}{\mathbf{n}} \text{ on } \partial \overset{\circ}{V},$$

where $w_o(t)$ is the current surface charge density per unit material surface aria in the configuration $\overset{\circ}{B}$.

Finally, we give the constitutive relations in the form:

$$\boldsymbol{\Pi}_o(t) = \frac{\partial \mathcal{H}}{\partial \mathbf{G}}(\mathbf{G}_o(t), \boldsymbol{\mathcal{E}}_o(t)), \quad \boldsymbol{\mathcal{P}}_o(t) = -\frac{\partial \mathcal{H}}{\partial \mathbf{E}}(\mathbf{G}_o(t), \boldsymbol{\mathcal{E}}_o(t)), \tag{3.18}$$

where

$$\mathbf{G}_o(t) = \frac{1}{2}(\mathbf{F}_o(t)^T \mathbf{F}_o(t) - \mathbf{1}). \tag{3.19}$$

Now, we define by $\mathbf{e}(t) = \mathbf{e}(\mathbf{x}, t)$ the infinitesimal perturbation of the initial applied electric field $\overset{\circ}{\mathbf{E}}$:

$$\mathbf{E}(t) = \overset{\circ}{\mathbf{E}} + \mathbf{e}(t), \tag{3.20}$$

and using (3.8) we derive the useful relation

$$\mathbf{F}(t) = \overset{\circ}{\mathbf{F}} + \mathbf{H}_o(t) \overset{\circ}{\mathbf{F}}. \tag{3.21}$$

In what follows, we suppose that the perturbations $\mathbf{H}_o(t)$ and $\mathbf{e}(t)$ are small, such that the products of all terms containing $\mathbf{H}_o(t)$ and $\mathbf{e}(t)$ may be neglect. In particular, we obtain

$$J_o(t) = 1 + \text{tr}\mathbf{H}_o(t), \quad \mathbf{F}_o^{-1}(t) = \mathbf{1} - \mathbf{H}_o(t). \tag{3.22}$$

Henceforward, we shall denote by a superposed bar the small perturbation of an arbitrary field. So, we have for Green tensor

$$\mathbf{G}(t) = \overset{\circ}{\mathbf{G}} + \bar{\mathbf{G}}(t), \quad \text{where} \quad \bar{\mathbf{G}}(t) = \overset{\circ}{\mathbf{F}}{}^T \mathbf{g}(t) \overset{\circ}{\mathbf{F}}. \tag{3.23}$$

Here

$$\mathbf{g}(t) = \frac{1}{2}(\mathbf{H}_o(t) + \mathbf{H}_o^T(t)) \tag{3.24}$$

is the associated infinitesimal strain tensor.

Similarly, we define the perturbation of the electric field $\bar{\boldsymbol{\mathcal{E}}}(t)$ by

$$\boldsymbol{\mathcal{E}}(t) = \overset{\circ}{\boldsymbol{\mathcal{E}}} + \bar{\boldsymbol{\mathcal{E}}}(t), \quad \text{where} \quad \bar{\boldsymbol{\mathcal{E}}}(t) = \overset{\circ}{\mathbf{F}}{}^T (\mathbf{e}(t) + \mathbf{H}_o^T(t) \overset{\circ}{\mathbf{E}}). \tag{3.25}$$

We also obtain that

$$\boldsymbol{\Theta}_o(t) = \overset{\circ}{\mathbf{S}} + \bar{\boldsymbol{\Theta}}_o(t),\ \boldsymbol{\Pi}(t) = \overset{\circ}{\boldsymbol{\Pi}} + \bar{\boldsymbol{\Pi}}(t),\ \bar{\boldsymbol{\Theta}}_o(t) = \overset{\circ}{J}^{-1} \overset{\circ}{\mathbf{F}}\, \bar{\boldsymbol{\Pi}}(t)\, \overset{\circ}{\mathbf{F}}^{T} + \overset{\circ}{\mathbf{S}}\, \mathbf{H}_o^T(t). \tag{3.26}$$

The previous relation shows that the stress perturbation $\bar{\boldsymbol{\Theta}}_o(t)$ is known if the stress perturbation $\bar{\boldsymbol{\Pi}}(t)$ is known, and vice-versa.

