A geometric space without conjugate points

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Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. From a spray space S on a manifold M we construct a new geometric space P of larger dimension with the following properties: (i) geodesics in P are in one-to-one correspondence with parallel Jacobi fields of M; (ii) P is complete if and only if S is complete; (iii) if two geodesics in P meet at one point, the geodesics coincide on their common domain, and P has no conjugate points; (iv) there exists a submersion that maps geodesics in P into geodesics on M.

The space P is constructed by first taking two complete lifts of spray S. This will give a spray S^{cc} on the second iterated tangent bundle TTM. Then space P is obtained by restricting tangent vectors of geodesics for S^{cc} onto a suitable $(2\dim M + 2)$-dimensional submanifold of $TTTM$. Due to the last restriction, the space P is not a spray space. However, the construction shows that conjugate points can be removed if we add dimensions and relax assumptions on the geometric structure.

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1 Introduction

Suppose S is a spray on a manifold M. In this paper we show how to construct a new geometric space P that is based on S, but such that P has no conjugate points. This is done in three steps:

(i) We start with a spray S on a manifold M. For example, S could be the geodesic spray for a Riemannian metric, a Finsler metric, or a non-linear connection [5, 7, 23, 24].

(ii) Next we take two *complete lifts* of S (see below). The first complete lift S^c gives a spray on TM whose geodesics are Jacobi fields on M. Similarly, the second complete lift gives a spray S^{cc} on TTM whose geodesics can be described as Jacobi fields for geodesics for S^c. That is, geodesics of S^c describe linear deviation of nearby geodesics in M, and geodesics of S^{cc} describe second order deviation of nearby geodesics in M.

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(iii) In the last step, we restrict tangent vectors of geodesics of S^{cc} onto a submanifold $\Delta \subset TTTM$ that is invariant under the geodesic flow of S^{cc}. By choosing Δ in a suitable way, we obtain a space P where geodesics are in one-to-one correspondence with parallel Jacobi fields in M.

In step (ii) the original spray S is lifted twice using the *complete lift*. Essentially, the complete lift can be seen as a geometrization of the Jacobi equation. For example, if we start with a (pseudo-)Riemannian metric g on M, the complete lift of g gives a pseudo-Riemannian metric g^c on TM whose geodesics are Jacobi fields on M. This means that Jacobi fields on M can be treated as solutions to a geodesic equation on TM, whence there is no need for a separate Jacobi equation. In this work we will use the complete lift of a spray. For affine sprays, this complete lift was introduced by A. Lewis [20]. In the Riemannian context, the complete lift is also known as the *Riemann extension*, and for a discussion about the complete lift in other contexts, see [6]. In step (ii), we need to study sprays on manifolds M, TM, and TTM and also complete lifts of sprays on M and TM. To avoid studying all these cases separately we first study sprays and complete lifts on iterated tangent bundles of arbitrary order. This is the topic of Sections 2-6. One advantage of studying the Jacobi equation using the complete lift is that it simplifies certain aspects of Jacobi fields. For example, using the complete lift the Jacobi equation for a spray is essentially a direct consequence of the chain rule and the canonical involutions on TTM and $TTTM$ (see remark after Proposition 6.3).

In step (iii) the phase space of spray S^{cc} on TTM is restricted to a submanifold $\Delta \subset TTTM$. By choosing Δ suitably, we define a geometry P where geodesics are in one-to-one correspondence with parallel Jacobi fields. The geometry of sprays that have been restricted in this way is described in Section 7. For previous work on sprays with restricted phase space, see [2, 18, 19]. The space P is constructed and discussed in Section 8. Here we show that P has no conjugate points. We also show that the canonical submersion $\pi\colon TTM \to M$ maps geodesics in P into geodesics in M. Hence the geometry of P can be used to study dynamical properties of M.

Let us emphasize that due to the restriction in step (iii), space P is not a spray space. It seems that to remove conjugate points, some relaxation of the underlying geometric structure is needed. For example, in Riemannian geometry the assumption that a manifold has no conjugate points can have strong implications.

(i) Suppose M is an n-torus with a Riemannian metric. Then the no-conjugate assumption implies that M is flat [8, 14].

(ii) Suppose M is a Riemannian manifold such that M is complete, simply connected, $\dim M \geq 3$, and M is flat outside a compact set. Then the no-conjugate assumption implies that M is isometric to $\mathbb{R}^n$ [10].

See also [9, 11, 22]. If one relaxes the assumption on the geometric structure, then the no-conjugate assumption becomes weaker; on the 2-torus, there are non-flat affine connections without conjugate points [15], and on the n-torus there are non-flat Finsler metrics without conjugate points [11].

We will not study applications. However, let us note that there are many problems in both mathematics and physics where a proper understanding of conjugate points and multi-path phenomena seem to be important. For example, in traveltime tomography a typical assumption is that the manifold has no conjugate points. See [9, 25]. Another example is geometric optics, where conjugate points are problematic since they lead to caustics, where the amplitude becomes infinite.

2 Preliminaries

We assume that M is a smooth manifold without boundary and with finite dimension $n \geq 1$. By smooth we mean that M is a topological Hausdorff space with countable base that is locally homeomorphic to $\mathbb{R}^n$, and transition maps are C^∞-smooth. All objects are assumed to be C^∞-smooth on their domains.

By (TM, π_0, M) we mean the tangent bundle of M. For $r \geq 1$, let $T^rM = T\cdots TM$ be the r:th *iterated tangent bundle*, and for $r = 0$ let $T^0M = M$. For example, when $r = 2$ we obtain the second tangent bundle TTM [4, 13], and in general $T^{r+1}M = TT^rM$ for $r \geq 0$.

For a tangent bundle $T^{r+1}M$ where $r \geq 0$, we denote the canonical projection operator by $\pi_r \colon T^{r+1}M \to T^rM$. Occasionally we also write $\pi_{TTM\to M}, \pi_{TM\to M}, \dots$ instead of $\pi_0 \circ \pi_1, \pi_0, \dots$. Unless otherwise specified, we always use canonical local coordinates (induced by local coordinates on M) for iterated tangent bundles. If x^i are local coordinates for T^rM for some $r \geq 0$, we denote induced local coordinates for $T^{r+1}M$, $T^{r+2}M$, and $T^{r+3}M$ by

$$(x, y), \quad (x, y, X, Y), \quad (x, y, X, Y, u, v, U, V).$$

As above, we usually leave out indices for local coordinates and write (x, y) instead of (x^i, y^i).

For $r \geq 1$, we treat T^rM as a vector bundle over the manifold $T^{r-1}M$ with the vector space structure induced by projection $\pi_{r-1} \colon T^rM \to T^{r-1}M$ unless otherwise specified. Thus, if $\{x^i : i = 1, \dots, 2^{r-1}n\}$ are local coordinates for $T^{r-1}M$, and (x, y) are local coordinates for T^rM, then vector addition and scalar multiplication are given by

$$(x, y) + (x, \tilde{y}) = (x, y + \tilde{y}), \tag{2.1}$$
$$\lambda \cdot (x, y) = (x, \lambda y). \tag{2.2}$$

If $x \in T^rM$ and $r \geq 0$ we define

$$T^{r+1}_xM = \{\xi \in T^{r+1}M : \pi_r(\xi) = x\}.$$

For $r \geq 0$, a *vector field* on an open set $U \subset T^rM$ is a smooth map $X \colon U \to T^{r+1}M$ such that $\pi_r \circ X = \mathrm{id}_U$. The set of all vector fields on U is denoted by $\mathfrak{X}(U)$.

Suppose that γ is a smooth map $\gamma \colon (-\varepsilon, \varepsilon)^k \to T^rM$ where $k \geq 1$ and $r \geq 0$. Suppose also that $\gamma(t^1, \dots, t^k) = (x^i(t^1, \dots, t^k))$ in local coordinates for T^rM. Then the *derivative* of γ with respect to variable t^j is the curve $\partial_{t^j}\gamma \colon (-\varepsilon, \varepsilon)^k \to T^{r+1}M$ defined by $\partial_{t^j}\gamma = (x^i, \partial x^i/\partial t^j)$. When $k = 1$ we also write $\gamma' = \partial_t\gamma$ and say that γ' is the *tangent of* γ.

