Tangent sphere bundles of natural diagonal lift type

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Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. We show that the tangent sphere bundles endowed with a Riemannian metric induced from the natural diagonal lifted metric of the tangent bundle are never space forms. Next we study the conditions under which these tangent sphere bundles are Einstein.

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1 Introduction

The tangent sphere bundle T_rM consisting of spheres of constant radius r centered in the null tangent vectors are hyper-surfaces of the tangent bundle TM, obtained by considering only the tangent vectors having the norm equal to r.

In the most part of the papers, like [3]-[6], [17], [25], [27], [28], the metric considered on the tangent bundle TM was the Sasaki metric (see [26]), but E. Boeckx noticed that the unit tangent bundle equipped with the induced Cheeger-Gromoll metric is isometric to the tangent sphere bundle $T_{\frac{1}{\sqrt{2}}}M$, of radius $\frac{1}{\sqrt{2}}$ endowed with the metric induced by the Sasaki metric. This suggested to O. Kowalski and M. Sekizawa the idea that the tangent sphere bundles with different constant radii and with the metrics induced from the Sasaki metric might possess different geometrical properties, and in the paper [11] from 2000 they showed how the geometry of the tangent sphere bundles depends on the radius.

Some other metrics on the tangent sphere bundles may be constructed by using a few lifts to the tangent bundle, which have been considered in [21]-[24], [29]-[31], and studied in some very recent papers, such as [2], [7] and [14].

In the last years some interesting results were obtained by endowing the tangent sphere bundles with Riemannian metrics induced by the natural lifted metrics from TM, which are not Sasakian (see [1], [16], [18] – [20]).

Roughly speaking, a natural operator (in the sense of [8] – [10], [13]) is a fibred manifold mapping, which is invariant with respect to the group of local diffeomorphisms of the base manifold.

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In the present paper we consider on the tangent bundle TM of a Riemannian manifold M a natural metric $\widetilde{G}$, obtained by the second author in [24] as the diagonal lift of the Riemannian metric g from the base manifold. We prove that the tangent sphere bundle T_rM endowed with the Riemannian metric G induced from $\widetilde{G}$ may never have constant sectional curvature, then we find the conditions under which (T_rM, G) is an Einstein manifold.

The manifolds, tensor fields and other geometric objects considered in this paper are assumed to be differentiable of class C^∞ (i.e. smooth). The Einstein summation convention is used throughout this paper, the range of the indices h, i, j, k, l, m, r, being always $\{1, \dots, n\}$.

2 Preliminary results

Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \to M$. The total space TM has a structure of a $2n$-dimensional smooth manifold, induced from the smooth manifold structure of M. This structure is obtained by using local charts on TM induced from usual local charts on M. If $(U, \varphi) = (U, x^1, \dots, x^n)$ is a local chart on M, then the corresponding induced local chart on TM is $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots, y^n)$, where the local coordinates $x^i, y^j,\ i, j = 1, \dots, n$, are defined as follows. The first n local coordinates of a tangent vector $y \in \tau^{-1}(U)$ are the local coordinates in the local chart (U, φ) of its base point, i.e. $x^i = x^i \circ \tau$, by an abuse of notation. The last n local coordinates $y^j,\ j = 1, \dots, n$, of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis in $T_{\tau(y)}M$ defined by the local chart (U, φ). Due to this special structure of differentiable manifold for TM, it is possible to introduce the concept of M-tensor field on it (see [15]). The vertical lift to TM of a vector field X defined on M will be denoted by X^V.

Denote by $\dot\nabla$ the Levi Civita connection of the Riemannian metric g on M. Then we have the direct sum decomposition

$$TTM = VTM \oplus HTM \tag{2.1}$$

of the tangent bundle to TM into the vertical distribution $VTM = \mathrm{Ker}\ \tau_*$ and the horizontal distribution HTM defined by $\dot\nabla$ (see [32]). The set of vector fields $\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\}$ on $\tau^{-1}(U)$ defines a local frame field for VTM and for HTM we have the local frame field $\{\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}\}$, where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i}\frac{\partial}{\partial y^h}$, $\quad \Gamma^h_{0i} = y^k\Gamma^h_{ki}$, and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of g.

The set $\{\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\}_{i,j=\overline{1,n}}$, denoted also by $\{\partial_i, \delta_j\}_{i,j=\overline{1,n}}$, defines a local frame on TM, adapted to the direct sum decomposition (2.1). The horizontal lift to TM of a vector field X defined on M will be denoted by X^H. Remark that $\delta_i = (\frac{\partial}{\partial x^i})^H$ and $\partial_i = (\frac{\partial}{\partial x^i})^V$.

The second author considered in [24] a Riemannian metric, which is obtained as a natural diagonal lift of the Riemannian metric g from the base manifold M to the tangent bundle TM, the coefficients being functions of the energy density defined by the tangent vector y as follows

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y, y) = \frac{1}{2}g_{ik}(x)y^iy^k, \quad y \in \tau^{-1}(U).$$

Obviously, we have $t \in [0,\infty)$ for all $y \in TM$.

Let us denote by $\widetilde{G}$ the natural diagonal Riemannian metric on TM, defined by:

$$
\begin{aligned}
&\widetilde{G}(X^H_y, Y^H_y) = c_1(t) g_{\tau(y)}(X,Y) + d_1(t) g_{\tau(y)}(X,y) g_{\tau(y)}(Y,y),\\
&\widetilde{G}(X^V_y, Y^V_y) = c_2(t) g_{\tau(y)}(X,Y) + d_2(t) g_{\tau(y)}(X,y) g_{\tau(y)}(Y,y),\\
&\widetilde{G}(X^V_y, Y^H_y) = \widetilde{G}(X^H_y, X^V_y) = 0,
\end{aligned}
\tag{2.2}
$$

$\forall X,Y \in \mathcal{T}^1_0(TM)$, $\forall y \in TM$, where $c_1,\ c_2,\ d_1,\ d_2$ are smooth functions of the energy density on TM. The conditions for $\widetilde{G}$ to be a Riemannian metric on TM (i.e. to be positive definite) are $c_1 > 0,\ c_2 > 0,\ c_1 + 2td_1 > 0,\ c_2 + 2td_2 > 0$ for every $t \geq 0$.