Next, we define

$$\boldsymbol{\mathcal{P}}_o(t) = \overset{\circ}{\mathbf{P}} + \bar{\boldsymbol{\mathcal{P}}}_o(t), \quad \boldsymbol{\mathcal{P}}(t) = \overset{\circ}{\boldsymbol{\mathcal{P}}} + \bar{\boldsymbol{\mathcal{P}}}(t), \quad \text{with} \quad \bar{\boldsymbol{\mathcal{P}}}_o(t) = \overset{\circ}{J}^{-1} \overset{\circ}{\mathbf{F}}\, \bar{\boldsymbol{\mathcal{P}}}(t). \tag{3.27}$$

Thus, the perturbation $\bar{\boldsymbol{\mathcal{P}}}_o(t)$ is known if the perturbation $\bar{\boldsymbol{\mathcal{P}}}(t)$ is known, and vice-versa.

Similarly, if we take

$$\boldsymbol{\mathcal{D}}_o(t) = \overset{\circ}{\mathbf{D}} + \bar{\boldsymbol{\mathcal{D}}}_o(t), \tag{3.28}$$

we obtain that

$$\bar{\boldsymbol{\mathcal{D}}}_o(t) = \mathbf{e}(t) + \bar{\boldsymbol{\mathcal{P}}}_o(t) + \overset{\circ}{\mathbf{E}}\, \mathrm{tr}\mathbf{H}_o(t) - \mathbf{H}_o(t)\, \overset{\circ}{\mathbf{E}}, \tag{3.29}$$

i.e. to know the perturbation $\bar{\boldsymbol{\mathcal{D}}}_o(t)$ we must know the perturbation $\bar{\boldsymbol{\mathcal{P}}}_o(t)$.

We also derive that

$$\boldsymbol{\mathcal{E}}_o(t) = \overset{\circ}{\mathbf{E}} + \bar{\boldsymbol{\mathcal{E}}}_o(t), \quad \text{with} \quad \bar{\boldsymbol{\mathcal{E}}}_o(t) = \mathbf{e}(t) + \mathbf{H}_o^T(t)\, \overset{\circ}{\mathbf{E}}\, . \tag{3.30}$$

Finally, we find that

$$\boldsymbol{\mathcal{T}}_o(t) = \overset{\circ}{\mathbf{T}} + \bar{\boldsymbol{\mathcal{T}}}_o(t), \quad \text{where} \quad \bar{\boldsymbol{\mathcal{T}}}_o(t) = \bar{\boldsymbol{\Theta}}_o(t) - \bar{\boldsymbol{\mathcal{P}}}_o(t) \otimes \overset{\circ}{\mathbf{E}} - \overset{\circ}{\mathbf{P}} \otimes \mathbf{e}(t). \tag{3.31}$$

At this stage is evident that all perturbations are known if the perturbations $\bar{\boldsymbol{\Pi}}(t)$ and $\bar{\boldsymbol{\mathcal{P}}}(t)$ are known. To obtain these perturbations we use the following constitutive equations:

$$\bar{\boldsymbol{\Pi}}(t) = \frac{\partial^2 \overset{\circ}{\mathcal{H}}}{\partial \mathbf{G} \partial \mathbf{G}} [\overset{\circ}{\mathbf{F}}^{T}\ \mathbf{g}(t)\, \overset{\circ}{\mathbf{F}}] + \frac{\partial^2 \overset{\circ}{\mathcal{H}}}{\partial \boldsymbol{\mathcal{E}} \partial \mathbf{G}} [\overset{\circ}{\mathbf{F}}^{T}\ (\mathbf{e}(t) + \mathbf{H}_o^T(t)\, \overset{\circ}{\mathbf{E}})], \tag{3.32}$$

$$\bar{\boldsymbol{\mathcal{P}}}(t) = -\frac{\partial^2 \overset{\circ}{\mathcal{H}}}{\partial \mathbf{G} \partial \boldsymbol{\mathcal{E}}} [\overset{\circ}{\mathbf{F}}^{T}\ \mathbf{g}(t)\, \overset{\circ}{\mathbf{F}}] - \frac{\partial^2 \overset{\circ}{\mathcal{H}}}{\partial \boldsymbol{\mathcal{E}} \partial \boldsymbol{\mathcal{E}}} [\overset{\circ}{\mathbf{F}}^{T}\ (\mathbf{e}(t) + \mathbf{H}_o^T(t)\, \overset{\circ}{\mathbf{E}})]. \tag{3.33}$$