Unless otherwise specified we always assume that I is an open interval of $\mathbb{R}$ that contains 0, and we do not exclude unbounded intervals. If $\phi \colon M \to N$ is a smooth map between manifolds, we denote the tangent map $TM \to TN$ by $D\phi$, and if $c\colon I \to M$ is a curve, then

$$(\phi \circ c)'(t) = D\phi \circ c'(t), \quad t \in I. \tag{2.3}$$

2.1 Transformation rules in T^rM

Suppose that $x = (x^i)$ and $\tilde{x} = (\tilde{x}^i)$ are overlapping coordinates for T^rM where $r \geq 0$. It follows that if $\xi \in T^{r+1}M$ has local representations (x, y) and $(\tilde{x}, \tilde{y})$, we have transformation rules

$$\tilde{x}^i = \tilde{x}^i(x), \quad \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^a}(x) y^a.$$

Now (x, y) and $(\tilde{x}, \tilde{y})$ are overlapping coordinates for $T^{r+1}M$. It follows that if $\xi \in T^{r+2}M$ has local representations (x, y, X, Y) and $(\tilde{x}, \tilde{y}, \tilde{X}, \tilde{Y})$, we have transformation rules

$$\begin{aligned}
\tilde{x}^i &= \tilde{x}^i(x), \\
\tilde{y}^i &= \frac{\partial \tilde{x}^i}{\partial x^a}(x) y^a, \\
\tilde{X}^i &= \frac{\partial \tilde{x}^i}{\partial x^a}(x) X^a, \\
\tilde{Y}^i &= \frac{\partial \tilde{x}^i}{\partial x^a}(x) Y^a + \frac{\partial^2 \tilde{x}^i}{\partial x^a \partial x^b}(x) y^a X^b.
\end{aligned}$$

3 Lifts on iterated tangent bundles

3.1 Canonical involution on T^rM

When $r \geq 2$ there are two canonical projections $T^rM \to T^{r-1}M$ given by

$$\pi_{r-1}\colon T^rM \to T^{r-1}M, \quad D\pi_{r-2}\colon T^rM \to T^{r-1}M. \tag{3.1}$$

This means that T^rM contains two copies of $T^{r-1}M$, and there are two ways to treat T^rM as a vector bundle over $T^{r-1}M$. Unless otherwise specified, we always assume that T^rM is vector bundle $(T^rM, \pi_{r-1}, T^{r-1}M)$, whence the vector structure of T^rM is locally given by equations (2.1)-(2.2). However, there is also another vector bundle structure induced by projection $D\pi_{r-2}\colon T^rM \to T^{r-1}M$. If x^i are local coordinates for $T^{r-2}M$ and (x, y, X, Y) are local coordinates for T^rM, this structure is given by

$$(x, y, X, Y) + (x, \tilde{y}, X, \tilde{Y}) = (x, y + \tilde{y}, X, Y + \tilde{Y}), \tag{3.2}$$

$$\lambda \cdot (x, y, X, Y) = (x, \lambda y, X, \lambda Y). \tag{3.3}$$

Next we define the canonical involution $\kappa_r\colon T^rM \to T^rM$ [6]. It is a linear isomorphism between the above two vector bundle structures for T^rM defined such that the following diagram commutes.

$$\begin{tikzcd}
T^r M \arrow[rr, leftrightarrow, "\kappa_r"] \arrow[dr, "\pi_{r-1}"'] & & T^r M \arrow[dl, "D\pi_{r-2}"] \\
& T^{r-1}M &
\end{tikzcd}$$

On TTM, this involution map is well known [4, 13, 16, 21, 23].

Definition 3.1 (Canonical involution on T^rM). For $r \geq 2$, the *canonical involution* $\kappa_r \colon T^rM \to T^rM$ is the unique diffeomorphism that satisfies

$$\partial_t \partial_s c(t,s) = \kappa_r \circ \partial_s \partial_t c(t,s) \tag{3.4}$$

for all maps $c\colon (-\varepsilon,\varepsilon)^2 \to T^{r-2}M$. For $r = 1$, we define $\kappa_1 = \mathrm{id}_{TM}$.

Let $r \geq 2$, let x^i be local coordinates for $T^{r-2}M$, and let (x,y,X,Y) be local coordinates for T^rM. Then

$$\kappa_r(x,y,X,Y) = (x,X,y,Y).$$

For example, in local coordinates for TTM and $TTTM$ we have

$$\begin{aligned}
\kappa_2(x,y,X,Y) &= (x,X,y,Y),\\
\kappa_3(x,y,X,Y,u,v,U,V) &= (x,y,u,v,X,Y,U,V).
\end{aligned}$$

For $r \geq 1$, we have identities

$$\kappa_r^2 = \mathrm{id}_{T^rM}, \tag{3.5}$$
$$\pi_r \circ D\kappa_r = \kappa_r \circ \pi_r, \tag{3.6}$$
$$D\pi_{r-1} = \pi_r \circ \kappa_{r+1}, \tag{3.7}$$
$$\pi_{r-1} \circ D\pi_{r-1} = \pi_{r-1} \circ \pi_r, \tag{3.8}$$
$$D\pi_{r-1} \circ \pi_{r+1} = \pi_r \circ DD\pi_{r-1}, \tag{3.9}$$
$$DD\pi_{r-1} \circ \kappa_{r+2} = \kappa_{r+1} \circ DD\pi_{r-1}, \tag{3.10}$$
$$\pi_{r-1} \circ \pi_r \circ \kappa_{r+1} = \pi_{r-1} \circ \pi_r. \tag{3.11}$$

Let us point out that the two projections in (3.1) are not the only projections from $T^{r+1}M \to T^rM$. For example, when $r = 3$, there are (at least) 6 projections $T^3M \to T^2M$; π_2, $\kappa_2 \circ \pi_2$, $D\pi_1$, $\kappa_2 \circ D\pi_1$, $DD\pi_0$, and $\kappa_2 \circ DD\pi_0$.

Let γ_0 be a curve $\gamma_0 \colon I \to T^{r-1}M$ for some $r \geq 1$, and let

$$\mathfrak{X}(\gamma_0) = \{\eta \colon I \to T^rM : \pi_{r-1} \circ \eta = \gamma_0\}.$$

Elements in $\mathfrak{X}(\gamma_0)$ are called *vector fields along* γ_0, and $\mathfrak{X}(\gamma_0)$ has a natural vector space structure induced by the vector bundle structure of T^rM in equations (2.1)-(2.2).

If $\eta \in \mathfrak{X}(\gamma_0)$ and $C \in \mathbb{R}$, then

$$\kappa_{r+1} \circ (C\eta)' = C(\kappa_{r+1} \circ \eta'), \tag{3.12}$$

and if $\eta_1, \eta_2 \in \mathfrak{X}(\gamma_0)$, then

$$\kappa_{r+1} \circ (\eta_1 + \eta_2)' = \kappa_{r+1} \circ \eta_1' + \kappa_{r+1} \circ \eta_2'. \tag{3.13}$$

It follows that $\kappa_{r+1} \circ \partial_t \colon \mathfrak{X}(\gamma_0) \to \mathfrak{X}(\gamma_0')$ is a linear map between vector spaces.

3.2 Slashed tangent bundles $T^rM \setminus \{0\}$

The *slashed tangent bundle* is the open set in TM defined as

$$TM \setminus \{0\} = \{y \in TM : y \neq 0\}.$$

For an iterated tangent bundle T^rM where $r \geq 2$ we define the *slashed tangent bundle* as the open set

$$T^rM \setminus \{0\} = \{\xi \in T^rM : (D\pi_{T^{r-1}M \to M})(\xi) \in TM \setminus \{0\}\}.$$

For example,

$$\begin{aligned} TTM \setminus \{0\} &= \{(x, y, X, Y) \in TTM : X \neq 0\}, \\ TTTM \setminus \{0\} &= \{(x, y, X, Y, u, v, U, V) \in TTTM : u \neq 0\}, \end{aligned}$$

where, say, $TTM \setminus \{0\} = T^2M \setminus \{0\}$. When $r = 0$, let us also define $T^rM \setminus \{0\} = M$, and for any set $A \subset T^rM$ where $r \geq 0$, let

$$A \setminus \{0\} = A \cap T^rM \setminus \{0\}.$$

For $r \geq 1$ we have

$$\kappa_{r+1}(T^{r+1}M \setminus \{0\}) = T(T^rM \setminus \{0\}), \tag{3.14}$$
$$(D\pi_{r-1})(T^{r+1}M \setminus \{0\}) = T^rM \setminus \{0\}, \tag{3.15}$$
$$(D\kappa_r)(T^{r+1}M \setminus \{0\}) = T^{r+1}M \setminus \{0\}. \tag{3.16}$$

Before proving these equations, we define the *Liouville vector field* $E_r \in \mathfrak{X}(T^rM)$. For $r \geq 1$, it is given by

$$E_r(\xi) = \partial_s((1+s)\xi)|_{s=0}, \quad \xi \in T^rM.$$

If $r \geq 1$, and (x, y) and (x, y, X, Y) are local coordinates for T^rM and $T^{r+1}M$, respectively, then

$$E_r(x, y) = (x, y, 0, y).$$

Equation (3.14) follows using equation (3.7) and by writing

$$\pi_{T^rM \to M} = \pi_{T^{r-1}M \to M} \circ \pi_{r-1}, \quad r \geq 1. \tag{3.17}$$

If $r \geq 1$, we have

$$\xi = D\pi_{r-1} \circ \kappa_{r+1} \circ E_r(\xi), \quad \xi \in T^rM, \tag{3.18}$$

and equation (3.15) follows using equations (3.14) and (3.17). Equation (3.16) follows using equations (3.8) and (3.17).

3.3 Lifts for functions

Suppose $f \in C^\infty(M)$ is a smooth function. Then we can lift f using the *vertical lift* or the *complete lift* and obtain functions $f^v, f^c \in C^\infty(TM)$ defined by

$$f^v(\xi) = f \circ \pi_0(\xi), \quad f^c(\xi) = df(\xi), \quad \xi \in TM. \tag{3.19}$$

Here df is the exterior derivative of f. In local coordinates (x, y) for TM, it follows that

$$f^v(x, y) = f(x), \quad f^c(x, y) = \frac{\partial f}{\partial x^i}(x) y^i.$$

Using these lifts one can define vertical and complete lift for tensor fields on M of arbitrary order. For a full development of these issues, see [26].