The symmetric matrix of type $2n \times 2n$

$$
\begin{pmatrix} \widetilde{G}^{(1)}_{ij} & 0 \\ 0 & \widetilde{G}^{(2)}_{ij} \end{pmatrix} = \begin{pmatrix} c_1(t) g_{ij} + d_1(t) g_{0i} g_{0j} & 0 \\ 0 & c_2(t) g_{ij} + d_2(t) g_{0i} g_{0j} \end{pmatrix},
$$

associated to the metric $\widetilde{G}$ in the adapted frame $\{\delta_j, \partial_i\}_{i,j=\overline{1,n}}$, has the inverse

$$
\begin{pmatrix} \widetilde{H}^{kl}_{(1)} & 0 \\ 0 & \widetilde{H}^{kl}_{(2)} \end{pmatrix} = \begin{pmatrix} p_1(t) g^{kl} + q_1(t) y^k y^l & 0 \\ 0 & p_2(t) g^{kl} + q_2(t) y^k y^l \end{pmatrix},
$$

where g^{kl} are the entries of the inverse matrix of $(g_{ij})_{i,j=\overline{1,n}}$, and $p_1,\ q_1,\ p_2,\ q_2$, are some real smooth functions of the energy density. More precisely, they may be expressed as rational functions of $c_1,\ d_1,\ c_2,\ d_2$:

$$
p_1 = \frac{1}{c_1},\ p_2 = \frac{1}{c_2},\ q_1 = -\frac{d_1}{c_1(c_1+2td_1)},\ q_2 = -\frac{d_2}{c_2(c_2+2td_2)}. \tag{2.3}
$$

Proposition 2.1. *The Levi-Civita connection $\widetilde{\nabla}$ associated to the Riemannian metric $\widetilde{G}$ from the tangent bundle TM has the form*

$$
\begin{cases}
\widetilde{\nabla}_{X^V} Y^V = Q(X^V, Y^V),\ \widetilde{\nabla}_{X^H} Y^V = (\dot{\nabla}_X Y)^V + P(Y^V, X^H) \\[2ex]
\widetilde{\nabla}_{X^V} Y^H = P(X^V, Y^H),\ \widetilde{\nabla}_{X^H} Y^H = (\dot{\nabla}_X Y)^H + S(X^H, Y^H),
\end{cases}
,\forall X,Y \in \mathcal{T}^1_0(M),
$$

where the M-tensor fields $Q,\ P,\ S$, have the following components with respect to the adapted frame $\{\partial_i, \delta_j\}_{i,j=\overline{1,n}}$:

$$
\begin{aligned}
Q^h_{ij} &= \tfrac{1}{2}(\partial_i \widetilde{G}^{(2)}_{jk} + \partial_j \widetilde{G}^{(2)}_{ik} - \partial_k \widetilde{G}^{(2)}_{ij}) \widetilde{H}^{kh}_{(2)},\\
P^h_{ij} &= \tfrac{1}{2}(\partial_i \widetilde{G}^{(1)}_{jk} + R^l_{0jk} \widetilde{G}^{(2)}_{li}) \widetilde{H}^{kh}_{(1)},\\
S^h_{ij} &= -\tfrac{1}{2}(\partial_k \widetilde{G}^{(2)}_{ij} + R^l_{0ij} \widetilde{G}^{(2)}_{lk}) \widetilde{H}^{kh}_{(2)},
\end{aligned}
\tag{2.4}
$$

R^h_{kij} being the components of the curvature tensor field of the Levi Civita connection $\dot{\nabla}$ from the base manifold (M,g) and $R^h_{0ij} = y^k R^h_{kij}$. The vector fields $Q(X^V, Y^V)$ and $S(X^H, Y^H)$ are vertically valued, while the vector field $P(Y^V, X^H)$ is horizontally valued.

Using the relations (2.4), we may easily prove that the M-tensor fields Q, P, S, have invariant expressions of the forms

$$\begin{aligned}
Q(X^V,Y^V) &= \tfrac{c_2'}{2c_2}[g(y,X)Y^V+g(y,Y)X^V]-\tfrac{c_2'-2d_2}{2(c_2+2td_2)}g(X,Y)y^V\\
&+\tfrac{c_2d_2'-2c_2'd_2}{2c_2(c_2+2td_2)}g(y,X)g(y,Y)y^V,\\
P(X^V,Y^H) &= \tfrac{c_1'}{2c_1}g(y,X)Y^H+\tfrac{d_1}{2c_1}g(y,Y)X^H+\tfrac{d_1}{2(c_1+2td_1)}g(X,Y)y^H\\
&+\tfrac{c_1d_1'-c_1'd_1-d_1^2}{2c_1(c_1+2td_1)}g(y,X)g(y,Y)y^H-\tfrac{c_2}{2c_1}(R(X,y)Y)^H\\
&-\tfrac{c_2d_1}{2c_1(c_1+2td_1)}g(X,R(Y,y)y)y^H,\\
S(X^H,Y^H) &= -\tfrac{d_1}{2c_2}[g(y,X)\ Y^V+g(y,Y)\ X^V]-\tfrac{c_1'}{2(c_2+2td_2)}g(X,Y)\ y^V\\
&-\tfrac{c_2d_1'-2d_1d_2}{2c_2(c_2+2td_2)}g(y,X)g(y,Y)y^V-\tfrac{1}{2}(R(X,Y)y)^V,
\end{aligned}$$

for every vector fields $X,Y\in\mathcal{T}^1_0(M)$ and every tangent vector $y\in TM$.