Here the symbol "$\circ$" superposed on $\mathcal{H}$ indicated that the corresponding second-order derivatives of the generalized Helmholtz free energy are taken at $\overset{\circ}{\mathbf{G}}$ and $\overset{\circ}{\boldsymbol{\mathcal{E}}}$.

The perturbations of the force, resp. of charge densities are defined by:

$$\mathbf{f}(t) = \overset{\circ}{\mathbf{f}} + \bar{\mathbf{f}}(t), \quad q_o(t) = \overset{\circ}{q} + \bar{q}(t), \quad w_o(t) = \overset{\circ}{w} + \bar{w}(t). \tag{3.34}$$

Concluding, from the relations obtained last section we derive that the incremental fields satisfy the following balance equations:

$$\overset{\circ}{\rho}\,\ddot{\mathbf{u}}(t) = \mathrm{div}_{\mathbf{X}}(\bar{\mathbf{\Theta}}_o(t) - \bar{\boldsymbol{\mathcal{P}}}_o(t)\otimes \overset{\circ}{\mathbf{E}} - \overset{\circ}{\mathbf{P}} \otimes\mathbf{e}(t)) + \overset{\circ}{\rho}\,\bar{\mathbf{f}}(t) + \bar{q}(t)\,\overset{\circ}{\mathbf{E}} + \tag{3.35}$$
$$+ \overset{\circ}{q}\,\mathbf{e}(t) + (\bar{\boldsymbol{\mathcal{P}}}_o(t)\cdot\nabla_{\mathbf{X}})\,\overset{\circ}{\mathbf{E}} + (\overset{\circ}{\mathbf{P}}\cdot\nabla_{\mathbf{X}})\mathbf{e}(t),$$

$$\mathrm{div}_{\mathbf{X}}(\mathbf{e}(t) + \bar{\boldsymbol{\mathcal{P}}}_o(t) + \overset{\circ}{\mathbf{E}}\,\mathrm{tr}\mathbf{H}_o(t) - \mathbf{H}_o(t)\,\overset{\circ}{\mathbf{E}}) = \bar{q}(t),\ \mathrm{rot}_{\mathbf{X}}(\mathbf{e}(t) + \mathbf{H}_o^T(t)\,\overset{\circ}{\mathbf{E}}) = \mathbf{0}. \tag{3.36}$$

Note that without further assumptions the differential balance equations satisfied by the incremental fields cannot be simplified.

The jump conditions for the involved incremental fields are:

$$\overset{\circ}{\mathbf{n}}\cdot[\mathbf{e}(t) + \bar{\boldsymbol{\mathcal{P}}}_o(t) + \overset{\circ}{\mathbf{E}}\,\mathrm{tr}\mathbf{H}_o(t) - \mathbf{H}_o(t)\,\overset{\circ}{\mathbf{E}}] = \bar{w}(t),\ \overset{\circ}{\mathbf{n}}\times[\mathbf{e}(t) + \mathbf{H}_o^T(t)\,\overset{\circ}{\mathbf{E}}] = \mathbf{0}. \tag{3.37}$$

Finally, we obtain that the incremental electromechanical surface stress vector of Piola-Kirchhoff type $\bar{\mathbf{t}}_{o\mathbf{n}}(t)$ reduces to

$$\bar{\mathbf{t}}_{o\mathbf{n}}(t) = (\bar{\mathbf{\Theta}}_o(t) - \bar{\boldsymbol{\mathcal{P}}}_o(t)\otimes\overset{\circ}{\mathbf{E}} - \overset{\circ}{\mathbf{P}}\otimes\mathbf{e}(t))\,\overset{\circ}{\mathbf{n}} \quad \text{on}\,\partial\overset{\circ}{V}. \tag{3.38}$$

4 Special cases: homogeneous initial state and non-polarizable environment

In this part we introduce two simplifying hypotheses, essential for the subsequent developments.