Next we generalize the vertical and complete lifts to functions defined on iterated tangent bundles $T^r M$ of arbitrary order $r \geq 0$.

Definition 3.2. For $r \geq 0$, the *vertical lift* of a function $f \in C^\infty(T^r M \setminus \{0\})$ is the function $f^v \in C^\infty(T^{r+1} M \setminus \{0\})$ defined by

$$f^v(\xi) \quad = \quad f \circ \pi_r \circ \kappa_{r+1}(\xi), \quad \xi \in T^{r+1}M \setminus \{0\}.$$

If $r = 0$, Definition 3.2 reduces to lift f^v in equation (3.19), and if $r \geq 1$, equation (3.7) implies that $f^v = f \circ D\pi_{r-1}$, and equation (3.15) implies that f^v is smooth. For $r \geq 1$, let x^i be local coordinates for $T^{r-1}M$, and let (x, y, X, Y) be local coordinates for $T^{r+1}M$. Then

$$f^v(x, y, X, Y) \quad = \quad f(x, X), \quad f \in C^\infty(T^r M \setminus \{0\}).$$

Definition 3.3. For $r \geq 0$, the *complete lift* of a function $f \in C^\infty(T^r M \setminus \{0\})$ is the function $f^c \in C^\infty(T^{r+1} M \setminus \{0\})$ defined by

$$f^c(\xi) \quad = \quad (df) \circ \kappa_{r+1}(\xi), \quad \xi \in T^{r+1}M \setminus \{0\}.$$

If $r = 0$, then Definition 3.3 reduces to lift f^c in equation (3.19), and if $r \geq 1$, then equation (3.14) implies that f^c is smooth. For $r \geq 1$, let x^i be local coordinates for $T^{r-1}M$, and let (x, y, X, Y) be local coordinates for $T^{r+1}M$. Then

$$f^c(x, y, X, Y) \quad = \quad \frac{\partial f}{\partial x^a}(x, X) y^a + \frac{\partial f}{\partial y^a}(x, X) Y^a, \quad f \in C^\infty(T^r M \setminus \{0\}).$$

Taking two complete lifts of $f \in C^\infty(T^r M \setminus \{0\})$ yields

$$\begin{aligned} f^{cc} \quad = \quad & f^c(x, Y, u, V) \\ & + \left(\frac{\partial f}{\partial x^a}\right)^c (x, y, u, v) X^a + \left(\frac{\partial f}{\partial y^a}\right)^c (x, y, u, v) U^a, \end{aligned} \tag{3.20}$$

where argument $(x, y, X, Y, u, v, U, V) \in T^{r+2}M \setminus \{0\}$ has been suppressed.

If $f \in C^\infty(T^r M \setminus \{0\})$ for some $r \geq 1$, then

$$f^{vv} \quad = \quad f^{vv} \circ (D\kappa_{r+1}), \tag{3.21}$$

$$f^{vc} \quad = \quad f^{cv} \circ (D\kappa_{r+1}), \tag{3.22}$$

$$f^{cc} \quad = \quad f^{cc} \circ (D\kappa_{r+1}). \tag{3.23}$$

In Section 6 we use these identities to study geodesics of iterated complete lifts of a spray.

4 Sprays

A *spray* on M is a vector field S on $TM\setminus\{0\}$ that satisfies two conditions. Essentially, these conditions state that (i) an integral curve of S is of the form $c'\colon I\to TM\setminus\{0\}$ for a curve $c\colon I\to M$, and (ii) integral curves of S are closed under affine reparametrizations $t\mapsto Ct+t_0$. Then curve $c\colon I\to M$ is a *geodesic* of S. The motivation for studying sprays is that they provides a unified framework for studying geodesics for Riemannian metrics, Finsler metrics, and non-linear connections. See [5, 7, 23, 24]. Next we generalize the definition of a spray to iterated tangent bundles T^rM for any $r\geq 0$.

4.1 Sprays on T^rM

Definition 4.1 (Spray space). Suppose S is a vector field $S\in\mathfrak{X}(T^{r+1}M\setminus\{0\})$ where $r\geq 0$. Then S is a *spray* on T^rM if

(i) $(D\pi_r)(S)=\mathrm{id}_{T^{r+1}M\setminus\{0\}}$,

(ii) $[E_{r+1},S]=S$ for Liouville vector field $E_{r+1}\in\mathfrak{X}(T^{r+1}M)$.

Let S be a vector field $S\in\mathfrak{X}(T^{r+1}M\setminus\{0\})$ where $r\geq 0$. Then condition (i) in Definition 4.1 states that if (x,y,X,Y) are local coordinates for $T^{r+2}M$, then locally

$$S(x,y) = \big(x^i,y^i,y^i,-2G^i(x,y)\big) = y^i\frac{\partial}{\partial x^i}\bigg|_{(x,y)} - 2G^i(x,y)\frac{\partial}{\partial y^i}\bigg|_{(x,y)}, \tag{4.1}$$

where G^i are locally defined functions $G^i\colon T^{r+1}M\setminus\{0\}\to\mathbb{R}$. Condition (ii) in Definition 4.1 states that functions G^i are *positively* 2*-homogeneous*; if $(x,y)\in T^{r+1}M\setminus\{0\}$, then

$$G^i(x,\lambda y) = \lambda^2G^i(x,y),\quad \lambda>0.$$

This is a consequence of Euler's theorem for homogeneous functions [3].

Conversely, if S is a vector field $S\in\mathfrak{X}(T^{r+1}M\setminus\{0\})$ that locally satisfies these two conditions, then S is a spray on T^rM. Functions G^i in equation (4.1) are called *spray coefficients* for S.

When $r=0$, Definition 4.1 is equivalent to the usual definition of a spray [7, 24]. However, when $r\geq 1$, Definition 4.1 makes a slightly stronger assumption on the smoothness of S. Namely, if $r\geq 1$ and S is a spray on T^rM (in the sense of Definition 4.1) then S is smooth on $T^{r+1}M\setminus\{0\}$, but if S is a spray on manifold T^rM (in the usual sense) then S is smooth on $T(T^rM)\setminus\{0\}$. Since $T(T^rM)\setminus\{0\}\supset T^{r+1}M\setminus\{0\}$, it follows that if S is a spray on T^rM (in the sense of Definition 4.1), then S is also a spray on manifold T^rM (in the usual sense). The stronger assumption on S will be needed in Section 5 to prove that the complete lift of a spray on T^rM is a spray on $T^{r+1}M$. In this work we only consider sprays on T^rM that arise from complete lifts of a spray on M. Therefore we do not distinguish between the weaker and stronger definitions of a spray. These comments motivate the slightly non-standard terminology in Definition 4.1.

The next proposition shows that a spray on T^rM induces sprays on all lower order tangent bundles $M,TM,\dots,T^{r-1}M$.

Proposition 4.2. *If S is a spray on $T^{r+1}M$ where $r \geq 0$, then*

$$S^* \quad = \quad (DD\pi_r) \circ S \circ \kappa_{r+2} \circ E_{r+1}$$

is a spray on $T^r M$.

Proof. Equations (3.7), (3.9), and (3.17) imply that maps

$$\begin{aligned} \kappa_{r+2} \circ E_{r+1} \colon T^{r+1}M \setminus \{0\} &\quad \to \quad T^{r+2}M \setminus \{0\}, \\ DD\pi_r \colon T(T^{r+2}M \setminus \{0\}) &\quad \to \quad T(T^{r+1}M \setminus \{0\}), \end{aligned}$$

are smooth, so $S^* \colon T^{r+1}M \setminus \{0\} \to T(T^{r+1}M \setminus \{0\})$ is a smooth map. Let (x, y) be local coordinates for $T^{r+1}M$, and let (x, y, X, Y) be local coordinates for $T^{r+2}M$. Then S can be written as

$$S(x, y, X, Y) \quad = \quad \big(x, y, X, Y, X, Y, -2G^i(x, y, X, Y), -2H^i(x, y, X, Y)\big)$$

for locally defined functions $G^i, H^i \colon T^{r+2}M\setminus\{0\} \to \mathbb{R}$ that are positively 2-homogeneous with respect to (X, Y). It follows that

$$S^*(x, y) \quad = \quad \big(x, y, y, -2G^i(x, 0, y, y)\big)\,,$$

whence S^* is a vector field $S^* \in \mathfrak{X}(T^{r+1}M \setminus \{0\})$, and $(D\pi_r)(S^*) = \mathrm{id}_{T^{r+1}M\setminus\{0\}}$. Since functions $(x, y) \mapsto G^i(x, 0, y, y)$ are positively 2-homogeneous, S^* is a spray. $\square$

4.2 Geodesics on $T^r M$

Suppose γ is a curve $\gamma \colon I \to T^r M$ where $r \geq 0$. Then we say that γ is *regular* if $\gamma'(t) \in T^{r+1}M \setminus \{0\}$ for all $t \in I$. When $r = 0$, this coincides with the usual definition of a regular curve, and when $r \geq 1$, curve γ is regular if and only if curve $\pi_{T^r M \to M} \circ \gamma \colon I \to M$ is regular.