Since in the following sections we shall work on the subset T_rM of TM consisting of spheres of constant radius r, we shall consider only the tangent vectors y for which the energy density t is equal to $\frac{r^2}{2}$, and the coefficients from the definition (2.2) of the metric $\widetilde{G}$ become constant. So we may consider them constant from the beginning. Then the M-tensor fields involved in the expression of the Levi-Civita connection become simpler:

$$\begin{aligned}
Q(X^V,Y^V) &= \tfrac{d_2}{c_2+r^2d_2}g(X,Y)y^V,\\
P(X^V,Y^H) &= \tfrac{d_1}{2c_1}g(Y,y)X^H+\tfrac{d_1}{2c_1(c_1+r^2d_1)}[c_1g(X,Y)-d_1g(X,y)g(Y,y)]y^H\\
&-\tfrac{c_2}{2c_1}(R(X,y)Y)^H-\tfrac{c_2d_1}{2c_1(c_1+r^2d_1)}g(X,R(Y,y)y)y^H,\\
S(X^H,Y^H) &= -\tfrac{d_1}{2c_2}[g(X,y)Y^V+g(Y,y)X^V]\\
&+\tfrac{d_1d_2}{c_2(c_2+2td_2)}g(y,X)g(y,Y)y^V-\tfrac{1}{2}(R(X,Y)y)^V.
\end{aligned}\tag{2.5}$$

3 Tangent sphere bundles of constant radius r

Let $T_rM=\{y\in TM : g_{\tau(y)}(y,y)=r^2\}$, with $r\in(0,\infty)$, and the projection $\overline{\tau}:T_rM\to M$, $\overline{\tau}=\tau\circ i$, where i is the inclusion map.

The horizontal lift of any vector field on M is tangent to T_rM, but the vertical lift is not always tangent to T_rM. The tangential lift of a vector X to $(p,y)\in T_rM$, used in some recent papers like [4], [11], [12], [18], [25], is defined by

$$X^T_y=X^V_y-\frac{1}{r^2}g_{\tau(y)}(X,y)y^V_y.$$

The tangent bundle to T_rM is spanned by δ_i and $\partial^T_j=\partial_j-\frac{1}{r^2}g_{0j}y^k\partial_k$, $i,j,k=\overline{1,n}$. Remark that the vector fields $\{\partial^T_j\}_{j=\overline{1,n}}$ are not independent; there is the relation $y^j\partial^T_j=0$, which can be directly checked. In any point of T_rM the vectors ∂^T_i; $i=1,\dots n$ span an $(n-1)$-dimensional subspace of TT_rM. The vector field

$$N=y^i\partial_i$$

is normal to T_rM in TM.

Denote by G the metric on T_rM induced from the metric $\widetilde{G}$ defined on TM. If we consider only the tangent vectors $y \in T_rM$, the energy density t defined by them is equal to $\frac{r^2}{2}$, and the coefficients from the definition (2.2) of the metric $\widetilde{G}$ become constants. Thus, the metric G from T_rM has the form

$$(3.1)\quad \begin{cases} G(X^H_y, Y^H_y) = c_1 g_{\tau(y)}(X,Y) + d_1 g_{\tau(y)}(X,y) g_{\tau(y)}(Y,y), \\ G(X^T_y, Y^T_y) = c_2[g_{\tau(y)}(X,Y) - \frac{1}{r^2} g_{\tau(y)}(X,y) g_{\tau(y)}(Y,y)], \\ G(X^H_y, Y^T_y) = G(Y^T_y, X^H_y) = 0, \end{cases}$$

for every vector fields X, Y on M and every tangent vector y, where $c_1,\ d_1,\ c_2$ are constants. The conditions for G to be positive are $c_1 > 0,\ c_2 > 0,\ c_1 + r^2 d_1 > 0$.

Proposition 3.1. *The Levi-Civita connection ∇, associated to the Riemannian metric G on the tangent sphere bundle T_rM of constant radius r has the expression*

$$\begin{cases} \nabla_{\partial^T_i} \partial^T_j = A^h_{ij} \partial^T_h,\ \nabla_{\delta_i} \partial^T_j = \Gamma^h_{ij} \partial^T_h + B^h_{ji} \delta_h, \\ \nabla_{\partial^T_i} \delta_j = B^h_{ij} \delta_h,\ \nabla_{\delta_i} \delta_j = \Gamma^h_{ij} \delta_h + C^h_{ij} \partial^T_h, \end{cases}$$

where the M-tensor fields involved as coefficients have the expressions:

$$\begin{aligned} A^h_{ij} &= -\tfrac{1}{r^2} g_{0j} \delta^h_i, \quad C^h_{ij} = -\tfrac{d_1}{2c_2}(\delta^h_j g_{0i} - \delta^h_i g_{0j}) - \tfrac{1}{2} R^h_{0ij}, \\ (3.2)\ B^h_{ij} &= P^h_{ij} - \tfrac{1}{r^2} g_{i0} P^h_{0j} = \tfrac{d_1}{2c_1} \delta^h_i g_{0j} + \tfrac{d_1}{2(c_1 + r^2 d_1)} (g_{ij} - \tfrac{2c_1 + r^2 d_1}{r^2 c_1} g_{0i} g_{0j}) y^h \\ &\quad - \tfrac{c_2}{2c_1} R^h_{jik} y^k - \tfrac{c_2 d_1}{2c_1(c_1 + r^2 d_1)} R_{ikjl} y^h y^k y^l, \end{aligned}$$

where $g_{0j} = g_{j0} = y^i g_{ij}$ and $P^h_{0j} = y^i P^h_{ij}$.

We mention that A and C have these quite simple expressions, since they are the coefficients of the tangential part of the Levi-Civita connection. All the terms containing $y^h \partial^T_h = 0$ have been canceled.

