Lagrangians and higher order tangent spaces

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Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. The aim of the paper is to prove that T^kM, the tangent space of order $k \geq 1$ of a manifold M, is diffeomorphic with T_k^1M, the tangent space of k^1–velocities, and also with $\left(T_k^1\right)^* M$, the cotangent space of k^1–covelocities, via suitable Lagrangians. One prove also that a hyperregular Lagrangian of first order on M can give rise to such diffeomorphisms.

M.S.C. 2000: 53C60, 53C80, 70H50.
Key words: Higher order tangent space; Lagrangian; Hamiltonian; semi-spray.

1 Introduction

Let M be a smooth manifold (all the objects considered in the paper are supposed to be of class C^∞). For every $k \in \mathbb{N}$ one can associate with M the differentiable manifolds T^kM, $T^{k*}M$, T_k^1M and $\left(T_k^1\right)^* M$, in a functorial manner.

First, T^kM is the *tangent space of order* k, $T^0M = M$, $T^1M = TM$ (see [4, 7]). Then T^kM can be considered as a locally trivial bundle $T^kM \stackrel{\pi_j}{\to} T^jM$ for every $j = \overline{0, k-1}$. The dual counterpart of T^kM, as considered in [8, 12], is $T^{k*}M = T^{k-1}M \times_M T^*M$, the *cotangent space of order* k, where $\times_M$ denotes the fibered products of bundles over the base M. For a *Lagrangian of order* k on M, $L : T^kM \to \mathbb{R}$, the dual counterpart definition proposed in [12] is the *affine Hamiltonian* $h : T^kM^\dagger \to T^{k*}M$; h is a section of the affine one-dimensional affine bundle $T^kM^\dagger \stackrel{\Pi}{\to} T^{k*}M$, where $T^{k\dagger}M \to T^{k-1}M$ is the affine dual of the affine bundle $T^kM \stackrel{\pi_{k-1}}{\to} T^{k-1}M$. Hyperregular Lagrangians and affine Hamiltonians are naturally related by Legendre transformations.

The manifold T_k^1M comes from the Whitney sum $T_k^1M = TM \oplus \cdots \oplus TM$ (k times); since T_k^1M can be identified with the manifold $J_0^1(\mathbb{R}^k, M)$ of the k^1-velocities of M, it is called the *tangent space of* k^1*-velocities* of M (see [5, 9]). The dual $\left(T_k^1\right)^* M = T^*M \oplus \cdots \oplus T^*M$ (k times) is the space of k^1*-covelocities* of M (see also [5, 9]).

A class of Lagrangians of order k, called *co-reducible* Lagrangians of order k, gives rise to a diffeomorphism of T^kM and $\left(T_k^1\right)^* M$ (Theorem 1). A co-reducible

Balkan Journal of Geometry and Its Applications, Vol.15, No.1, 2010, pp. 142-148.

Lagrangian induces a Hamiltonian $\tilde{H}$ on $\left(T_k^1\right)^* M$. If $\tilde{H}$ is hyperregular one say that L is co-hyperreducible.

An example is given by the lift of a hyperregular Lagrangian of first order L to a Lagrangian $\tilde{L}$ of order k, constructed in Proposition 4, that is co-hyperreducible. The Lagrangian L gives rise also to a diffeomorphism of $T^k M$ and $T_k^1 M$ (Proposition 2).

We use local coordinates as in [7], but in spite of their local forms, the main objects are global ones.

2 The main results and constructions

A semispray of order k is a section $S : T^k M \to T^{k+1} M$ of the affine bundle $\pi_k : T^{k+1} M \to T^k M$. Since $T^{k+1} M \subset TT^k M$ (in fact $T^{k+1} M$ is an affine subbundle of the tangent bundle of $TT^k M$), then S can be regarded as well as a vector field on $T^k M$. The local form of S is

$$(x^i, y^{(1)i}, \ldots, y^{(k)i}) \xrightarrow{S} (x^i, y^{(1)i}, \ldots, y^{(k)i}, S^i(x^i, y^{(1)i}, \ldots, y^{(k)i}));$$

viewed as a vector field,

$$S = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + k y^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} + (k+1) S^i \frac{\partial}{\partial y^{(k)i}}.$$

Let us denote by $T^{k-1,1} M = T^{k-1} M \times_M TM$; more general, if $0 \leq r \leq k$, then $T^{r,k-r} M = T^r M \times_M T^1_{k-r} M$, where $T^0 M = M = T^1_0 M$.