Definition 4.3 (Geodesic). Suppose S is a spray on $T^r M$ where $r \geq 0$. Then a regular curve $\gamma \colon I \to T^r M$ is a *geodesic* of S if and only if

$$\gamma'' \quad = \quad S \circ \gamma'.$$

Suppose S is a spray on $T^r M$ and locally S is given by equation (4.1). Then a regular curve $\gamma \colon I \to T^r M$, $\gamma = (x^i)$, is a geodesic of S if and only if

$$\ddot{x}^i \quad = \quad -2G^i \circ \gamma'. \tag{4.2}$$

In Definition 4.3 we have defined geodesics on open intervals. If γ is a curve on a closed interval we say that γ is a geodesic if γ can be extended into a geodesic defined on an open interval.

5 Complete lifts for a spray

Let S be a spray on M. Then the complete lift of S is a spray S^c on TM. That is, if S determines a geometry on M, then S^c determines a geometry on TM. The characteristic feature of spray S^c is that its geodesics are essentially in one-to-one correspondence with Jacobi fields of S. This correspondence will be the topic of Section 6. In this section, we define the complete lift for a spray on an iterated tangent bundle T^rM of arbitrary order $r \geq 0$. This makes it possible to take iterated complete lifts; if we start with a spray S on M we can take iterated lifts $S^c, S^{cc}, S^{ccc}, \dots$ and lift S onto an arbitrary iterated tangent bundle.

The definition below for the complete lift of a spray can essentially be found in [20, Remark 5.3]. For a further discussion about related lifts, see [6].

Definition 5.1 (Complete lift of spray). Suppose S is a spray on T^rM for some $r \geq 0$. Then the *complete lift* of S is the spray S^c on $T^{r+1}M$ defined by

$$S^c \quad = \quad D\kappa_{r+2} \circ \kappa_{r+3} \circ DS \circ \kappa_{r+2},$$

where DS is the tangent map of S,

$$DS\colon T(T^{r+1}M \setminus \{0\}) \quad \to \quad T^2(T^{r+1}M \setminus \{0\}).$$

Let us first note that equations (3.6), (3.7), (3.11), and (3.17) imply that

$$D\kappa_{r+2} \circ \kappa_{r+3}\colon T^2(T^{r+1}M \setminus \{0\}) \quad \to \quad T(T^{r+2}M \setminus \{0\})$$

is a smooth map. Thus S^c is a smooth map $T^{r+2}M \setminus \{0\} \to T(T^{r+2}M \setminus \{0\})$, and by equations (3.6), (3.7), and (3.14), S^c is a vector field $S^c \in \mathfrak{X}(T^{r+2}M \setminus \{0\})$. If S is the spray in equation (4.1), then locally

$$\begin{aligned} S^c &= \left(x, y, X, Y, X, Y, -2(G^i)^v, -2\left(G^i\right)^c\right) \\ &= X^i\frac{\partial}{\partial x^i} + Y^i\frac{\partial}{\partial y^i} - 2(G^i)^v\frac{\partial}{\partial X^i} - 2(G^i)^c\frac{\partial}{\partial Y^i}, \end{aligned} \tag{5.1}$$

and S^c is a spray on $T^{r+1}M$.

Suppose that S is a spray on T^rM for some $r \geq 0$, and suppose that γ is a regular curve $\gamma\colon I \to T^{r+1}M$, $\gamma = (x, y)$. Then γ is a geodesic of S^c if and only if

$$\ddot{x}^i \quad = \quad -2G^i \circ (\pi_r \circ \gamma)', \tag{5.2}$$

$$\ddot{y}^i \quad = \quad -2(G^i)^c \circ \gamma'. \tag{5.3}$$

It follows that $\pi_r \circ \gamma = (x^i)$ is a geodesic of S. In fact, if S^* is the spray in Proposition 4.2, then

$$S \quad = \quad (S^c)^*. \tag{5.4}$$

Thus a spray can always be recovered from its complete lift. What is more, if S is a spray on $T^{r+1}M$ for $r \geq 0$, then $S^{*c} = S$ if and only if $S = A^c$ for a spray A on T^rM.

The *geodesic flow* of a spray S is defined as the flow of S as a vector field.

Proposition 5.2 (Geodesic flow for the complete lift of a spray). *Suppose S is a spray on T^rM where $r \geq 0$ and S^c is the complete lift of S. Suppose furthermore that*

$$\phi\colon \mathcal{D}(S) \to T^{r+1}M \setminus \{0\}, \quad \phi^c\colon \mathcal{D}(S^c) \to T^{r+2}M \setminus \{0\},$$

are the geodesic flows of sprays S and S^c, respectively, with maximal domains

$$\mathcal{D}(S) \subset T^{r+1}M \setminus \{0\} \times \mathbb{R}, \quad \mathcal{D}(S^c) \subset T^{r+2}M \setminus \{0\} \times \mathbb{R}.$$

Then

$$((D\pi_r) \times \mathrm{id}_{\mathbb{R}})\,\mathcal{D}(S^c) = \mathcal{D}(S), \tag{5.5}$$

and

$$\phi^c_t(\xi) = \kappa_{r+2} \circ D\phi_t \circ \kappa_{r+2}(\xi), \quad (\xi, t) \in \mathcal{D}(S^c), \tag{5.6}$$

where $D\phi_t$ is the tangent map of the map $\xi \mapsto \phi_t(\xi)$ where t is fixed.

Proof. To prove inclusion "$\subset$" in equation (5.5), let $(\xi, t) \in \mathcal{D}(S^c)$, and let $\gamma\colon I \to T^{r+2}M \setminus \{0\}$ be an integral curve of S^c such that $\gamma(0) = \xi$ and $t \in I$. Then

$$DD\pi_r \circ S^c = S \circ D\pi_r,$$

so $D\pi_r \circ \gamma\colon I \to T^{r+1}M \setminus \{0\}$ is an integral curve of S, and $((D\pi_r)(\xi), t) \in \mathcal{D}(S)$. The other inclusion follows similarly since γ' is an integral curve of S^c when γ is an integral curve of S.

Suppose v is a curve $v\colon (-\varepsilon, \varepsilon) \to T^{r+1}M \setminus \{0\}$, and suppose that curve $\xi\colon I \times (-\varepsilon, \varepsilon) \to T^{r+2}M \setminus \{0\}$,

$$\xi(t, s) = \kappa_{r+2} \circ \partial_s\,(\phi_t \circ v(s)) \tag{5.7}$$

is defined for some interval I and $\varepsilon > 0$. For $(t, s) \in I \times (-\varepsilon, \varepsilon)$ we then have

$$\begin{aligned} S^c \circ \xi(t, s) &= D\kappa_{r+2} \circ \kappa_{r+3} \circ DS \circ \partial_s\,(\phi_t \circ v(s)) \\ &= D\kappa_{r+2} \circ \kappa_{r+3} \circ \partial_s (S \circ \phi_t \circ v(s)) \\ &= D\kappa_{r+2} \circ \kappa_{r+3} \circ \partial_s \partial_t (\phi_t \circ v(s)) \\ &= D\kappa_{r+2} \circ \partial_t \partial_s (\phi_t \circ v(s)) \\ &= D\kappa_{r+2} \circ \partial_t\,(\kappa_{r+2} \circ \xi(t, s)) \\ &= \partial_t \xi(t, s). \end{aligned}$$

To prove equation (5.6), let $(\xi_0, t_0) \in \mathcal{D}(S^c)$. Let $j(t) = \phi^c_t(\xi_0)$ be the integral curve $j\colon I^* \to T^{r+2}M \setminus \{0\}$ of S^c with maximal domain $I^* \subset \mathbb{R}$. Then $t_0 \in I^*$. For a compact subset $K \subset I^*$ with $0 \in K$ we show that

$$j(t) = \kappa_{r+2} \circ D\phi_t \circ \kappa_{r+2}(\xi_0), \quad t \in K \tag{5.8}$$

whence equation (5.6) follows. Since $\xi_0 \in T^{r+2}M \setminus \{0\}$, it follows that $\kappa_{r+2}(\xi_0) = \partial_s v(s)|_{s=0}$ for a curve $v\colon (-\varepsilon, \varepsilon) \to T^{r+1}M \setminus \{0\}$. Suppose $\tau \in K$. Then $(\xi_0, \tau) \in \mathcal{D}(S^c)$, and by equation (5.5), $(v(0), \tau) \in \mathcal{D}(S)$. Since $\mathcal{D}(S)$ is open [1], there is an open interval $I \ni \tau$ and an $\varepsilon > 0$ such that curve $\xi(t, s)$ in equation (5.7) is defined on $I \times (-\varepsilon, \varepsilon)$. Then, as K is compact, we can shrink ε and assume that $K \subset I$. Now equation (5.8) follows since $\xi(t, 0) = \kappa_{r+2} \circ D\phi_t \circ \kappa_{r+2}(\xi_0)$, $S^c \circ \xi(t, 0) = \partial_t \xi(t, 0)$ for $t \in I$, and $\xi(0, 0) = \xi_0$. □

6 Jacobi fields for a spray

Definition 6.1 (Jacobi field). Suppose S is a spray on T^rM where $r \geq 0$, and suppose that $\gamma\colon I \to T^rM$ is a geodesic of S. Then a curve $J\colon I \to T^{r+1}M$ is a *Jacobi field along* γ if

(i) J is a geodesic of S^c,

(ii) $\pi_r \circ J = \gamma$.