The invariant form of the Levi-Civita connection from T_rM is

$$\begin{cases} \nabla_{X^T} Y^T = -\frac{1}{r^2} g(Y,y) X^T, \\ \nabla_{X^T} Y^H = \frac{d_1}{2c_1} g(Y,y)\ X^H + \frac{d_1}{2(c_1 + d_1 r^2)} g(X,Y) y^H \\ -\frac{d_1(2c_1 + r^2 d_1)}{2r^2 c_1(c_1 + r^2 d_1)} g(X,y) g(Y,y) y^H - \frac{c_2}{2c_1}(R(X,y)Y)^H - \frac{c_2 d_1}{2c_1(c_1 + r^2 d_1)} g(X, R(Y,y)y) y^H, \\ \nabla_{X^H} Y^T = (\dot\nabla_X Y)^T + \frac{d_1}{2c_1} g(X,y)\ Y^H + \frac{d_1}{2(c_1 + r^2 d_1)} g(X,Y) y^H \\ -\frac{d_1(2c_1 + r^2 d_1)}{2r^2 c_1(c_1 + r^2 d_1)} g(X,y) g(Y,y) y^H - \frac{c_2}{2c_1}(R(Y,y)X)^H - \frac{c_2 d_1}{2c_1(c_1 + r^2 d_1)} g(Y, R(X,y)y) y^H, \\ \nabla_{X^H} Y^H = (\dot\nabla_X Y)^H - \frac{d_1}{2c_2}[g(X,y)\ Y^T - g(Y,y)\ X^T] - \frac{1}{2}(R(X,Y)y)^T. \end{cases}$$

The covariant derivatives $\dot\nabla_k Q^h_{ij},\ \dot\nabla_k P^h_{ij},\ \dot\nabla_k S^h_{ij}$ of the M-tensor fields Q, P, S are defined by some formulas similar to those from the classical cases. The covariant derivative of P is given by

$$\dot\nabla_k P^h_{ij} = \delta_k P^h_{ij} + \Gamma^h_{kl} P^l_{ij} - \Gamma^l_{ki} P^h_{lj} - \Gamma^l_{kj} P^h_{il}.$$

Remark that instead of the partial derivative $\frac{\partial}{\partial x^k}$ we use the operator $\delta_k = \frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - \Gamma^h_{k0}\frac{\partial}{\partial y^k}$. Similar expressions are obtained for $\dot\nabla_k Q^h_{ij}$ and for $\dot\nabla_k S^h_{ij}$. It can be proved by a straightforward computation that $\dot\nabla_k Q^h_{ij},\ \dot\nabla_k P^h_{ij},\ \dot\nabla_k S^h_{ij}$ are M-tensors of type $(1,3)$. The expression of the covariant derivative of an arbitrary M-tensor can be obtained easily. It follows that $\dot\nabla_k y^i = 0,\ \dot\nabla_k g_{ij} = 0$. In the case where Q, P, S are given by (2.5), their covariant derivatives are:

$$\dot\nabla_k Q^h_{ij} = 0,\ \dot\nabla_k S^h_{ij} = -\tfrac{1}{2}\dot\nabla_k R^h_{lij} y^l$$
$$\dot\nabla_k P^h_{ij} = -\tfrac{c_2}{2c_1}\dot\nabla_k R^h_{jil} y^l - \tfrac{c_2 d_1}{2c_1(c_1+r^2 d_1)}\dot\nabla_k R_{iljr} y^l y^r y^h,$$

where $\dot\nabla_k R^h_{kij},\ \dot\nabla_k R_{hkij}$ are the usual local coordinate expressions of the covariant derivatives of the curvature tensor field and the Riemann-Christoffel tensor field of the Levi Civita connection $\dot\nabla$ on the base manifold M. If the base manifold (M, g) is locally symmetric then, obviously, $\dot\nabla_k Q^h_{ij} = 0,\ \dot\nabla_k P^h_{ij} = 0,\ \dot\nabla_k S^h_{ij} = 0$.

In a similar way we obtain

$$\dot\nabla_k A^h_{ij} = 0,\quad \dot\nabla_k C^h_{ij} = -\tfrac{1}{2}\dot\nabla_k R^h_{lij} y^l,$$
$$\dot\nabla_k B^h_{ij} = -\tfrac{c_2}{2c_1}\dot\nabla_k R^h_{jil} y^l - \tfrac{c_2 d_1}{2c_1(c_1+r^2 d_1)}\dot\nabla_k R_{iljr} y^l y^r y^h,$$

The curvature tensor field K of the connection ∇ from $T_r M$ is defined by the formula

$$K(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,\quad X, Y, Z \in \mathcal{T}^1_0(T_r M).$$

We obtain by a standard straightforward computation the horizontal and tangential components of the curvature tensor field K

$$K(\delta_i,\delta_j)\delta_k = HHHH^h_{kij}\delta_h + HHHT^h_{kij}\partial^T_h,$$
$$K(\delta_i,\delta_j)\partial^T_k = HHTH^h_{kij}\delta_h + HHTT^h_{kij}\partial^T_h,$$
$$K(\partial^T_i,\partial^T_j)\delta_k = TTHH^h_{kij}\delta_h,\ K(\partial^T_i,\partial^T_j)\partial^T_k = TTTT^h_{kij}\partial^T_h,$$
$$K(\partial^T_i,\delta_j)\delta_k = THHH^h_{kij}\delta_h + THHT^h_{kij}\partial^T_h,\ K(\partial^T_i,\delta_j)\partial^T_k = THTH^h_{kij}\delta_h,$$