H1: The body is homogeneous, the initial deformation gradient $\overset{\circ}{\mathbf{F}}$ is constant in the domain V_R and the initial applied electric field $\overset{\circ}{\mathbf{E}}$ is constant in all of space.

H2: The environment (i.e. the vacuum) of the body is not polarizable.

As regards the second assumption, he is justified since the dielectric constants of electroelastic materials are significantly larger than the dielectric constant of the vacuum. Then, we have that $\overset{\circ}{\mathbf{P}} = \mathbf{0}$ and $\mathbf{e}(t) = \mathbf{0}$ in the exterior of the body $\overset{\circ}{V}$. Thus, the associated limit values on $\partial\overset{\circ}{V}$ satisfy the relations $\overset{\circ}{\mathbf{P}}{}^{+} = \mathbf{0}$ and $\mathbf{e}^{+}(t) = \mathbf{0}$. It is evident that the second assumption leads to an important simplification of the problem, since, by neglecting the surroundings of the body, our problem is transformed into one of a hyperelastic dielectric.

Now, if we consider the first assumption, we observe that $\overset{\circ}{\mathbf{S}}$, $\overset{\circ}{\mathbf{P}}$, $\overset{\circ}{\mathbf{E}}$, $\overset{\circ}{\mathbf{T}}$ and $\overset{\circ}{\mathbf{D}}$ are constant fields in the domain $\overset{\circ}{V}$. Consequently, the balance equations (3.3) take place in the assumed homogeneous state, only if

$$\overset{\circ}{\mathbf{f}} = \mathbf{0}, \quad \overset{\circ}{q} = 0 \quad \text{in} \quad \overset{\circ}{V}. \tag{4.1}$$

Since the initial applied electric field is constant in all the space, and tacking into account the previous remarks, we find that the second jump condition (3.4) is satisfied if $\overset{\circ}{\mathbf{P}}$ and $\overset{\circ}{w}$ are related by

$$\overset{\circ}{\mathbf{P}} \cdot \overset{\circ}{\mathbf{n}} = -\overset{\circ}{w}, \quad \text{on} \quad \partial \overset{\circ}{V}. \tag{4.2}$$

Further, the electromechanical stress vector $\overset{\circ}{\mathbf{t}}_{\mathbf{n}}$ is given by the third relation (3.4).

An important consequence of assumption **H1**, together with relations (4.1) and equations (3.35-3.36), is that the differential balance equations take the form:

$$\overset{\circ}{\rho}\, \ddot{\mathbf{u}}(t) = \mathrm{div}_{\mathbf{X}}(\bar{\mathbf{\Theta}}_o(t) - \bar{\boldsymbol{\mathcal{P}}}_o(t) \otimes \overset{\circ}{\mathbf{E}}) + \overset{\circ}{\rho}\, \bar{\mathbf{f}}(t) + \bar{q}(t)\, \overset{\circ}{\mathbf{E}}, \tag{4.3}$$

$$\mathrm{div}_{\mathbf{X}}(\mathbf{e}(t) + \bar{\boldsymbol{\mathcal{P}}}_o(t)) = \bar{q}(t), \quad \mathrm{rot}_{\mathbf{X}}\mathbf{e}(t) = \mathbf{0}. \tag{4.4}$$

Moreover, the second assumption **H2** implies that the boundary conditions (3.37-3.38) reduce to:

$$\begin{aligned} &\overset{\circ}{\mathbf{n}} \cdot (\mathbf{e}(t) + \bar{\boldsymbol{\mathcal{P}}}_o(t)) = -\bar{w}(t), \quad \overset{\circ}{\mathbf{n}} \times \mathbf{e}(t) = \mathbf{0}, \\ &\bar{\mathbf{t}}_{o\mathbf{n}}(t) = (\bar{\mathbf{\Theta}}_o(t) - \bar{\boldsymbol{\mathcal{P}}}_o(t) \otimes \overset{\circ}{\mathbf{E}})\, \overset{\circ}{\mathbf{n}} \quad \text{on}\, \partial \overset{\circ}{V}. \end{aligned} \tag{4.5}$$

The system (4.3-4.5) takes place whenever the initial state of the body is homogeneous and the environment of the body is not polarizable.