In Proposition 6.3 we will show that Definition 6.1 is equivalent with the usual characterization of a Jacobi field in terms of geodesic variations. In view of Proposition 5.2, this should not be surprising. For example, in Riemannian geometry it is well known that Jacobi fields are closely related to the tangent map of the exponential map.

Definition 6.2 (Geodesic variation). Suppose S is a spray on T^rM where $r \geq 0$, and suppose that $\gamma\colon I \to T^rM$ is a geodesic of S. Then a *geodesic variation* of γ is a smooth map $V\colon I \times (-\varepsilon, \varepsilon) \to T^rM$ such that

(i) $V(t, 0) = \gamma(t)$ for all $t \in I$,

(ii) $t \mapsto V(t, s)$ is a geodesic for all $s \in (-\varepsilon, \varepsilon)$.

Suppose that I is a closed interval. Then we say that a curve $J\colon I \to T^rM$ is a Jacobi field if we can extend J into a Jacobi field defined on an open interval. Similarly, a map $V\colon I \times (-\varepsilon, \varepsilon) \to T^rM$ is a geodesic variation if there is a geodesic variation $V^*\colon I^* \times (-\varepsilon^*, \varepsilon^*) \to T^rM$ such that $V = V^*$ on the common domain of V and V^* and $I \subset I^*$.

The next proposition motivates the above non-standard definition for a Jacobi field using the complete lift of a spray.

Proposition 6.3 (Jacobi fields and geodesic variations). *Let S be a spray on T^rM where $r \geq 0$, let $J\colon I \to T^{r+1}M$ be a curve, where I is open or closed, and let $\gamma\colon I \to T^rM$ be the curve $\gamma = \pi_r \circ J$.*

(i) If J can be written as

$$J(t) = \partial_s V(t, s)|_{s=0}, \quad t \in I \tag{6.1}$$

for a geodesic variation $V\colon I \times (-\varepsilon, \varepsilon) \to T^rM$, then J is a Jacobi field along γ.

(ii) If J is a Jacobi field along γ and I is compact, then there exists a geodesic variation $V\colon I \times (-\varepsilon, \varepsilon) \to T^rM$ such that equation (6.1) *holds.*

Proof. For (i), let us first assume that I is open. For $t \in I$ we then have

$$\begin{aligned}
S^c \circ \partial_t J(t) &= D\kappa_{r+2} \circ \kappa_{r+3} \circ DS \circ \kappa_{r+2} \circ \partial_t \partial_s V(t, s)|_{s=0} \\
&= D\kappa_{r+2} \circ \kappa_{r+3} \circ \partial_s (S \circ \partial_t V(t, s))|_{s=0} \\
&= D\kappa_{r+2} \circ \kappa_{r+3} \circ \partial_s \partial_t \partial_t V(t, s)|_{s=0} \\
&= D\kappa_{r+2} \circ \partial_t \partial_s \partial_t V(t, s)|_{s=0} \\
&= \partial_t \partial_t \partial_s V(t, s)|_{s=0} \\
&= J''(t).
\end{aligned}$$

If I is closed, we can extend V and J so that I is open and the result follows from the case when I is open.

For (ii), we have $J'(0) \in T^{r+2}M \setminus \{0\}$, so we can find a curve $w\colon (-\varepsilon, \varepsilon) \to T^{r+1}M \setminus \{0\}$ such that $\kappa_{r+2}(J'(0)) = \partial_s w(s)|_{s=0}$. Then $w(0) = \gamma'(0)$. Since I is compact and $\mathcal{D}(S)$ is open, we can extend I into an open interval I^* and find an $\varepsilon > 0$ such that $V(t,s) = \pi_r \circ \phi_t \circ w(s)$ is a map $V\colon I^* \times (-\varepsilon, \varepsilon) \to T^rM$. We have $V(t,0) = \gamma(t)$ for $t \in I$, and for each $s \in (-\varepsilon, \varepsilon)$, the map $t \mapsto V(t,s)$ is a geodesic of S. Proposition 5.2 and equations (3.7) and (2.3) imply that for $t \in I$,

$$\begin{aligned} J(t) &= \pi_{r+1} \circ \phi_t^c \circ J'(0) \\ &= \pi_{r+1} \circ \kappa_{r+2} \circ D\phi_t \circ \partial_s w(s)|_{s=0} \\ &= \partial_s(\pi_r \circ \phi_t \circ w(s))|_{s=0}. \end{aligned}$$

We have shown that V is a geodesic variation for Jacobi field J. □

Remark 6.4. Suppose $c\colon I \to M$ is a geodesic for a Riemannian metric, where I is compact. Then one can characterize Jacobi fields along c using geodesic variations as in Proposition 6.3 [12]. Using the complete lift, we can therefore write the traditional Jacobi equation in Riemannian geometry as $J'' = S^c \circ J'$. It is interesting to note that the derivation of the latter equation only uses the definition of S^c, the geodesic equation for S, the commutation rule (3.4) for κ_r, and the chain rule in equation (2.3). In particular, there is no need for local coordinates, covariant derivatives, nor curvature. For comparison, see the derivations of the Jacobi equations in Riemannian geometry [23], in Finsler geometry [3], and in spray spaces [24]. All of these derivations are considerably more involved than the proof of Proposition 6.3 (i). For semi-sprays, see also [6] and [7].

6.1 Geodesics of S^{cc}

Let S be a spray on T^rM for some $r \geq 0$. We know that a regular curve $\gamma\colon I \to T^{r+1}M$, $\gamma = (x,y)$, is a geodesic of S^c if and only if γ locally solves equations (5.2)-(5.3). Let us next derive corresponding geodesic equations for spray S^{cc}.

Let S be given by equation (4.1) in local coordinates (x,y) for $T^{r+1}M$. Then the complete lift of S^c is the spray S^{cc} on $T^{r+2}M$ given by

$$\begin{aligned} S^{cc} &= \big(x, y, X, Y, u, v, U, V, u, v, U, V, \\ &\qquad -2(G^i)^{vv}, -2(G^i)^{cv}, -2(G^i)^{vc}, -2(G^i)^{cc}\big). \end{aligned}$$

Suppose J is a regular curve $J\colon I \to T^{r+2}M$, $J = (x,y,X,Y)$. By equations (3.20) and (3.22), J is a geodesic of S^{cc} if and only if

$$\begin{aligned} \ddot{x}^i &= -2G^i \circ c', \\ \ddot{y}^i &= -2(G^i)^c \circ J_1', \\ \ddot{X}^i &= -2(G^i)^c \circ J_2', \\ \ddot{Y}^i &= -2(G^i)^{cc} \circ J' \\ &= -2(G^i)^c(x^i, Y^i, \dot{x}^i, \dot{Y}^i) \\ &\quad -2\left(\left(\frac{\partial G^i}{\partial x^a}\right)^c (J_1')X^a + \left(\frac{\partial G^i}{\partial y^a}\right)^c (J_1')\dot{X}^a\right), \end{aligned}$$

where curves $c\colon I \to T^rM$, $J_1\colon I \to T^{r+1}M$, and $J_2\colon I \to T^{r+1}M$ are given by

$$c = \pi_{T^{r+2}M\to T^rM} \circ J, \quad J_1 = \pi_{r+1} \circ J, \quad J_2 = (D\pi_r)(J),$$

and in local coordinates $c = (x^i)$, $J_1 = (x^i, y^i)$, and $J_2 = (x^i, X^i)$.

We have shown that if J is a geodesic of spray S^{cc}, then J contains two independent Jacobi fields J_1 and J_2 along c. The interpretation of this is seen by writing $J = (x, y, X, Y)$ using a geodesic variation. Then $J_1 = (x, y)$ is the base geodesic of S^c, and $J_2 = (x, X)$ describes the variation of geodesic $c\colon I \to M$. A geometric interpretation of components Y^i seems to be more complicated. For example, (x, Y) does not define a vector field along c. However, for fixed local coordinates, Y^i describe the variation of the vector components of Jacobi field $J_1 = (x, y)$. If $J_2 = 0$, that is, the variation does not vary the base geodesic c, then equations for Y^i simplify and (x^i, Y^i) is a Jacobi field. In this case, curve (x^i, Y^i) is also independent of local coordinates (see transformation rules in Section 2.1).

6.2 Iterated complete lifts

Let S^0 be a spray on M. For $r \geq 1$, let S^r be the rth iterated complete lift of S^0, that is, for $r \geq 1$, let

$$S^r = (S^{r-1})^c.$$

Then $S^0, S^1, S^2, \dots$ are sprays on $M, TM, TTM, \dots$, and in general, S^r is a spray on T^rM.

Equation (5.4) shows that each S^r contains all geometry of the original spray S^0. A more precise description is given by equation (5.1). It shows that sprays $S^1, S^2, \dots$ also contain new geometry obtained from derivatives of spray coefficients G^i of S^0. Namely, the rth complete lift S^r depends on derivatives of G^i to order r. This phenomena can also be seen from the geodesic flows of higher order lifts. If ϕ is the flow of S^0, then up to a permutation of coordinates, the flow of S^1 is $D\phi$, the flow of S^2 is $DD\phi$, and, in general, the flow of S^r is the rth iterated tangent map $D \cdots D\phi$. This means that the flow of S^1 describes the linear deviation of nearby geodesic of S. That is, the flow of S^1 describes the evolution of Jacobi fields. Similarly, flows of higher order lifts describe higher order derivatives of geodesic deviations.