where the non-zero M-tensor fields which appear as coefficients are given by

$$HHHH^h_{kij} = R^h_{kij} + B^h_{li}C^l_{jk} - B^h_{lj}C^l_{ik} + B^h_{lk}R^l_{0ij},$$
$$HHHT^h_{kij} = \tfrac{1}{2}(\dot\nabla_j R^h_{lik} y^l - \dot\nabla_i R^h_{ljk} y^l),$$
$$HHTH^h_{kij} = \tfrac{c_2}{2c_1}(\dot\nabla_j R^h_{ikl} - \dot\nabla_i R^h_{jkl}) y^l + \tfrac{c_2 d_1}{2c_1(c_1+r^2 d_1)}(\dot\nabla_j R_{ilkm} - \dot\nabla_i R_{jlkm}) y^l y^m y^h,$$
$$HHTT^h_{kij}\partial^T_h = (R^h_{kij} + C^h_{il}B^l_{kj} - C^h_{jl}B^l_{ki} + A^h_{lk}R^l_{0ij})\partial^T_h,$$
$$TTHH^h_{kij} = \partial^T_i B^h_{jk} - \partial^T_j B^h_{ik} + B^h_{il}B^l_{jk} - B^h_{jl}B^l_{ik} - \tfrac{1}{r^2}(g_{i0}B^h_{jk} - g_{j0}B^h_{ik}),$$
$$TTTT^h_{kij} = \tfrac{1}{r^2}(g_{jk}\delta^h_i - g_{ik}\delta^h_j) + \tfrac{1}{r^4}(\delta^h_j g_{0i}g_{0k} - \delta^h_i g_{0j}g_{0k}),$$
$$THHH^h_{kij} = \tfrac{c_2}{2c_1}\dot\nabla_j R^h_{kil} y^l + \tfrac{c_2 d_1}{2c_1(c_1+r^2 d_1)}\dot\nabla_j R_{ilkr} y^l y^r y^h,$$
$$THHT^h_{kij}\partial^T_h = (\partial^T_i C^h_{jk} + A^h_{il}C^l_{jk} - C^h_{jl}B^l_{ik} - \dot\nabla_j R^h_{lik} y^l)\partial^T_h,$$
$$THTH^h_{kij} = \partial^T_i B^h_{kj} + B^l_{kj}B^h_{il} - A^l_{ik}B^h_{lj},\ THTT^h_{kij} = -\dot\nabla_j A^h_{ik} = 0.$$

We mention that from the final values of $HHTT^h_{kij}\partial^T_h$ and $THHT^h_{kij}\partial^T_h$, obtained by replacing the expressions (3.2) of $A^h_{ij}, B^h_{ij}, C^h_{ij}$, we shall eliminate the terms containing $y^h\partial^T_h$, because they vanish.

We want to find the manifolds (T_rM, G) of constant sectional curvature k, i.e the manifolds for which the curvature tensor field K satisfies the relation

$$K(\overline{X},\overline{Y})\overline{Z} - k[G(\overline{Y},\overline{Z})\overline{X} - G(\overline{X},\overline{Z})\overline{Y}] = 0, \forall \overline{X},\overline{Y},\overline{Z} \in \mathcal{T}^1_0(T_rM). \tag{3.3}$$

Let us consider $\overline{X},\overline{Y},\overline{Z} \in \mathcal{T}^1_0(T_rM)$ as being the tangential lifts of $X, Y, Z \in \mathcal{T}^1_0(M)$. We have to study the vanishing conditions for the difference

$$K(X^T, Y^T)Z^T - k[G(Y^T, Z^T)X^T - G(X^T, Z^T)Y^T] = 0, \forall X, Y, Z \in \mathcal{T}^1_0(M),$$

which has the detailed expression

$$\begin{aligned} &\tfrac{1}{r^2}[g(Y,Z)X^T - g(X,Z)Y^T] + \tfrac{1}{r^4}g(y,Z)[g(y,X)Y^T - g(y,Y)X^T] \\ &\qquad -k[G(Y^T, Z^T)X^T - G(X^T, Z^T)Y^T] = 0, \end{aligned} \tag{3.4}$$

for all the vector fields X, Y, Z and for every vector y tangent to M.

The local coordinate form of (3.4) is

$$\frac{kr^2c_2 - 1}{r^2}\left[g_{jk}\delta^h_i - g_{ik}\delta^h_j - \frac{1}{r^2}(\delta^h_j g_{0i}g_{0k} - \delta^h_i g_{0j}g_{0k})\right] = 0,$$

from which we obtain a first necessary condition for the manifold (T_rM, G) to have constant sectional curvature k:

$$c_2 = \frac{1}{kr^2}. \tag{3.5}$$

If in the relation (3.3) we take instead of $\overline{X},\overline{Y},\overline{Z} \in \mathcal{T}^1_0(T_rM)$ the horizontal lifts of $X, Y, Z \in \mathcal{T}^1_0(M)$, we obtain that the following vanishing condition must be satisfied:

$$K(X^H, Y^H)Z^H - k[G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H] = 0, \tag{3.6}$$

for every tensor fields $X, Y, Z \in \mathcal{T}^1_0(M)$.

Since

$$\begin{aligned} &K(X^H, Y^H)Z^H = (R(X,Y)Z)^H - \tfrac{d_1^2}{4c_1c_2}[g(y,X)Y^H - g(y,Y)X^H]g(y,Z) \\ &+\tfrac{d1^2}{4c_2(c_1+r^2d_1)}[g(Y,Z)g(y,X) - g(X,Z)g(y,Y)]y^H + \tfrac{d1}{4c_1}[(R(y,Y)Z)^H g(y,X) \\ &-(R(y,X)Z)^H g(y,Y) + (R(X,Y)y)^H g(y,Z)] + \tfrac{3d_1}{4(c_1+r^2d_1)}g(X, R(Z,y)Y)y^H \\ &-\tfrac{d_1^2}{4c_1(c_1+r^2d_1)}[g(Y, R(Z,y)y)g(X,y) - g(X, R(Z,y)y)g(Y,y)]y^H \\ &+\tfrac{c_2}{4c_1}[R(R(Y,Z)y,y)X - R(R(X,Z)y,y)Y]^H - \tfrac{c_2}{2c_1}[R(R(X,Y)y,y)Z]^H \\ &+\tfrac{c_2d_1}{4c_1(c_1+r^2d_1)}[g(X, R(R(Y,Z)y,y)y) - g(X, R(R(Y,y)y,y)Z) \\ &-2g(X, R(R(Z,y)y,y)Y)]y^H + \tfrac{1}{2}(\dot{\nabla}_Z R(X,Y)y)^T, \end{aligned}$$

the coefficient of the horizontal part of the difference (3.6) has the following local coordinate expression:

