Connections on k-symplectic manifolds

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Abstract. On a k-symplectic manifold will be defined a canonical connection which induces on the reduced manifold a canonical connection, too. Two reduced standard k-symplectic manifolds with respect to the action of a Lie group G are considered, and the relation between the induced canonical connections is established.

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1 Introduction

Having a k-symplectic manifold, one can obtain, by Marsden-Weinstein reduction, other k-symplectic manifolds. This procedure is well known and important in the symplectic mechanics, having many applications in fluids [8], electromagnetism and plasma physics [7], etc. We proved that under certain assumptions [2], a k-symplectic manifold can be reduced to a k-symplectic manifold, too.

In the present paper, using a momentum map for an appropriate action of a Lie group G on the standard k-symplectic manifold $(T_k^1)^*\mathbb{R}^n$ endowed with the canonical k-symplectic structure induced from $(\mathbb{R}^n, \omega_0)$ [1], we shall describe the Marsden-Weinstein reduction in this case. Then, by the mean of a diffeomorphism between $T_k^1\mathbb{R}^n$ and $(T_k^1)^*\mathbb{R}^n$ (for instance, the Legendre transformation associated to a regular Lagrangian), we can define a k-symplectic structure on the k-tangent bundle $T_k^1\mathbb{R}^n$, that will be reduced, too. We proved that on a k-symplectic manifold, there exists a canonical connection [3]. This canonical connection induces a canonical connection on the reduced manifold. Finally, we shall discuss the relation between the two induced canonical connections on the reduced manifolds.

2 k-symplectic structures

Definition 2.1. [1] A k-symplectic manifold $(M, \omega_i, V)_{1\leq i\leq k}$ is an $(n+nk)$-dimensional smooth manifold M together with k 2-forms ω_i, $1 \leq i \leq k$, and an nk-dimensional distribution V that satisfy the conditions:

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1. ω_i is closed, for every $1 \leq i \leq k$;
2. $\bigcap_{i=1}^{k} \ker\ \omega_i = \{0\}$;
3. $\omega_{i|_{V\times V}} = 0$, for every $1 \leq i \leq k$.

The canonical model for this structure is the k-cotangent bundle $(T_k^1)^*N$ of an arbitrary manifold N, which can be identified with the vector bundle $J^1(N, \mathbb{R}^k)_0$ whose total space is the manifold of 1-jets of maps with target $0 \in \mathbb{R}^k$, and projection $\tau^*(j^1_{x,0}\sigma) = x$. Identify $(T_k^1)^*N$ with the Whitney sum of k copies of T^*N [6],

$$(T_k^1)^*N \cong T^*N \oplus \overset{k}{\ldots} \oplus T^*N,\ j_{x,0}\sigma \mapsto (j^1_{x,0}\sigma^1, \ldots, j^k_{x,0}\sigma^k),$$

where $\sigma^i = \pi_i \circ \sigma : N \longrightarrow \mathbb{R}$ is the i-th component of σ. The k-symplectic structure on $(T_k^1)^*N$ is given by $\omega_i = (\tau_i^*)^*(\omega_0)$ and $V_{j^1_{x,0}\sigma} = \ker(\tau^*)_*(j^1_{x,0}\sigma)$, where $\tau_i^* : (T_k^1)^*N \longrightarrow T^*N$ is the projection on the i-th copy T^*N of $(T_k^1)^*N$ and ω_0 is the standard symplectic structure on T^*N.

Let $(M, \omega_i, V)_{1\leq i\leq k}$ be a k-symplectic manifold. Consider the bundle morphism

$$\Omega^{\#} : T_k^1 M \longrightarrow T^*M,\ \Omega^{\#}(X_1, \ldots, X_k) := \sum_{j=1}^{k} i_{X_j}\omega_j.$$

Definition 2.2. A k-Hamiltonian system is an ordered k-tuple of vector fields $(X_1, \ldots, X_k) \in T_k^1 M$ such that there exists a smooth function $H : M \longrightarrow \mathbb{R}$, called the Hamiltonian of $(X_1, \ldots, X_k)$, with the property

$$\Omega^{\#}(X_1, \ldots, X_k) = dH. \tag{2.1}$$

We will denote by $((X_1)_H, \ldots, (X_k)_H)$ the k-Hamiltonian system corresponding to H.