Proposition 6.5 (New Jacobi fields from old ones). *Suppose $S^0, S^1, S^2, \dots$ are defined as above, and suppose that $j\colon I \to T^rM$ is a geodesic for some S^r.*

(i) If $r \geq 0$, $t_0 \in \mathbb{R}$, and $C > 0$, then $j(Ct + t_0)$ is a geodesic of S^r.

(ii) If $r \geq 1$ and $k\colon I \to T^rM$ is another geodesic of S^r such that $\pi_{r-1} \circ j(t) = \pi_{r-1} \circ k(t)$, then

$$\alpha j + \beta k, \quad \alpha, \beta \in \mathbb{R}$$

is a geodesic of S^r.

(iii) If $r \geq 1$, then $\kappa_r \circ j\colon I \to T^rM$ is a geodesic of S^r.

(iv) If $r \geq 1$, then $\pi_{r-1} \circ j\colon I \to T^{r-1}M$ is a geodesic of S^{r-1}.

(v) If $r \geq 2$, then $(D\pi_{r-2})(j) \colon I \to T^{r-1}M$ is a geodesic of S^{r-1}.

(vi) If $r \geq 0$, then $j' \colon I \to T^{r+1}M$ is a geodesic of S^{r+1}.

(vii) If $r \geq 0$, then $tj'(t) \colon I \to T^{r+1}M$ is a geodesic of S^{r+1}.

(viii) If $r \geq 1$, then $E_r \circ j \colon I \to T^{r+1}M$ is a geodesic of S^{r+1}.

Proof. Properties (i), (ii), and (iv) follow using equations (4.2), (5.2), and (5.3). Properties (vi), (vii), and (viii) follow by locally studying geodesic variations

$$\begin{aligned} V(t,s) &= j(t+s), \\ V(t,s) &= j((1+s)t), \\ V(t,s) &= (1+s)j(t), \end{aligned}$$

and using Proposition 6.3 (i). Property (iii) follows using geodesic equations for S^{cc} in Section 6.1 and equation (3.23). Property (v) follows using equation (3.7). □

6.3 Conjugate points

Suppose S is a spray on T^rM for some $r \geq 0$. If a, b are distinct points in T^rM that can be connected by a geodesic $\gamma \colon [0, L] \to T^rM$, then a and b are *conjugate points* if there is a Jacobi field $J \colon [0, L] \to T^{r+1}M$ along γ that vanishes at a and b, but J is not identically zero (with respect to vector space structure in equations (2.1)-(2.2)).

The next proposition shows that S has conjugate points if and only if S^c has conjugate points. Thus the complete lift alone does not remove conjugate points.

Proposition 6.6 (Conjugate points and complete lift). *Suppose S is a spray on T^rM for some $r \geq 0$.*

(i) If $a, b \in T^rM$ are conjugate points for S, then zero vectors in T^{r+1}_aM and T^{r+1}_bM are conjugate points for S^c.

(ii) If $a, b \in T^rM$ are conjugate points for S, then there are non-zero conjugate points in T^{r+1}_aM and T^{r+1}_bM for S^c.

(iii) If $a, b \in T^{r+1}M$ are conjugate points for S^c, then $\pi_r(a), \pi_r(b)$ are conjugate points for S.

Proof. For property (i), suppose $J \colon [0, L] \to T^{r+1}M$ is a Jacobi field of S that shows that a and b are conjugate points. Then the claim follows by studying Jacobi field $E_{r+1} \circ J$. For property (ii), suppose that $J \colon [0, L] \to T^{r+1}M$ is as in (i), and let $\gamma \colon [0, L] \to T^rM$ be the geodesic $\gamma = \pi_r \circ J$ for S. We will show that $\gamma'(0), \gamma'(L) \in T^{r+1}M \setminus \{0\}$ are conjugate points for S^c. This follows by considering Jacobi field $j \colon [0, L] \to T^{r+2}M$,

$$j(t) = \partial_s \left(\gamma'(t) + sJ(t)\right)|_{s=0}.$$

For property (iii), suppose $J \colon [0, L] \to T^{r+2}M$ is a Jacobi field of S^c that shows that a and b are conjugate points. Then J is a geodesic of S^{cc}, and locally J satisfy equations in Section 6.1. If Jacobi field $j = (D\pi_r)(J)$ does not vanish identically, the claim follows. Otherwise $(D\pi_r)(J)$ vanishes identically, and the result follows by the last comment in Section 6.1. □

7 Sprays restricted to a semi-distribution

From a spray S on M one can construct a new geometric space by restricting the spray to a geodesically invariant distribution $\Delta \subset TM$. This is done by requiring that all geodesics are tangent to the distribution. For example, geodesics in Euclidean space $\mathbb{R}^3$ can in this way be constrained to xy-planes. See [2, 18, 19].

In this section we study a slightly more general geometry, where one can not only restrict possible directions, but also basepoints for geodesics. For example, geodesics in $\mathbb{R}^3$ can in this way be constrained to one line or one plane. For a spray on T^rM, this is done by requiring that geodesics are tangent to a suitable geodesically invariant submanifold $\Delta \subset T^{r+1}M$. Such a submanifold will be called a *semi-distribution* and the restricted geometry will be called a *sub-spray.* There does not seem to be any work on this type of geometry. The terms semi-distribution and sub-spray neither seem to have been used before.

Definition 7.1. A set $\Delta \subset T^{r+1}M$ where $r \geq 0$ is a *semi-distribution on* T^rM if

(i) $\pi_r(\Delta)$ is a submanifold in T^rM.

(ii) $B = \pi_r \circ \kappa_{r+1}(\Delta)$ is a submanifold in T^rM.

(iii) There is a $k \geq 1$ such that every $b \in B$ has an open neighborhood $U \subset B$, and there are k maps $V_1, \ldots, V_k \colon U \to T^{r+1}M$ such that

  (a) $\pi_r \circ V_i = \iota$ for $i = 1, \ldots, k$, where ι is inclusion $U \hookrightarrow T^rM$,

  (b) V_i are pointwise linearly independent,

  (c) for all $u \in U$ we have

$$\kappa_{r+1}(\Delta) \cap \pi_r^{-1}(u) \quad = \quad \operatorname{span}\{V_1(u), \ldots, V_k(u)\}.$$

  (In (b) and (c), the linear structure of $T^{r+1}M$ is with respect to equations (2.1)-(2.2).)

We say that k is the *rank* of Δ and write $\operatorname{rank} \Delta = k$.

In condition (ii), $B = \pi_0(\Delta)$ when $r = 0$, and $B = (D\pi_{r-1})(\Delta)$ when $r \geq 1$. Thus, if $r = 0$ and $\pi_0(\Delta) = M$, a semi-distribution is a distribution in the usual sense.

Condition (iii) states that there is a k dimensional vector space associated to each $b \in B$, and $1 \leq k \leq 2^r \dim M$. When $r = 0$, the structure of these vector spaces in Δ is given by equations (2.1)-(2.2), and when $r \geq 1$, the structure is given by equations (3.2)-(3.3). The next example motivates the use of vector space structure in equations (3.2)-(3.3) when $r \geq 1$. Namely, these equations describe the natural vector space structure for tangents to Jacobi fields.

Example 7.2. Let S be a spray on T^rM for some $r \geq 0$, let $\gamma \colon I \to T^rM$ be a geodesic of S, and let $\mathfrak{X}(\gamma)$ be the set of vector fields along γ with the vector space structure defined by equations (2.1)-(2.2). Furthermore, let $J_1, J_2 \in \mathfrak{X}(\gamma)$ be Jacobi fields along γ, such that locally $\gamma = (x)$, $J_1 = (x, y)$, and $J_2 = (x, z)$. For $\alpha, \beta \in \mathbb{R}$ we then have

$$\begin{aligned}\alpha J_1 + \beta J_2 &= (x, \alpha y + \beta z),\\ (\alpha J_1 + \beta J_2)' &= (x, \alpha y + \beta z, \dot{x}, \alpha \dot{y} + \beta \dot{z})\\ &= \alpha \cdot J_1' + \beta \cdot J_2',\end{aligned}$$

where on the last line, $+$ and $\cdot$ are as in equations (3.2)-(3.3). Thus, if we define the vector space structure for Jacobi fields by equations (2.1)-(2.2), then the natural vector structure for tangents (and initial values) is given by equations (3.2)-(3.3). On the other hand, the multiplication operator in equation (2.2) appears naturally when reparametrizing a curve. If $J\colon I \to T^r M$ is a curve for $r \geq 0$, and $j(t) = J(Ct + t_0)$, then $j'(t) = C \cdot J'(Ct + t_0)$, where multiplication $\cdot$ is as in equation (2.2).

Proposition 7.3. *Suppose Δ is a semi-distribution on $T^r M$ and $B = \pi_r \circ \kappa_{r+1}(\Delta)$. Then Δ is a sub-manifold in $T^{r+1} M$ and*

$$\dim \Delta = \dim B + \operatorname{rank} \Delta.$$

The proof of Proposition 7.3 follows by setting $A = \kappa_{r+1}(\Delta)$ in the lemma below. We also use this lemma to prove Proposition 8.3.