To complete the description of the incremental behaviour of the body, we analyze the constitutive equations (3.32-3.33), which give the perturbations $\bar{\mathbf{\Pi}}(t)$ and $\bar{\boldsymbol{\mathcal{P}}}(t)$, under the present assumptions. Moreover, if we use the relations (3.26-3.27), we obtain the perturbations $\bar{\mathbf{\Theta}}_o(t)$ and $\bar{\boldsymbol{\mathcal{P}}}_o(t)$ in the form:

$$\bar{\Theta}_{okl} = (\overset{\circ}{c}_{klmn} + \overset{\circ}{S}_{kn}\, \delta_{lm} - \overset{\circ}{e}_{nkl} \overset{\circ}{E}_m) u_{m,n} - \overset{\circ}{e}_{mkl}\, e_m \tag{4.6}$$

$$\bar{\mathcal{P}}_{ok} = (\overset{\circ}{e}_{kmn} + \overset{\circ}{\eta}_{km} \overset{\circ}{E}_n) u_{n,m} + \overset{\circ}{\eta}_{kl}\, e_l. \tag{4.7}$$

Here

$$\begin{aligned} &\overset{\circ}{c}_{klmn} = \overset{\circ}{J}^{-1} \overset{\circ}{F}_{kp} \overset{\circ}{F}_{lq} \overset{\circ}{F}_{mr} \overset{\circ}{F}_{ns} \frac{\partial^2 \overset{\circ}{\mathcal{H}}}{\partial G_{rs} \partial G_{pq}}, \\ &\overset{\circ}{e}_{mkl} = -\overset{\circ}{J}^{-1} \overset{\circ}{F}_{mp} \overset{\circ}{F}_{kq} \overset{\circ}{F}_{lr} \frac{\partial^2 \overset{\circ}{\mathcal{H}}}{\partial \mathcal{E}_p \partial G_{qr}}, \quad \overset{\circ}{\eta}_{kl} = -\overset{\circ}{J}^{-1} \overset{\circ}{F}_{km} \overset{\circ}{F}_{ln} \frac{\partial^2 \overset{\circ}{\mathcal{H}}}{\partial \mathcal{E}_m \partial \mathcal{E}_n}. \end{aligned} \tag{4.8}$$

are the *instantaneous material moduli* (elastic, piezoelectric and dielectric moduli). The constitutive relations (4.8) are valid even the simplifying assumptions **H1** and **H2** are not satisfied.

The instantaneous material moduli possesses the following symmetry properties:

$$\overset{\circ}{c}_{klmn} = \overset{\circ}{c}_{lkmn} = \overset{\circ}{c}_{klnm} = \overset{\circ}{c}_{mnkl}, \quad \overset{\circ}{e}_{mkl} = \overset{\circ}{e}_{mlk}, \quad \overset{\circ}{\eta}_{kl} = \overset{\circ}{\eta}_{lk}. \tag{4.9}$$

It follows that, for general anisotropy, there exist 21 independent instantaneous moduli $\overset{\circ}{c}_{klmn}$, 18 independent instantaneous moduli $\overset{\circ}{e}_{mkl}$ and 6 independent instantaneous moduli $\overset{\circ}{\eta}_{kl}$.