$$
\begin{aligned}
&HHHH^h_{kij} - k[(c_1 g_{jk} + d_1 g_{0j} g_{0k})\delta^h_i - (c_1 g_{ik} + d_1 g_{0i} g_{0k})\delta^h_j] = \\
&= R^h_{kij} + c_1 k(g_{ik}\delta^h_j - g_{jk}\delta^h_i) + \tfrac{d_1(4c_1c_2k - d_1)}{4c_1c_2}(\delta^h_j g_{0i} g_{0k} - \delta^h_i g_{0j} g_{0k}) \\
&+ \tfrac{d_1^2}{4c_2(c_1+r^2d_1)}(g_{jk}g_{0i} - g_{ik}g_{0j})y^h + \tfrac{d_1}{4c_1}(R^h_{k0j}g_{0i} - R^h_{k0i}g_{0j} + R^h_{0ij}g_{0k}) \\
&+ \tfrac{3d_1}{4(c_1+r^2d_1)}R_{ijk0}y^h - \tfrac{d_1^2}{4c_1(c_1+r^2d_1)}(R_{j0k0}g_{0i} - R_{i0k0}g_{0j})y^h \\
&+ \tfrac{c_2}{4c_1}(R^l_{0jk}R^h_{il0} - R^l_{0ik}R^h_{jl0} - 2R^l_{0ij}R^h_{kl0}) \\
&+ \tfrac{c_2 d_1}{4c_1(c_1+r^2d_1)}(R^l_{0i0}R_{jkl0} - R^l_{0ik}R_{j0l0} - 2R^l_{0ij}R_{k0l0})y^h.
\end{aligned} \tag{3.7}
$$

For the indices h, i, j, k fixed, the condition for $HHHH^h_{kij} - k[(c_1 g_{jk} + d_1 g_{0j} g_{0k})\delta^h_i - (c_1 g_{ik} + d_1 g_{0i} g_{0k})\delta^h_j]$ to be zero leads to an equation of the type $A + A_{l_1l_2}y^{l_1}y^{l_2} + A_{l_1l_2l_3l_4}y^{l_1}y^{l_2}y^{l_3}y^{l_4} = 0$, in the tangential coordinates y^i; $i = \overline{1,n}$, where the coefficients A's are obtained from the above expression by full symmetrization. Differentiating the obtained expression four times, it follows that $A_{l_1l_2l_3l_4} = 0$. Then differentiating the remaining expression $A + A_{l_1l_2}y^{l_1}y^{l_2}$ two times, it follows $A_{l_1l_2} = 0$. We conclude that the constant term A is zero, whence the base manifold must be a space form, with the curvature of the form:

$$R^h_{kij} = c_1 k(g_{jk}\delta^h_i - g_{ik}\delta^h_j). \tag{3.8}$$

Replacing this expression and the value (3.5) of c_2 into (3.7), we obtain the following vanishing condition:

$$\frac{k(3c_1 - r^2d_1)}{4r^2}\Big[\frac{c_1 + r^2d_1}{c_1}(\delta^h_j g_{0i} - \delta^h_i g_{0j})g_{0k} - (g_{jk}g_{0i} - g_{ik}g_{0j})y^h\Big] = 0,$$

which is satisfied if and only if c_1 has the form

$$c_1 = \frac{r^2 d_1}{3}. \tag{3.9}$$

The same values (3.5) of c_2 and (3.9) of c_1 lead to the identity $TTHH^h_{kij} = 0$, which is another necessary condition for (T_rM, G) to be a space form. More precisely, after imposing the expression (3.8) for the curvature of the base manifold, and the value (3.5) of c_2 into the expression of $TTHH^h_{kij}$, this becomes

$$TTHH^h_{kij} = \frac{3c_1 - r^2d_1}{4c_1r^2}[g_{jk}\delta^h_i - g_{ik}\delta^h_j + (\delta^h_j g_{0i} - \delta^h_i g_{0j})g_{0k} - (g_{jk}g_{0i} - g_{ik}g_{0j})y^h],$$

which vanishes if and only if c_1 takes the value (3.9).

Due to the condition for the base manifold to be a space form, the components $HHHT^h_{kij}, HHTH^h_{kij}, THHH^h_{kij}$ of the curvature tensor field of (T_r, G) and the corresponding components of the difference from (3.3) become zero. Since $HHTT^h_{kij}$ is also vanishing when we substitute the expression of the curvature (3.8) and the values (3.9) for c_1 and (3.5) for c_2, we have immediately that

$$K(X^H, Y^H)Z^T = k[G(Y^H, Z^T)X^H - G(X^H, Z^T)Y^H], \ \forall X, Y, Z \in \mathcal{T}^1_0(M).$$

The identity

$$K(X^T,Y^H)Z^H=k[G(Y^H,Z^H)X^T-G(X^T,Z^H)Y^H],\ \forall X,Y,Z\in\mathcal{T}^1_0(M),$$

reduces, after imposing the conditions (3.8), (3.9), (3.5), to the relation

$$\begin{aligned}&-\tfrac{d_1kr^2}{3}[g(Y,Z)X^T+g(X,Z)Y^T+g(X,Y)Z^T\\&-g(y,X)g(y,Y)Z^T-g(y,X)g(y,Z)Y^T-g(y,Y)g(y,Z)X^T]=0,\end{aligned}$$

which is true for every vector fields $X,Y,Z\in\mathcal{T}^1_0(M)$, and for every tangent vector $y\in TM$, if and only if $d_1=0$ or $k=0$ (i.e the manifold (T_r,G) is flat).