Lemma 7.4. *Suppose A is a subset $A \subset T^{r+1} M$ for some $r \geq 0$ such that*

(i) *$\pi_r(A)$ is a submanifold in $T^r M$.*

(ii) *There is a $k \geq 1$ such that every $b \in \pi_r(A)$ has an open neighborhood $U \subset \pi_r(A)$, and there are k maps $V_1, \ldots, V_k\colon U \to T^{r+1} M$ such that*

(a) *$\pi_r \circ V_i = \iota$ for $i = 1, \ldots, k$, where ι is inclusion $U \hookrightarrow T^r M$,*

(b) *$V_1, \ldots, V_k$ are pointwise linearly independent in U,*

(c) *for all $u \in U$ we have*

$$A \cap \pi_r^{-1}(u) = \operatorname{span}\{V_1(u), \ldots, V_k(u)\}.$$

(In (b) and (c), the linear structure of $T^{r+1} M$ is with respect to equations (2.1)-(2.2).*)*

Then A is a submanifold of $T^{r+1} M$ of dimension $\dim \pi_r(A) + k$. *Moreover, if we can assume that $U = \pi_r(A)$, then A is diffeomorphic to $\pi_r(A) \times \mathbb{R}^k$.*

Proof. Let $\xi \in A$. Then $\pi_r(\xi)$ has an open neighborhood $U \subset \pi_r(A)$ with k maps $V_1, \ldots, V_k\colon U \to T^{r+1} M$ such that (a), (b), and (c) hold. By possibly shrinking U we can find maps $V_{k+1}, \ldots, V_N\colon U \to T^{r+1} M$, where $N = \dim T^{r+1}_{\pi_r(\xi)} M$, such that $\pi_r \circ V_i = \iota$ for $i = 1, \ldots, N$, and

$$\pi_r^{-1}(u) = \operatorname{span}\{V_1(u), \ldots, V_N(u)\}.$$

Let f be the diffeomorphism $f\colon U \times \mathbb{R}^N \to \pi_r^{-1}(U)$ defined as

$$f(u, \alpha_1, \ldots, \alpha_N) = \alpha_1 V_1(u) + \cdots + \alpha_N V_N(u).$$

Let $g\colon U\times\mathbb{R}^k \to \pi_r^{-1}(U)$ be the restriction of f onto $U\times\mathbb{R}^k$. Then g is a smooth injection and immersion such that $g(U\times\mathbb{R}^k) = A\cap\pi_r^{-1}(U)$, and map $f^{-1}\circ g\colon U\times\mathbb{R}^k \to U\times\mathbb{R}^N$ is the inclusion $(u,\alpha_1,\ldots,\alpha_k)\mapsto(u,\alpha_1,\ldots,\alpha_k,0,\ldots,0)$. Since a closed set in a compact Hausdorff space is compact, $f^{-1}\circ g$ is proper. Thus g is proper, and the claim follows from the following result: If $f\colon M\to N$ is a smooth map between manifolds that is proper, injective, and an immersion, then $f(M)$ is a submanifold in N of dimension $\dim M$, and f restricts to a diffeomorphism $f\colon M\to F(M)$. See results 7.4, 8.3, and 8.25 in [17]. $\square$

7.1 Geodesics in a sub-spray

Definition 7.5 (Geodesically invariant set). Let S be a spray on T^rM where $r\geq 0$. Then a set $\Delta\subset T^{r+1}M$ is a *geodesically invariant set for S* provided that:

> If $\gamma\colon I\to T^rM$ is a geodesic of S with $\gamma'(t_0)\in\Delta$ for some $t_0\in I$, then $\gamma'(t)\in\Delta$ for all $t\in I$.

Definition 7.6 (Sub-spray). Suppose S is a spray on T^rM for some $r\geq 0$, and Δ is a geodesically invariant semi-distribution on T^rM. Then we say that triple $\Sigma=(S,T^rM,\Delta)$ is a *sub-spray.* A curve $\gamma\colon I\to T^rM$ is a *geodesic in Σ* if

(i) $\gamma\colon I\to T^rM$ is a geodesic of S,

(ii) $\gamma'(t_0)\in\Delta$ for some $t_0\in I$ (whence $\gamma'(t)\in\Delta$ for all $t\in I$).

By taking $\Delta=T^{r+1}M$, we may treat any spray as a sub-spray. Let us also note that if $\Delta\subset T^{r+1}M\setminus\{0\}$ where $r\geq 0$, then

$$\pi_r(\Delta)\subset T^rM,\quad \pi_r\circ\kappa_{r+1}(\Delta)\subset T^rM\setminus\{0\}.$$

Then

$$\begin{aligned}\Delta\setminus\{0\} &= \{\gamma'(0) : \gamma\colon(-\varepsilon,\varepsilon)\to T^rM \text{ is a geodesic in } \Sigma\},\\ \pi_r(\Delta\setminus\{0\}) &= \{\gamma(0) : \gamma\colon(-\varepsilon,\varepsilon)\to T^rM \text{ is a geodesic in } \Sigma\}.\end{aligned}$$

In other words, a vector $\xi\in T^{r+1}M$ is in $\Delta\setminus\{0\}$ if and only if there is a geodesic in Σ whose tangent passes through ξ, and a point $x\in T^rM$ is in $\pi_r(\Delta\setminus\{0\})$ if and only if there is a geodesic in Σ that passes through x. We therefore say that $\Delta\setminus\{0\}$ is *phase space* for Σ, and $\pi_r(\Delta\setminus\{0\})$ is *configuration space* for Σ. When $r\geq 1$, the set $B=(D\pi_{r-1})(\Delta)$ satisfies

$$B\setminus\{0\} = \{(\pi_{r-1}\circ\gamma)'(0) : \gamma\colon(-\varepsilon,\varepsilon)\to T^rM \text{ is a geodesic in } \Sigma\}.$$

and we can interpret $B\setminus\{0\}$ as phase space of geodesics in Σ that have been projected onto $T^{r-1}M$.

Example 7.7 (Geodesics through a point). Let $\Sigma=(S,T^rM,\Delta)$ be a sub-spray for some $r\geq 0$, and let $z\in\pi_r(\Delta\setminus\{0\})$ be a point in configuration space. Then the set

$$\Delta(z) = \Delta\cap(T_z^{r+1}M\setminus\{0\})$$

parameterizes initial values for geodesics that pass through z. Let us study the structure and the degrees of freedom for $\Delta(z)$.

When $r = 0$, the structure of $\Delta(z)$ is easy to understand; the set $\Delta(z)$ is a punctured vector subspace of $T_z M$ whose dimension is the rank of Δ.

When $r \geq 1$, the structure of $\Delta(z)$ becomes more complicated. For example, in Section 8, we construct a sub-spray where configuration space and phase space are diffeomorphic, and $\Delta(z)$ contains only one vector. To understand this, let us assume that Δ is represented in canonical local coordinates (x, y, X, Y) for $T^{r+1}M$. That is, we here only consider coordinates (x, y, X, Y) that belong to Δ. Then coordinates (x, y, X, Y) have $\dim \Delta = \dim B + \operatorname{rank} \Delta$ degrees of freedom. Coordinates (x, X) represent submanifold B. They have $\dim B$ degrees of freedom, and once $(x, X) \in B$ is fixed, coordinates (y, Y) parameterize the rank Δ dimensional vector space associated with (x, X). If $z = (x_0, y_0)$, then geodesics that pass through z are parameterized by (x_0, y_0, X, Y), but very little can be said about possible values for (X, Y). Coordinates (x, X) have $\dim B$ degrees of freedom, but we do not know how these divide between x- and X-coordinates. Similarly, coordinates (y, Y) have $\operatorname{rank} \Delta$ degrees of freedom, but we do not know how these divide between y- and Y-coordinates.

The next proposition shows that geodesics in a sub-spray on $T^r M$ have a linear structure when $r \geq 1$, but geodesics are not necessarily invariant under affine reparametrizations.

Proposition 7.8. *Let $\Sigma = (S, T^r M, \Delta)$ be a sub-spray where $r \geq 0$.*

(i) Suppose that $j \colon I \to T^r M$ is a geodesic in Σ. If $t_0 \in \mathbb{R}$, and $C > 0$, then $j(Ct + t_0)$ is a geodesic in Σ if $r = 0$ or $C = 1$.

(ii) Suppose that $r \geq 1$. If $j, k \colon I \to T^r M$ are geodesics in Σ such that $\pi_{r-1} \circ j(t) = \pi_{r-1} \circ k(t)$, then

$$\alpha j + \beta k, \quad \alpha, \beta \in \mathbb{R}$$

is a geodesic in Σ.

Proof. Property (i) follows since reparametrizations scale tangent vectors as in equation (2.2), and this multiplication is only compatible with the vector structure of Δ when $r = 0$ or $C = 1$. Property (ii) follows using equations (3.12)-(3.13). $\square$

7.2 Jacobi fields for a sub-spray

Proposition 6.3 shows that for sprays, Jacobi fields on compact intervals can be characterized using geodesic variations. For sub-sprays, we take this characterization as the definition of a Jacobi field.