Definition 2.3. A k-symplectic action of a Lie group G on M is an action $\Phi : G \times M \longrightarrow M$ such that

$$(\Phi_g)^*\omega_i = \omega_i, \quad \forall\ g \in G, \forall i \in \overline{1,k}, \tag{2.2}$$

where $\Phi_g : M \longrightarrow M, \Phi_g(x) := \Phi(g, x)$.

Let $\mathcal{G}^k = \mathcal{G} \times \overset{k}{\ldots} \times \mathcal{G}$ and $\mathcal{G}^{*^k} = \mathcal{G}^* \times \overset{k}{\ldots} \times \mathcal{G}^*$, where $\mathcal{G}^*$ is the dual of the Lie algebra $\mathcal{G}$ of G.

Definition 2.4. A momentum map for the k-symplectic action $\Phi : G \times M \longrightarrow M$ is a map $J : M \longrightarrow \mathcal{G}^{*^k}$ defined by

$$(X_i)_{\widehat{J}(\xi_1,\ldots,\xi_k)} := (\xi_i)_M, \quad \forall(\xi_1, \ldots, \xi_k) \in \mathcal{G}^k, \forall i \in \overline{1,k}, \tag{2.3}$$

where $\widehat{J}(\xi_1, \ldots, \xi_k) : M \longrightarrow \mathbb{R}$, $\widehat{J}(\xi_1, \ldots, \xi_k)(x) := J(x)(\xi_1, \ldots, \xi_k)$ and $(\xi_i)_M$ are the fundamental vector fields on M corresponding to the elements $\xi_i \in \mathcal{G}$, $i \in \{1, \ldots, k\}$.

For $g \in G$, define ${Ad_g}^k : \mathcal{G}^k \longrightarrow \mathcal{G}^k$, ${Ad_g}^k(\xi_1, \dots, \xi_k) := (Ad_g\xi_1, \dots, Ad_g\xi_k)$, where $Ad : G \longrightarrow Aut(G)$ denotes the adjoint representation and $Ad_g = Ad(g)$, and $Ad_g^{*^k} : \mathcal{G}^{*^k} \longrightarrow \mathcal{G}^{*^k}$, $Ad_g^{*^k}(\mu) = \mu \circ {Ad_g}^k$.

A momentum map $J : M \longrightarrow \mathcal{G}^{*^k}$ is called (Φ, Ad^{*^k})-*equivariant* if

$$J(\Phi_g(x)) = Ad^{*^k}_{g^{-1}} J(x), \ \forall g \in G, \ \forall x \in M. \tag{2.4}$$

Consider G a Lie group and $\Phi : G \times M \longrightarrow M$ a k-symplectic action of G on the k-symplectic manifold $(M, \omega_i, V)_{1 \leq i \leq k}$. Let $J : M \longrightarrow \mathcal{G}^{*^k}$ be a (Φ, Ad^{*^k})-equivariant momentum map for Φ and $\mu \in \mathcal{G}^{*^k}$ a regular value of J. Then $J^{-1}(\mu)$ is a smooth manifold. The isotropy subgroup of μ with respect to the k-coadjoint action, $G_\mu := \{g \in G \mid Ad^{*^k}_{g^{-1}}(\mu) = \mu\} \subset G$, leaves invariant $J^{-1}(\mu)$. Assume that G_μ acts freely and properly on $J^{-1}(\mu)$. Then the quotient space $M_\mu := J^{-1}(\mu)/G_\mu$ is also a smooth manifold. A reduction type theorem for k-symplectic manifolds holds:

Theorem 2.5. [2] *Under the hypotheses above, on $M_\mu := J^{-1}(\mu)/G_\mu$ there exists a unique k-symplectic structure $((\omega_\mu)_i, V_\mu)_{1 \leq i \leq k}$, such that*

$$\pi_\mu{}^*(\omega_\mu)_i = i_\mu{}^*\omega_i, \ \forall i \in \overline{1, k}, \tag{2.5}$$

where $\pi_\mu : J^{-1}(\mu) \longrightarrow M_\mu$ is the canonical projection and $i_\mu : J^{-1}(\mu) \longrightarrow M$ the canonical inclusion.