Proposition 1. *If $S : T^{k-1} M \to T^k M$ is a semispray of order $k-1$, then there is a diffeomorphism $\Phi : T^k M \to T^{k-1,1} M$; more general, if $0 \leq r \leq k-1$ and $S^{(\alpha)} : T^{\alpha-1} M \to T^\alpha M$, $\alpha = \overline{r+1, k}$ are semisprays (of order $\alpha - 1$), then there is a diffeomorphism $\Phi^{(r)} : T^k M \to T^{r,k-r} M$. In the particular case $r = 1$, if $S^{(\alpha)} : T^{\alpha-1} M \to T^\alpha M$, $\alpha = \overline{2, k}$ are semisprays, then there is a diffeomorphism $\Phi^{(1)} : T^k M \to T^{1,k-1} = T^1_k M$.*

Proof. If $S : T^{k-1} M \to T^k M$ is a semispray having the local form

$$(x^i, y^{(1)i}, \ldots, y^{(k-1)i}) \xrightarrow{S} (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, S^{(k)i}(x^i, y^{(1)i}, \ldots, y^{(k-1)i})),$$

then the diffeomorphism $\Phi : T^k M \to T^{k-1,1} M$ is given by

$$(x^i, y^{(1)i}, \ldots, y^{(k)i}) \xrightarrow{\Phi} (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, y^{(k)i} - S^{(k)i}).$$

For $0 \leq r \leq k$, then $\Phi^{(r)} : T^k M \to T^{r,k-r} M$ is given by

$$\begin{aligned}(x^i, y^{(1)i}, \ldots, y^{(k)i}) \xrightarrow{\Phi^{(r)}} (x^i, y^{(1)i}, \ldots, y^{(r)i}, \quad & y^{(r+1)i} - S^{(r+1)i}(x^i, y^{(1)i}, \ldots, y^{(r)i}), \ldots, \\ & y^{(k)i} - S^{(k)i}(x^i, y^{(1)i}, \ldots, y^{(k-1)i})).\end{aligned}$$

In the particular case $r = 1$, the diffeomorphism $\Phi^{(1)} : T^k M \to T^1_k M$ is given by $(x^i, y^{(1)i}, \ldots, y^{(k)i}) \xrightarrow{\Phi^{(1)}} (x^i, y^{(1)i}, y^{(2)i} - S^{(2)i}(x^i, y^{(1)i}), \ldots, y^{(k)i} - S^{(k)i})$. □

We say that a diffeomorphism $\Phi : T^kM \to T^1_kM$ is a *semi-spray type diffeomorphism* if it has the form $\Phi = \Phi^{(1)}$ as above.

There is a semispray of order $k \geq 1$ canonically associated with a k-order Lagrangian L (see, for example, [7, 2]), given by a section $S : T^kM \to T^{k+1}M$ that in local coordinates has the form $(x^i, y^{(1)j}, \dots, y^{(k)j}) \xrightarrow{S} (x^i, y^{(1)j}, \dots, y^{(k)j}, S^i(x^i, y^{(1)j}, \dots, y^{(k)j}))$, where

$$(k+1)S^i = \frac{1}{2}g^{ij}\left(d_T^{(k)}\left(\frac{\partial L}{\partial y^{(k)j}}\right) - \frac{\partial L}{\partial y^{(k-1)j}}\right)$$

and

$$d_T^{(k)} = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \dots + (k+1)y^{(k)i}\frac{\partial}{\partial y^{(k-1)i}}$$

is the Tulczyjew local operator (it is not a global vector field, but called a vector pseudofield in [12]).

Proposition 2. *Let $L : TM \to \mathbb{R}$ be a hyperregular Lagrangian (of first order). Then there is a semi-spray type diffeomorphism $\Phi : T^kM \to T^1_kM$ canonically associated with L.*

Proof. Let us consider a regular Lagrangian of first order $L : TM \to \mathbb{R}$ and its canonical semispray $S : TM \to T^2M$.