Definition 7.9 (Jacobi field in a sub-spray). Let $\gamma \colon I \to T^r M$ be a geodesic in a sub-spray $\Sigma = (S, T^r M, \Delta)$ where $r \geq 0$. Suppose that $J \colon K \to T^{r+1} M$ is a curve where $K \subset I$ is compact, and V is a map $V \colon I \times (-\varepsilon, \varepsilon) \to T^r M$ such that

(i) $t \mapsto V(t, s)$, $t \in I$ is a geodesic in sub-spray Σ for all $s \in (-\varepsilon, \varepsilon)$,

(ii) $V(t, 0) = \gamma(t)$ for $t \in I$,

(iii) $J(t) = \partial_s V(t, s)|_{s=0}$ for $t \in K$.

Then $J\colon K \to T^{r+1}M$ is a *Jacobi field along* γ.

By Proposition 6.3 (ii), a Jacobi field for a sub-spray (S, T^rM, Δ) is a Jacobi field for the spray S. The converse also holds when $\Delta = T^{r+1}M$.

8 A sub-spray for parallel Jacobi fields

This section contains the main results of this paper. We construct a sub-spray P whose geodesics are in one-to-one correspondence with parallel Jacobi fields, and study its geodesics.

Definition 8.1 (Parallel Jacobi field). Let S be a spray on M, and let $J\colon I \to TM$ be a curve. Then J is called a *parallel Jacobi field for* S if there are $\alpha, \beta \in \mathbb{R}$, and a geodesic $c\colon I \to M$ such that

$$J(t) \quad = \quad \alpha c'(t) + \beta t c'(t), \quad t \in I. \tag{8.1}$$

If I is closed we say that a curve $J\colon I \to TM$ is a parallel Jacobi field if J can be extended into a parallel Jacobi defined on an open interval. Proposition 6.5 shows that a parallel Jacobi field is a Jacobi field.

Lemma 8.2. *Suppose $J\colon I \to TM$ is a parallel Jacobi field.*

(i) If $C > 0$ and $t_0 \in \mathbb{R}$, then $J(Ct + t_0)$ is a parallel Jacobi field.

(ii) J can be extended to the maximal domain of geodesic $c = \pi_{TM\to M} \circ J$, and the extension is a parallel Jacobi field.

To construct sub-spray P, let S be a spray on a manifold M, let S^{cc} be the second complete lift of S, and let Δ be the geodesically invariant semi-distribution on TTM defined in Proposition 8.3. Then we define sub-spray P as

$$P \quad = \quad (S^{cc}, TTM, \Delta).$$

Proposition 8.3. *Suppose S is a spray on a manifold M, and let Δ be the set*

$$\Delta \quad = \quad \{ (\kappa_2 \circ J')'(0) : J\colon (-\varepsilon, \varepsilon) \to TM \ \text{ is parallel Jacobi field for } S \}.$$

Then

(i) $\Delta \subset TTTM \setminus \{0\}$,

(ii) Δ is a geodesically invariant semi-distribution on TTM of rank 2,

(iii) phase space Δ and configuration space $\pi_2(\Delta)$ are diffeomorphic.

Proof. Let us first note that Δ consists of points

$$\begin{aligned}&(x(0), \dot{x}(0), \alpha\dot{x}(0), \alpha\ddot{x}(0) + \beta\dot{x}(0),\\ &\quad \dot{x}(0), \ddot{x}(0), \alpha\ddot{x}(0) + \beta\dot{x}(0), \alpha\dddot{x}(0) + 2\beta\ddot{x}(0)),\end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$ and $c\colon (-\varepsilon, \varepsilon) \to M$ is a geodesic $c(t) = (x(t))$. By Lemma 7.4 (and by the result used to prove Lemma 7.4), it follows that sets

$$\begin{aligned} \pi_2(\Delta) &= \{\alpha S(y) + \beta E_1(y) : y \in TM \setminus \{0\}, \alpha, \beta \in \mathbb{R}\}, \\ (D\pi_1)(\Delta) &= \{S(y) : y \in TM \setminus \{0\}\}, \end{aligned}$$

are submanifolds in TTM diffeomorphic to $TM \setminus \{0\} \times \mathbb{R}^2$ and $TM \setminus \{0\}$, respectively. Let ι be the inclusion $B \hookrightarrow TTM$, where $B = (D\pi_1)(\Delta)$, and let $\widehat{S}$ be the diffeomorphism $\widehat{S}\colon TM \setminus \{0\} \to B$ such that $S = \iota \circ \widehat{S}$ and $\widehat{S}^{-1} = \pi_1 \circ \iota$. By the geodesic equation for S^c and the definition of S^c it follows that

$$\begin{aligned} \kappa_3(\Delta) &= (DS)(\pi_2(\Delta)) \\ &= \{\alpha V_1(\xi) + \beta V_2(\xi) : \xi \in B, \alpha, \beta \in \mathbb{R}\}, \end{aligned}$$

where $V_1, V_2\colon B \to TTTM$ are smooth maps

$$V_1 = DS \circ \iota, \quad V_2 = DS \circ E_1 \circ \pi_1 \circ \iota.$$

Now $\pi_2 \circ V_i = \iota$ for $i = 1, 2$. A local calculation shows that V_1 and V_2 are pointwise linearly independent. Hence Δ is a semi-distribution on TTM, and by Lemma 7.4, $\kappa_3(\Delta)$ is diffeomorphic to $B \times \mathbb{R}^2$.

To prove that Δ is geodesically invariant, let $\gamma\colon I \to TTM$ be a geodesic of S^{cc} with $\gamma'(0) \in \Delta$. By Proposition 6.5, it follows that

$$\gamma(t) = \kappa_2 \circ J'(t), \quad t \in (-\varepsilon, \varepsilon) \tag{8.2}$$

for a parallel Jacobi field $J\colon (-\varepsilon, \varepsilon) \to TM$. By Lemma 8.2 (ii) we can extend J into a parallel Jacobi field $J\colon I \to TM$ such that (8.2) holds for all $t \in I$. If $t_0 \in I$, we have $\gamma'(t_0) = (\kappa_2 \circ \tilde{J}')'(0)$ for parallel Jacobi field $\tilde{J}(t) = J(t + t_0)$, and (ii) follows. Property (iii) follows since both submanifolds are diffeomorphic to $B \times \mathbb{R}^2$. □

Let us note that configuration space $\pi_2(\Delta)$ is a proper subset of TTM, and

$$\dim \pi_2(\Delta) = 2n + 2, \ \dim \Delta = 2n + 2, \ \dim (D\pi_1)(\Delta) = 2n.$$

The next proposition shows that geodesics in P are in one-to-one correspondence with parallel Jacobi fields for M.

Proposition 8.4 (Geodesics in P). *Suppose $\gamma\colon I \to TTM$ is a curve. Then the following are equivalent:*

(i) γ is a geodesic in P.

(ii) There is a parallel Jacobi field $J\colon I \to TM$ such that

$$\gamma(t) = \kappa_2 \circ J'(t), \quad t \in I.$$

(iii) There is a geodesic $c\colon I \to M$ and $\alpha, \beta \in \mathbb{R}$ such that

$$\gamma(t) = (\alpha + \beta t) c''(t) + \beta E_1 \circ c'(t), \quad t \in I.$$

Moreover, in (ii) and (iii) J and c, α, β are uniquely determined by γ.

The next proposition shows that the geometry of P can be used to study dynamical properties of M.

Proposition 8.5. *The projection $\pi_{TTM \to M} \colon TTM \to M$ is a submersion that maps geodesics in P into geodesics on M.*

A sub-spray $(S, T^r M, \Delta)$ where $r \geq 0$ is *complete* if any geodesic $\gamma \colon I \to T^r M$ can be extended into a geodesic $\gamma \colon \mathbb{R} \to T^r M$.

Proposition 8.6. *Sub-spray P is complete if and only if M is complete.*

Proof. Suppose P is complete. By Proposition 8.4, any geodesic $c \colon I \to M$ can be lifted into a geodesic $c'' \colon I \to TTM$ for P. The converse direction follows by Proposition 8.4 and Lemma 8.2 (ii). □

In general, a geodesic $c \colon I \to M$ for a spray S on M is uniquely determined by $c'(0)$. The next proposition shows that in sub-spray P, a geodesic $\gamma \colon I \to TTM$ is uniquely determined by $\gamma(0)$. This is not surprising in view of Proposition 8.3 (iii).

Proposition 8.7. *If $\gamma_1 \colon I_1 \to TTM$ and $\gamma_2 \colon I_2 \to TTM$ are geodesics in P with $\gamma_1(0) = \gamma_2(0)$, then $\gamma_1 = \gamma_2$ on their common domain.*

Proof. By Proposition 8.4 we have that $\gamma_i = \kappa_2 \circ J_i'$ for parallel Jacobi fields $J_i \colon I \to TM$, $i = 1, 2$. Hence $J_1'(0) = J_2'(0)$, and the claim follows. □

Proposition 8.7 imposes a strong restriction on the behavior of geodesics in P. For example, if two points in TTM can be connected with a geodesic in P, then the geodesic is unique (up to loops). Also, any piece-wise geodesic curve that is continuous must be smooth. Therefore P has no broken geodesics nor geodesic triangles.

For a sub-spray we define conjugate points as for sprays (see Section 6.3).

Proposition 8.8. *Sub-spray P has no conjugate points.*

Proof. If a Jacobi field vanishes once, Proposition 8.7 implies that the corresponding geodesic variation is trivial. □

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