3 Canonical connections on k-symplectic manifolds

Let $(M, \omega_i, V)_{1 \leq i \leq k}$ be a k-symplectic manifold. For every $1 \leq i \leq k$, define

$$V_{i_x} := \bigcap_{j \neq i} \ker(\omega_{j_x}). \tag{3.1}$$

Denote by $\mathcal{F}$ the foliation integral to the distribution V and by $\mathcal{F}_i$ the foliation integral to V_i. It follows that [3]:

(a) for each $i \in \{1, \dots, k\}$ the distribution $V_i = (V_{i_x})_{x \in M}$ is integrable;

(b) $V = V_1 \oplus \dots \oplus V_k$;

(c) for each $j \in \{1, \dots, k\}$ the map

$$i_j : V_j \longrightarrow (N\mathcal{F})^*, \ X \mapsto i_X\omega_j \tag{3.2}$$

is an isomorphism, where $N\mathcal{F}$ denotes the normal bundle of $\mathcal{F}$.

Consider Q an n-dimensional integrable distribution on M transversal to $\mathcal{F}$ (and denote by $\mathcal{G}$ the foliation integral to Q), such that

(1) $\omega_i(Y, Y') = 0$ for any $Y, Y' \in \Gamma(Q)$ and for every $1 \leq i \leq k$;

(2) $[X, Y] \in \Gamma(V_i \oplus Q)$ for any $X \in \Gamma(V_i)$ and for any $Y \in \Gamma(Q)$.

Lemma 3.1. [3] *Let* $Y, Y' \in \Gamma(Q)$. *For each* $j \in \{1, \ldots, k\}$, *the map*

$$\psi_j^{YY'} : W \mapsto (\mathcal{L}_Y i_{Y'} \omega_j)(W) \tag{3.3}$$

$$(\mathcal{L}_Y i_{Y'} \omega_j)(W) = Y(\omega_j(Y', W)) - \omega_j(Y', [Y, W]),$$

for any $W \in \Gamma(TM)$, *belongs to* V_j^*.

Theorem 3.2. [3] *Let* $(M, \omega_i, V)_{1 \leq i \leq k}$ *be a k-symplectic manifold and let Q be an integrable distribution supplementary to V verifying the above conditions (1), (2) and such that*

$$(i_1^*)^{-1}(\psi_1^{YY'}) = \cdots = (i_k^*)^{-1}(\psi_k^{YY'}) \tag{3.4}$$

for any $Y, Y' \in \Gamma(Q)$, *where* $\psi_1^{YY'}, \ldots, \psi_k^{YY'}$ *are the maps defined in Lemma 3.1. Then there exists a unique connection* ∇ *on M satisfying the following properties:*

1. $\nabla \mathcal{F}_i \subset \mathcal{F}_i$ *for each* $i \in \{1, \ldots, k\}$, *and* $\nabla Q \subset Q$,
2. $\nabla \omega_1 = \cdots = \nabla \omega_k = 0$,
3. $T(X, Y) = 0$ *for any* $X \in \Gamma(V)$ *and for any* $Y \in \Gamma(Q)$,

where T denotes the torsion tensor field of ∇.

Remark that the splitting

$$TM = V \oplus Q = V_1 \oplus \cdots \oplus V_k \oplus Q$$

induces a canonical isomorphism between Q and $N\mathcal{F} := TM/V$, the normal bundle to the foliation $\mathcal{F}$. So, we shall define a connection ∇^{V_i} on each subbundle V_i, a connection ∇^Q on Q and then we take the sum of these connections for defining a global connection on M: for any $V, W \in \Gamma(TM)$, let

$$\nabla_V W := \nabla_V^{V_1} W_{V_1} + \cdots + \nabla_V^{V_k} W_{V_k} + \nabla_V^Q W_Q. \tag{3.5}$$

Proposition 3.3. [3] *The connection* ∇ *defined in Theorem 3.2. is torsion free along the leaves of the foliations* $\mathcal{F}$ *and* $\mathcal{G}$.

Proposition 3.4. [3] *The curvature tensor field of the connection* ∇ *defined in Theorem 3.2. vanishes along the leaves of the foliations* $\mathcal{F}$ *and* $\mathcal{G}$.