Finally, we have that the manifold (T_rM,G) may never be a space form, since the relation

$$K(X^T,Y^H)Z^T=k[G(Y^H,Z^T)X^T-G(X^T,Z^T)Y^H],\ \forall X,Y,Z\in\mathcal{T}^1_0(M),$$

which has the final form

$$\begin{aligned}&\tfrac{1}{r^2}[g(Y,Z)X^T+g(X,Z)Y^T+g(X,Y)Z^T]\\&-\tfrac{1}{r^4}\{g(y,X)g(y,Y)Z^T+g(y,X)g(y,Z)Y^T+g(y,Y)g(y,Z)X^T\\&+[g(Y,Z)g(y,X)+g(X,Z)g(y,Y)+g(X,Y)g(y,Z)\\&-\tfrac{3}{r^2}g(y,X)g(y,Y)g(y,Z)]y^h\}=0,\end{aligned}\tag{3.10}$$

may never be satisfied. Hence we may state

Theorem 3.2. *The tangent sphere bundle T_rM, with the Riemannian metric G induced from the metric $\widetilde{G}$ of diagonal lift type on the tangent bundle TM, has never constant sectional curvature.*

Corollary 3.3. *The tangent sphere bundle T_rM, endowed with the metric induced by the Sasaki metric g^s from the tangent bundle TM is never a space form.*

In fact, the Sasaki metric can be obtained as a particular case of our natural diagonal lifted metric, with the coefficients $c_1=c_2=1,\ d_1=0$.

4 Einstein tangent sphere bundles

We shall find the conditions under which the manifold (T_rM,G), with G given by the relations (3.1), is Einstein. To this aim, we shall compute the Ricci tensor of the manifold (T_rM,G).

First, let us remark some facts concerning the obtaining of the Ricci tensor field for the tangent bundle TM. We have the well known formula

$$Ric(Y,Z)=trace(X\to K(X,Y)Z),$$

where X,Y,Z are vector fields on TM. Then we get easily the components of the Ricci tensor field on TM

$$\widetilde{Ric}HH_{jk}=\widetilde{Ric}(\delta_j,\delta_k)=HHHH^h_{khj}+VHHV^h_{khj},$$

$$\widetilde{Ric}VV_{jk}=\widetilde{Ric}(\partial_j,\partial_k)=VVVV^h_{khj}-VHVH^h_{kjh},$$

where the components $VVVV^h_{kij}, VHVH^h_{kij}, VHHV^h_{kij}$, are obtained from the curvature tensor field on TM in a similar way as the components $TTTT^h_{kij}, THTH^h_{kij}$, $THHT^h_{kij}$ are obtained from the curvature tensor field on T_rM. In the expression of $\widetilde{Ric}HV_{jk} = \widetilde{Ric}(\delta_j, \partial_k) = \widetilde{Ric}VH_{jk} = \widetilde{Ric}(\partial_j, \delta_k)$ there are involved the covariant derivatives of the curvature tensor field R. If (M, g) is locally symmetric then $RicHV_{jk} = RicVH_{jk} = 0$. In particular we have $RicHV_{jk} = RicVH_{jk} = 0$ in the case where (M, g) has constant sectional curvature.

Equivalently, we may use an orthonormal frame $(E_1, \ldots, E_{2n})$ on TM and we may use the formula

$$\widetilde{Ric}(Y, Z) = \sum_{i=1}^{2n} G(E_i, K(E_i, Y)Z).$$

We may choose the orthonormal frame $(E_1, \ldots, E_{2n})$ such that the first n vectors $E_1, \ldots, E_n$ are the vectors of a (orthonormal) frame in HTM and the last n vectors $E_{n+1}, \ldots E_{2n}$ are the vectors of a (orthonormal) frame in VTM. Moreover, we may assume that the last vector E_{2n} is the unitary vector of the normal vector $N = y^i\partial_i$ to T_rM.

The components of the Ricci tensor field of T_rM can be obtained in a similar way by using the above traces. However the vector fields $\partial^T_1, \ldots, \partial^T_n$ are not independent. On the open set from T_rM, where $y^n \neq 0$ we can consider the basis $\delta_1, ..., \delta_n, \partial^T_1, ..., \partial^T_{n-1}$ for TT_rM. The last vector ∂^T_n is expressed as

$$\partial^T_n = -\frac{1}{y^n}\sum_{i=1}^{n-1} y^i \partial^T_i.$$

Remark that the basis $\delta_1, ..., \delta_n, \partial^T_1, ..., \partial^T_{n-1}$ can be completed with the normal vector $N = y^V = y^h\partial_h$.

For the components $HHHH^h_{kij}$ and $HTTH^h_{kij}$ the traces can pe computed easily, just like in the case of TM. The components for which we must compute more carefully the traces are $TTTT^h_{kij}$, and $THHT^h_{kij}$. We have:

$$\begin{aligned} THHT^h_{kij}\partial^T_h &= \textstyle\sum_{l=1}^{n-1} THHT^l_{kij}\partial^T_l + THHT^n_{kij}\partial^T_n \\ &= \textstyle\sum_{l=1}^{n-1} THHT^l_{kij}\partial^T_l - THHT^n_{kij}\frac{1}{y^n}\sum_{l=1}^{n-1} y^l\partial^T_l \\ &= THHT^h_{kij}\partial^T_h - THHT^n_{kij}\frac{1}{y^n}y^h\partial^T_h. \end{aligned}$$

Thus the trace involved in the definition of $RicHH$ on T_rM is

$$THHT^h_{khj} - \frac{1}{y^n}y^h THHT^n_{khj}.$$

A short computation made by using the above expression of of $THHT^h_{kij}$ gives

$$y^i THHT^h_{kij} = \frac{d_1(4c_1 + r^2 d_1)}{4c_1c_2r^2} g_{0j}g_{0k}y^h,$$

thus we get

$$\frac{1}{y^n}y^i THHT^n_{kij} = \frac{d_1(4c_1 + r^2 d_1)}{4c_1c_2r^2} g_{0j}g_{0k}.$$

It follows that

$$RicHH_{jk} = HHHH^h_{khj} + THHT^h_{khj} - \frac{d_1(4c_1 + r^2 d_1)}{4c_1 c_2 r^2} g_{0j} g_{0k}.$$