Using local coordinates, $(x^i, y^i = y^{(1)i}) \to (x^i, y^{(1)i}, 2S^i(x^j, y^{(1)j}))$, where

$$S^i(x^j, y^{(1)j})) = \frac{1}{4}g^{ij}\left(y^{(1)p}\frac{\partial^2 L}{\partial x^p \partial y^{(1)j}} - \frac{\partial L}{\partial x^j}\right) = \frac{1}{4}g^{ij}\left(d_T^{(1)}\left(\frac{\partial L}{\partial y^{(1)j}}\right) - \frac{\partial L}{\partial x^j}\right).$$

Denoting by $z^{(2)i} = y^{(2)i} - S^i(x^j, y^{(1)j})$, we have $z^{(2)i'} = \dfrac{\partial x^{i'}}{\partial x^i} z^{(2)i}$.

It follows that the association $(x^i, y^{(1)i}, y^{(2)i}) \to (x^i, y^{(1)i}, z^{(2)i})$ defines a global diffeomorphism $T^2M \to TM \times_M TM = T^1_2M$ of T^2M with the tangent space of 2^1-velocities on M.

The above construction can be given for any higher order $k \geq 1$. Finally one can consider the k-Lagrangian $L^{(k)} : T^kM \to \mathbb{R}$ having the local form

$$L^{(k)}(x^i, y^{(1)i}, y^{(2)i}, \dots, y^{(k)i}) = L(x^i, y^{(1)i}) + L(x^i, z^{(2)i}) + \dots + L(x^i, z^{(k)i}). \tag{2.1}$$

So, one construct inductively a semi-spray type diffeomorphism $T^kM \to T^1_kM = TM \times_M \dots \times_M TM$ (k times) of T^kM with the tangent space of k^1-velocities on M, $k \geq 1$. Notice that this diffeomorphism has the local form

$$(x^i, y^{(1)i}, \dots, y^{(k)i}) \to (x^i, y^{(1)i}, z^{(2)i}, \dots, z^{(k)i}),$$

where $z^{(\alpha)i} = y^{(\alpha)i} - S^{(\alpha)i}(x^j, y^{(1)j}, \dots, y^{(\alpha-1)i})$, $\alpha = \overline{2,k}$. $\square$

Notice that in particular the Lagrangian L can be a Finslerian if it is 2–homogeneous, or it is possible that L comes from a Riemannian metric if it is quadratic in velocities.

If $\varepsilon_1, \dots, \varepsilon_k$ are real numbers, $\varepsilon_i \neq 0$, $i = \overline{1,k}$, one can consider also a k-Lagrangian $L^{(k)} : T^kM \to \mathbb{R}$ having the local form

$$L^{(k)}(x^i, y^{(1)i}, y^{(2)i}, \dots, y^{(k)i}) = \varepsilon_1 L(x^i, y^{(1)i}) + \varepsilon_2 L(x^i, z^{(2)i}) + \dots + \varepsilon_k L(x^i, z^{(k)i});$$

using the coordinates $(x^i, y^{(1)i}, z^{(2)i}, \ldots, z^{(k)i})$ on T^kM, it is easy to see that $L^{(k)}$ a Lagrangian in the multisymplectic sense (see [4, 7]). More general, one can prove the following result.

Proposition 3. *Let $\{L_\alpha\}_{\alpha=\overline{1,k}}$, $L_\alpha : TM \to \mathbb{R}$ be hyperregular Lagrangians of order $k \in \mathbb{N}^*$. Then there is a semi-spray type diffeomorphism $\Phi : T^kM \to T^1_kM$ canonically associated with $\{L_\alpha\}_{\alpha=\overline{1,k}}$.*

Proof. The diffeomorphism Φ can be given using Proposition 1; one can construct inductively the Lagrangians $\{L^{(\alpha)}\}_{\alpha=\overline{1,k}}$ by formula

$$L^{(\alpha)}(x^i, y^{(1)i}, y^{(2)i}, \ldots, y^{(\alpha)i}) = L_1(x^i, y^{(1)i}) + L_2(x^i, z^{(2)i}) + \cdots + L_\alpha(x^i, z^{(\alpha)i}),$$

where $z^{(\alpha)i}$ are constructed successively as in Proposition 2, using (2). $\square$

According to [12], an *affine Hamiltonian* of order k on M is a differentiable map $h : \widetilde{T^{k*}M} \to \widetilde{T^kM^\dagger}$, such that $\Pi \circ h = 1_{\widetilde{T^{k*}M}}$, where $\Pi : \widetilde{T^kM^\dagger} \to \widetilde{T^{k*}M}$. Thus h has the local form

$$h(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i, -H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)).$$