Generalizing the result obtained by I. Vaisman in [10], we shall give a reduction type theorem for the canonical connection on a k-symplectic manifold as follows.

Let ∇ be the canonical connection defined in Theorem 3.2. Assume that the k-symplectic action Φ is a ∇-*affine action*, that is, it preserves the connection ∇ and that $J^{-1}(\mu)$ is ∇-*self-parallel*, that is, $TJ^{-1}(\mu)$ is preserved by ∇-parallel translations along paths in $J^{-1}(\mu)$.

Theorem 3.5. *Let* $(M, \omega_i, V)_{1 \leq i \leq k}$ *be a k-symplectic manifold on which we have a* ∇-*affine k-symplectic action* Φ *of a Lie group G and there exists a* $(\Phi, {Ad^*}^k)$-*equivariant momentum map* $J : M \longrightarrow {\mathcal{G}^*}^k$. *Assume that* $\mu \in {\mathcal{G}^*}^k$ *is a regular value of J and that the isotropy group* G_μ *under the* ${Ad^*}^k$-*action on* ${\mathcal{G}^*}^k$ *acts freely and properly on* $J^{-1}(\mu)$. *Assume that* $J^{-1}(\mu)$ *is* ∇-*self-parallel. Then the canonical connection* ∇ *defined in Theorem 3.2. induces a canonical connection* ∇_μ *on* $M_\mu = J^{-1}(\mu)/G_\mu$.

4 The standard k-symplectic manifolds

For an arbitrary action $\Phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ of a Lie group G on $\mathbb{R}^n$, define the lifted action $\Phi^{T_k^*}$ to the standard k-symplectic manifold $(T_k^1)^*\mathbb{R}^n$:

$$\Phi^{T_k^*} : G \times (T_k^1)^*\mathbb{R}^n \to (T_k^1)^*\mathbb{R}^n,$$

$$\Phi^{T_k^*}(g, \alpha_{1q}, \ldots, \alpha_{kq}) := (\alpha_{1q} \circ (\Phi_{g^{-1}})_{*\Phi_g(q)}, \ldots, \alpha_{kq} \circ (\Phi_{g^{-1}})_{*\Phi_g(q)}), \tag{4.1}$$

$g \in G$, $(\alpha_1, \ldots, \alpha_k) \in (T_k^1)^*\mathbb{R}^n$, $q \in \mathbb{R}^n$, which is a k-symplectic action [9] and respectively, the lifted action Φ^{T_k} to $T_k^1\mathbb{R}^n$:

$$\Phi^{T_k} : G \times T_k^1\mathbb{R}^n \to T_k^1\mathbb{R}^n,$$

$$\Phi^{T_k}(g, v_{1q}, \ldots, v_{kq}) := ((\Phi_g)_{*q}v_{1q}, \ldots, (\Phi_g)_{*q}v_{kq}), \tag{4.2}$$

$g \in G$, $(v_1, \ldots, v_k) \in T_k^1\mathbb{R}^n$, $q \in \mathbb{R}^n$.

If $F : T_k^1\mathbb{R}^n \to (T_k^1)^*\mathbb{R}^n$ is a diffeomorfism, equivariant with respect to these actions, that is $\Phi_g^{T_k^*} \circ F = F \circ \Phi_g^{T_k}$, for any $g \in G$, then by taking the pull-back of the k-symplectic structure $(\omega_i, V)_{1\le i\le k}$ on the standard k-symplectic manifold $(T_k^1)^*\mathbb{R}^n$, we can define a k-symplectic structure $((\omega_F)_i, V_F)_{1\le i\le k}$ on $T_k^1\mathbb{R}^n$ [6]:

$$(\omega_F)_i := F^*\omega_i, \quad V_F := \ker(\pi_F)_*$$

for any $1 \le i \le k$, where $\pi_F : T_k^1\mathbb{R}^n \to \mathbb{R}^n, \pi_F(v_{1q}, \ldots, v_{kq}) := q$. Then F becomes a symplectomorphism between $(T_k^1\mathbb{R}^n, (\omega_F)_i, V_F)_{1\le i\le k}$ and $((T_k^1)^*\mathbb{R}^n, \omega_i, V)_{1\le i\le k}$.