At this point, we introduce the incremental electromechanical stress tensor $\mathbf{\Sigma}$ and the incremental electric displacement vector $\mathbf{\Delta}$ by the relations:

$$\mathbf{\Sigma}(t) = \bar{\mathbf{\Theta}}_o(t) - \bar{\boldsymbol{\mathcal{P}}}_o(t) \otimes \overset{\circ}{\mathbf{E}}, \quad \mathbf{\Delta}(t) = \mathbf{e}(t) + \bar{\boldsymbol{\mathcal{P}}}_o(t). \tag{4.10}$$

It follows that, according to relations (4.6-4.7), the constitutive relations describing the behaviour of the incremental fields, under the previous hypotheses, are:

$$\Sigma_{kl} = \overset{\circ}{\Omega}_{klmn}\, u_{m,n} - \overset{\circ}{\Lambda}_{mkl}\, e_m, \quad \Delta_k = \overset{\circ}{\Lambda}_{kmn}\, u_{n,m} + \overset{\circ}{\epsilon}_{kl}\, e_l, \tag{4.11}$$

where

$$\begin{gathered} \overset{\circ}{\Omega}_{klmn} = \overset{\circ}{c}_{klmn} + \overset{\circ}{S}_{kn}\, \delta_{lm} - \overset{\circ}{e}_{kmn} \overset{\circ}{E}_l - \overset{\circ}{e}_{nkl} \overset{\circ}{E}_m - \overset{\circ}{\eta}_{kn} \overset{\circ}{E}_l \overset{\circ}{E}_m, \\ \overset{\circ}{\Lambda}_{mkl} = \overset{\circ}{e}_{mkl} + \overset{\circ}{\eta}_{mk} \overset{\circ}{E}_l, \quad \overset{\circ}{\epsilon}_{kl} = \delta_{kl} + \overset{\circ}{\eta}_{kl} \end{gathered} \tag{4.12}$$

are the components of the instantaneous elasticity tensor $\overset{\circ}{\mathbf{\Omega}}$, of the instantaneous coupling tensor $\overset{\circ}{\mathbf{\Lambda}}$, resp. of the instantaneous dielectric tensor $\overset{\circ}{\boldsymbol{\epsilon}}$.

From relations (4.9) we find the symmetry relations

$$\overset{\circ}{\Omega}_{klmn} = \overset{\circ}{\Omega}_{nmlk}, \quad \overset{\circ}{\epsilon}_{kl} = \overset{\circ}{\epsilon}_{lk}\ . \tag{4.13}$$

Moreover, we see that $\overset{\circ}{\Omega}_{klmn}$ is not symmetric according to indices (k,l) and (m,n) and $\overset{\circ}{\Lambda}_{mkl}$ is not symmetric relative to indices (k,l). It follows that, generally, there are 45 independent instantaneous elastic moduli $\overset{\circ}{\Omega}_{klmn}$, 27 independent instantaneous coupling moduli $\overset{\circ}{\Lambda}_{mkl}$ and 6 independent instantaneous dielectric moduli $\overset{\circ}{\epsilon}_{kl}$. These moduli are constant parameters depending on the considered hyperelastic material, and on the initial electric and mechanical applied fields. We note at this stage that, even if this problem is linearized, the solution depends non-linearly on the initial applied electric field.

In this frame, using the incremental electromechanical stress tensor $\mathbf{\Sigma}$ and the incremental electric displacement vector $\mathbf{\Delta}$, we derive from (4.3-4.4) the differential balance equations in the final form:

$$\overset{\circ}{\rho}\, \ddot{\mathbf{u}}(t) = \mathrm{div}_{\mathbf{X}} \mathbf{\Sigma} + \overset{\circ}{\rho}\, \bar{\mathbf{f}}(t) + \bar{q}(t)\, \overset{\circ}{\mathbf{E}}, \quad \mathrm{div}_{\mathbf{X}} \mathbf{\Delta} = \bar{q}(t), \quad \mathrm{rot}_{\mathbf{X}} \mathbf{e}(t) = \mathbf{0}. \tag{4.14}$$

The associated boundary conditions are:

$$\overset{\circ}{\mathbf{n}} \cdot \mathbf{\Delta} = -\bar{w}(t), \quad \overset{\circ}{\mathbf{n}} \times \mathbf{e}(t) = \mathbf{0}, \quad \bar{\mathbf{t}}_{o\mathbf{n}}(t) = \mathbf{\Sigma}\, \overset{\circ}{\mathbf{n}} \quad \text{on}\, \partial \overset{\circ}{V}\ . \tag{4.15}$$

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