The local functions H_0 change according to the rules

$$H_0'(x^{i'}, y^{(1)i'}, \ldots, y^{(k-1)i'}, p_{i'}) = H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) + \frac{1}{k}\Gamma_U^{(k-1)}(y^{(k-1)i'})\frac{\partial x^i}{\partial x^{i'}} p_i.$$

It is easy to see that one has $\frac{\partial H_0'}{\partial p_{i'}} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial H_0}{\partial p_i} + \frac{1}{k}\Gamma_U^{(k-1)}(y^{(k-1)i'})$. Thus there is a map $\mathcal{H} : T^{k*}M \to T^kM$, given in local coordinates by

$$\mathcal{H}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \frac{\partial H_0}{\partial p_i}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)),$$

called the *co-Legendre map* of the affine Hamiltonian h. We say also that h is *regular* if $\mathcal{H}$ is a local diffeomorphism and h is *hiperregular* if $\mathcal{H}$ is a global diffeomorphism. Since $\frac{\partial^2 H_0'}{\partial p_{i'}\partial p_{j'}} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial^2 H_0}{\partial p_i \partial p_j}$, it follows that $h^{ij} = \frac{\partial^2 H_0}{\partial p_i \partial p_j}$ is a symmetric 2-contravariant d-tensor, which is non-degenerate iff h is regular. There exists a real function $H : T^{k*}M \to \mathbb{R}$ defined by the formula

$$H(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = p_i \frac{\partial H_0}{\partial p_i} - H_0.$$

We call H the *pseudo-energy* of h.

Let $L : T^kM \to \mathbb{R}$ be a hyperregular k-Lagrangian. The Legendre map $\mathcal{L} : T^kM \to T^{k*}M$ is a diffeomorphism and there is an affine Hamiltonian h defined using L, as follows. Let

$$(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) \to (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, H^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i))$$

be the local form of the inverse of $\mathcal{L}$. Then the local function H_0 on $T^{k*}M$, defined by the formula

$$H_0(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = p_j H^j - L(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, H^i)$$

gives a global affine Hamiltonian of order k on M. Let us consider the real function on $T^{k*}M$: $\tilde{H}_0^{(k)} = \frac{\partial H_0}{\partial p_j} p_j - H_0$.

We denote $\mathcal{L} = \mathcal{L}_{(k)}$ and we define $\mathcal{L}_{(k-1)} : T^{k*}M \to T^{(k-1)*}M \times_M T^*M$ using the formula

$$\mathcal{L}_{(k-1)}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = (x^i, y^{(1)i}, \ldots, y^{(k-2)i}, \frac{\partial \tilde{H}_0^{(k)}}{\partial y^{(k-1)i}}, p_i).$$

We denote $p_i = p_{(k)i}$ and $H_0 = H_0^{(k)}$. We suppose that $\mathcal{L}_{(k-1)}$ is a diffemorphism, then $\mathcal{L}_{(k-1)}^{-1}$ has the local form $(x^i,\ y^{(1)i},\ \ldots,\ y^{(k-2)i}, p_{(k-1)i},\ p_{(k)i}) \overset{\mathcal{L}_{(k-1)}^{-1}}{\longrightarrow}$ $(x^i,\ y^{(1)i},\ \ldots,\ y^{(k-2)i},\ H^i(x^i,\ y^{(1)i},\ \ldots,\ y^{(k-2)i},\ p_{(k-1)i},\ p_{(k)i}),\ p_{(k)i})$. We consider $H_0^{(k-1)}(x^i, y^{(1)i}, \ldots, y^{(k-2)i}, p_{(k-1)i}, p_{(k)i}) = p_{(k-1)j}H^i - \tilde{H}_0^{(k)}(x^i, y^{(1)i}, \ldots, y^{(k-2)i}, H^i,$ $p_{(k)i})$, where $H^i = H^i(x^i, y^{(1)i}, \ldots, y^{(k-2)i}, p_{(k-1)i}, p_{(k)i})$. Consider the real function on $T^{(k-1)*}M \times_M T^*M$ given by $\tilde{H}_0^{(k-2)} = \frac{\partial H_0^{(k-1)}}{\partial p_{(k-1)j}} p_{(k-1)j} - H_0^{(k-1)}$.

Following the above idea, we give a procedure that descends the degree of the higher order Hamiltonians.