On the two standard k-symplectic manifolds described above, consider the two canonical connections ∇ on $(T_k^1)^*\mathbb{R}^n$ and $\bar{\nabla}$ on $T_k^1\mathbb{R}^n$ which induce, naturally, on the reduced manifolds $((T_k^1)^*\mathbb{R}^n)_\mu$ and $(T_k^1\mathbb{R}^n)_\mu$ respectively the reduced canonical connections ∇_μ and $\bar{\nabla}_\mu$ (see Theorem 3.5.). Then we have

Proposition 4.1. *The two reduced connections are connected by the relation*

$$[F]_* \circ \bar{\nabla}_\mu = \nabla_\mu \circ ([F]_* \times [F]_*). \tag{4.3}$$

Proof. Since F is a diffeomorphism compatible with the equivalence relations that define the quotient manifolds $((T_k^1)^*\mathbb{R}^n)_\mu$ and $(T_k^1\mathbb{R}^n)_\mu$, for any $\bar{X}, \bar{Y} \in \Gamma(T(((T_k^1)\mathbb{R}^n)_\mu))$, if π^{T_k} and $\pi^{T_k^*}$ denote the canonical projections, we obtain:

$$\begin{aligned}
(\nabla_\mu \circ ([F]_* \times [F]_*))(\bar{X}, \bar{Y}) &= \nabla_\mu(([F]_* \circ \pi_*^{T_k})(X), ([F]_* \circ \pi_*^{T_k})(Y)) \\
&= \nabla_\mu((\pi_*^{T_k^*} \circ F_*)(X), (\pi_*^{T_k^*} \circ F_*)(Y)) \\
&= (\nabla_\mu \circ (\pi_*^{T_k^*} \times \pi_*^{T_k^*}))(F_*(X), F_*(Y)) \\
&= (\pi_*^{T_k^*} \circ \nabla)(F_*(X), F_*(Y)) \\
&= (\pi_*^{T_k^*} \circ \nabla \circ (F_* \times F_*))(X, Y) \\
&= (\pi_*^{T_k^*} \circ F_* \circ \bar{\nabla})(X, Y) \\
&= ([F]_* \circ \pi_*^{T_k} \circ \bar{\nabla})(X, Y) \\
&= ([F]_* \circ \bar{\nabla}_\mu \circ (\pi_*^{T_k} \times \pi_*^{T_k}))(X, Y) \\
&= ([F]_* \circ \bar{\nabla}_\mu)(\bar{X}, \bar{Y}). \ \square
\end{aligned}$$

References

[1] A. Awane, *k-symplectic structures*, J. Math. Phys. 33 (1992), 4046-4052.

[2] A. M. Blaga, *The reduction of a k-symplectic manifold*, Mathematica, Cluj-Napoca, 50 (73), 2 (2008), 149-158.

[3] B. Cappelletti Montano, A. M. Blaga, *Some geometric structures associated with a k-symplectic manifold*, J. Phys. A: Math. Theor., 41 (2008).

[4] S. Deshmukh, F. R. Al-Solany, *Hopf hypersurfaces in nearly Kaehler 6-sphere*, Balkan Journal of Geometry and Its Applications, 13, 1 (2008), 38-46.

[5] M. Girtu, *A framed* $f(3,-1)$ *structure on a GL-tangent manifold*, Balkan Journal of Geometry and Its Applications, 13, 1 (2008), 47-54.

[6] M. De Leon, E. Merino, J. Qubiña, P. Rodrigues, M. R. Salgado, *Hamiltonian systems on k-cosymplectic manifolds*, J. Math. Phys. 39 (1998), 876-893.

[7] J. M. Marsden, A. Weinstein, *The Hamiltonian structure of the Maxwell-Vlasov equations*, Physica D4 (1982), 394-406.

[8] J. M. Marsden, A. Weinstein, *Coadjoint orbits, vortices and Clebsch variables for incompressible fluids*, Physica D4 (1983), 305-323.

[9] F. Munteanu, A. M. Rey, M. R. Salgado, *The Günther's formalism in classical field theory: momentum map and reduction*, J. Math. Phys. 45 (2004), 1730-1751.

[10] I. Vaisman, *Connections under symplectic reduction*, arXiv: math. SG / 0006023, 2000.

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