Inductively, let us suppose that the diffeomorphisms $\mathcal{L}_{(k)}, \ldots, \mathcal{L}_{(k-q)}$ have been constructed for $1 < q < k-1$. We have that

$$\mathcal{L}_{(k-q)} : T^{k-q}M \times_M (T^*M)^q \to T^{(k-q)*}M \times_M (T^*M)^q = T^{(k-q-1)}M \times_M (T^*M)^{q+1},$$

where $(T^*M)^q = T^*M \oplus \cdots \oplus T^*M$, ($q$ times) is a diffemorphism, given by the formula

$$\mathcal{L}_{(k-q)}(x^i, y^{(1)i}, \ldots, y^{(k-q)i}, p_{(k-q+1)i}, \ldots, p_{(k)i}) = (x^i, y^{(1)i}, \ldots, y^{(k-q-1)i},$$
$$\frac{\partial \tilde{H}_0^{(q)}}{\partial y^{(k-q)i}}(x^i, y^{(1)i}, \ldots, y^{(k-q)i}, p_{(k-q+1)i}, \ldots, p_{(k)i}), p_{(k-q+1)i}, \ldots, p_{(k)i}).$$

Let $\mathcal{L}_{(k-q)}^{-1}$ having the local form

$$(x^i, y^{(1)i}, \ldots, y^{(k-q-1)i}, p_{(k-q)i}, \ldots, p_{(k)i}) \overset{\mathcal{L}_{(k-q)}^{-1}}{\longrightarrow} (x^i, y^{(1)i}, \ldots, y^{(k-q-1)i},$$
$$H^i(x^i, y^{(1)i}, \ldots, y^{(k-q-1)i}, p_{(k-q)i}, \ldots, p_{(k)i}), p_{(k-q+1)i}, \ldots, p_{(k)i}).$$

We consider

$$\begin{aligned} &H_0^{(k-q-1)}(x^i, y^{(1)i}, \ldots, y^{(k-q-1)i}, p_{(k-q)i}, \ldots, p_{(k)i}) \\ &\quad = p_{(k-q)j}H^j(x^i, y^{(1)i}, \ldots, y^{(k-q-1)i}, p_{(k-q)i}, \ldots, p_{(k)i}) \\ &\quad\quad -\tilde{H}_0^{(k-q+1)}(x^i, y^{(1)i}, \ldots, y^{(k-q-1)i}, H^i, p_{(k-q+1)i}, \ldots, p_{(k)i}). \end{aligned}$$

If $k-q-1 > 1$, we consider the real function on $T^{(k-q-1)*}M \times_M (T^*M)^{q+1}$ given by $\tilde{H}_0^{(k-q-1)} = \frac{\partial H_0^{(k-q-1)}}{\partial p_{(k-q-1)j}} p_{(k-q-1)j} - H_0^{(k-q-1)}$ and we define $\mathcal{L}_{(k-q-1)} : T^{k-q-1}M \times_M (T^*M)^{q+1} \to T^{(k-q-1)*}M \times_M (T^*M)^{q+1} = T^{(k-q-2)}M \times_M (T^*M)^{q+2}$ using the formula $\mathcal{L}_{(k-q-1)}(x^i, y^{(1)i}, \ldots, y^{(k-q-1)i}, p_{(k-q)i}, \ldots, p_{(k)i}) = (x^i, y^{(1)i}, \ldots, y^{(k-q-2)i},$ $\frac{\partial \tilde{H}_0^{(k-q-1)}}{\partial y^{(k-q-1)i}}(x^i, y^{(1)i}, \ldots, y^{(k-q-1)i}, p_{(k-q)i}, \ldots, p_{(k)i}), p_{(k-q)i}, \ldots, p_{(k)i})$. We suppose that $\mathcal{L}_{(k-q-1)}$ is a diffeomorphism. If $k-q-1=1$, we skip $\tilde{H}_0^{(1)}$ and we define directly

$$\mathcal{L}_{(1)} : TM \times_M (T^*M)^{k-1} \to T^*M \times_M (T^*M)^{k-1} = (T^*M)^k$$

by the formula

$$\mathcal{L}_{(1)}(x^i, y^{(1)i}, p_{(2)i}, \dots, p_{(k)i}) = (x^i, \frac{\partial H_0^{(1)}}{\partial y^{(1)i}}(x^i, y^{(1)i}, p_{(2)i}, \dots, p_{(k)i}), p_{(2)i}, \dots, p_{(k)i}).$$

We suppose also that $\mathcal{L}_{(1)}$ is a diffeomorphism and its inverse has the local form $\mathcal{L}_{(1)}(p_{(1)i}, \dots,\ p_{(k)i}) = (H^i(p_{(1)i}, \dots, p_{(k)i}), p_{(2)i}, \dots, p_{(k)i})$. We define the multi-Hamiltonian $\tilde{H}^{(0)} : (T^*M)^k \to \mathbb{R}$ using the formula

$$\tilde{H}^{(0)}(p_{(1)i}, \dots, p_{(k)i}) = p_{(1)i} H^i(p_{(1)i}, \dots, p_{(k)i}) - H_0^{(1)}(H^i, p_{(2)i}, \dots, p_{(k)i}).$$

If we suppose that all the applications $\mathcal{L}_{(k)}, \dots, \mathcal{L}_1$ are diffeomorphisms, we say that the Lagrangian L of order k is *co-reducible.* Let us denote $\Psi = \mathcal{L}_{(1)} \circ \cdots \circ \mathcal{L}_{(k)}$. The above construction can be synthesized in the following main result.

Theorem 1. *If the Lagrangian L of order $k \geq 1$ is co-reducible, then there is a diffeomorphism $T^kM \overset{\Psi}{\to} \left(T_k^1\right)^* M = TM^* \times_M \cdots \times_M TM^*$ (k times) such that $L = \tilde{H}^{(0)} \circ \Psi$.*

We prove below that the lift (2.1) gives rise to a completely regular Lagrangian of order k.

Proposition 4. *Let $L : TM \to \mathbb{R}$ be a hyperregular Lagrangian and $L^{(k)} : T^kM \to \mathbb{R}$ be the Lagrangian given by (2.1). Then $L^{(k)}$ is a co-reducible Lagrangian of order k.*

Proof. The inverse of the Legendre map is given by

$$H^{(k)i}(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i) = S^i(x^i, y^{(1)i}, \dots, y^{(k-1)i}) + H^i(x^j, p_j),$$

i.e.,

$$\frac{\partial L^{(k)}}{\partial y^{(k)i}}(x^j, y^{(1)j}, \dots, y^{(k-1)j}, H^{(k)j}(x^j, y^{(1)j}, \dots, y^{(k-1)j}, p_j) = p_i.$$

One has

$$\begin{aligned} &H_0^{(k)}(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i) \\ &\quad = p_i(H^i(x^j, p_j) + S^i) - L^{(k)}(x^i, y^{(1)i}, \dots, y^{(k-1)i}, H^i + S^i) \\ &\quad = p_i(H^i(x^j, p_j) + S^i) - L(x^i, y^{(1)i})(x^i, H^i) - \cdots - L(x^i, z^{(k-1)i}) - L(x^i, H^i), \end{aligned}$$

and thus

$$\frac{\partial H_0^{(k)}}{\partial p_i} = H^i + S^i + p_j \frac{\partial H^j}{\partial p_i} - \frac{\partial L}{\partial y^j}(x^i, H^i) \frac{\partial H^j}{\partial p_i} = H^i + S^i.$$

One also has

$$\begin{aligned} H_0^{(k-1)}(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_{(k)i}) &= \frac{\partial H_0^{(k)}}{\partial p_i} p_i - \bar{H}_0^{(k)} \\ &= L(x^i, y^{(1)i}) + L(x^i, z^{(2)i}) + \cdots + L(x^i, z^{(k-1)i}) + L(x^i, H^i(x^j, p_{(k)j})). \end{aligned}$$

Then $\tilde{H}_0^{(1)}(x^i, p_{(1)i}, \dots, p_{(k)i}) = L(x^i, H^i(x^j, p_{(1)j})) + \cdots + L(x^i, H^i(x^j, p_{(k)j}))$. $\square$

Notice that all the above constructions and properties can be adapted to the case when the differentiable Lagrangian $L : T^kM \to \mathbb{R}$ is replaced by a differentiable Lagrangian $L : \widetilde{T^kM} \to \mathbb{R}$, where $\widetilde{T^kM} = T^kM\backslash\{0\}$ is T^kM without the image of the „null" section $y^{(\alpha)i} = 0$, $\alpha = \overline{0, k}$.

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