In a similar way we get that the trace involved in the definition of $RicTT$ on T_rM is

$$TTTT^h_{khj} - \frac{1}{y^n} y^h TTTT^n_{khj}.$$

Then

$$y^h TTTT^n_{khj} = \frac{1}{r^2} g_{jk} y^n - \frac{1}{r^4} g_{0j} g_{0k} y^n.$$

Hence

$$RicTT_{jk} = Ric(\partial^T_j, \partial^T_k) = HTTH^h_{khj} + TTTT^h_{khj} - \frac{1}{r^2} g_{jk} + \frac{1}{r^4} g_{0j} g_{0k}.$$

After the computations we obtain the detailed expressions of $RicHH_{jk}$ and $RicTT_{jk}$:

$$\begin{aligned} RicHH_{jk} = \quad & Ric_{jk} - \tfrac{d_1(2c_1+r^2d_1)}{2c_2(c_1+r^2d_1)} g_{jk} + \tfrac{d_1(2c_1+r^2d_1)[c_1n+r^2d_1(n-1)]}{2r^2c_1c_2(c_1+r^2d_1)} g_{0j}g_{0k} \\ & -\tfrac{c_2}{2c_1} R^h{}_{0kl} R^l{}_{jh0} + \tfrac{c_2 d_1}{2c_1(c_1+r^2d_1)} R^h{}_{0k0} R_{h0j0}, \\ RicTT_{jk} = \quad & \tfrac{r^4d_1^2 - 2c_1(c_1+r^2d_1)(n-2)}{2c_1r^2(c_1+d_1r^2)} (\tfrac{1}{r^2} g_{0j}g_{0k} - g_{jk}) + \tfrac{c_2^2}{4c_1^2} R_{hik0} R^{hi}{}_{j0} \\ & -\tfrac{c_2^2 d_1}{2c_1^2(c_1+r^2d_1)} R^h{}_{0j0} R_{h0k0}. \end{aligned}$$

The components $R^h{}_{0kl}, R^l{}_{jh0}, R^h{}_{0k0}, R_{h0j0}$, etc. are obtained as usual from the components of the curvature tensor field R of $\dot\nabla$ by transvecting with y's. E.g. $R^h{}_{0kl} = R^h{}_{ikl} y^i$, $R^l{}_{jh0} = R^l{}_{jhi} y^i$, $R^h{}_{0k0} = R^h{}_{ikj} y^i y^j$, $R_{h0j0} = R_{hijk} y^i y^k$. Recall that $R_{hijk} = g_{hl} R^l{}_{ijk}$ are the components of the Riemann-Christoffel tensor field of $\dot\nabla$. Taking the above relations into account, we obtain that the differences between the components of the Ricci tensor and the corresponding components of the metric G multiplied by a constant ρ have the following forms:

$$\begin{aligned} RicHH_{jk} - \rho G^{(1)}_{jk} &= Ric_{jk} - \tfrac{d_1(2c_1+r^2d_1)+2\rho c_1c_2(c_1+r^2d_1)}{2c_2(c_1+r^2d_1)} g_{jk} \\ &+ \tfrac{d_1(2c_1+r^2d_1)[c_1n+r^2d_1(n-1)] - 2\rho r^2 c_1c_2d_1(c_1+r^2d_1)}{2c_1c_2r^2(c_1+r^2d_1)} g_{0j}g_{0k} \\ &+ \tfrac{c_2}{4c_1}(R^l_{kh0}R^h_{0jl} + R^l_{jh0}R^h_{0kl}) + \tfrac{c_2d_1}{2c_1(c_1+r^2d_1)} R_{h0j0} R^h_{0k0}, \\ RicTT_{jk} - \rho G^{(2)}_{jk} &= \tfrac{c_2^2}{4c_1^2} R_{hik0} R^{hi}_{j0} - \tfrac{c_2^2 d_1}{2c_1^2(c_1+r^2d_1)} R_{h0j0} R^h_{0k0} \\ &+ \tfrac{r^4d_1^2 + 2c_1(c_1+r^2d_1)(n-2) + \rho 2c_1c_2r^2)}{2r^2c_1(c_1+r^2d_1)} (\tfrac{1}{r^2} g_{0j}g_{0k} - g_{jk}). \end{aligned}$$

When the base manifold (M, g) has constant sectional curvature c, i.e. when

$$R^h_{kij} = c(g_{jk}\delta^h_i - g_{ik}\delta^h_j), \tag{4.1}$$

the above differences take the forms

$$RicHH_{jk} - \rho G^{(1)}_{jk} = \tfrac{cc_2[2(n-1)(c_1+r^2d_1)-cc_2r^2]-d_1(2c_1+r^2d_1)-2\rho c_1c_2(c_1+r^2d_1)}{2c_2(c_1+d_1r^2)} g_{jk}$$
$$+\tfrac{d_1[2c_1^2n+r^4(d_1^2-c^2c_2^2)(n-1)]+r^2c_1[d_1^2(3n-2)-c^2c_2^2(n-2)]-2\rho r^2c_1c_2d_1(c_1+r^2d_1)}{2r^2c_1c_2(c_1+d_1r^2)} g_{0j}g_{0k},$$
$$RicTT_{jk} - \rho G^{(2)}_{jk} = \tfrac{r^4(d_1^2-c^2c_2^2)+2c_1(c_1+r^2d_1)(2-n+\rho c_2r^2)}{2r^2c_1(c_1+r^2d_1)} \left(\tfrac{1}{r^2}g_{0j}g_{0k} - g_{jk}\right)$$

The difference $RicTT_{jk} - \rho G^{(2)}_{jk}$ vanishes if and only if the constant ρ has the value

$$\rho = \tfrac{2c_1^2(n-2)+r^2[2c_1d_1(n-2)+r^2(c^2c_2^2-d_1^2)]}{2r^2c_1c_2(c_1+r^2d_1)}.$$

After replacing this expression, the difference $RicHH_{jk} - \rho G^{(1)}_{jk}$ becomes

$$\begin{aligned} RicHH_{jk} - \rho G^{(1)}_{jk} &= \tfrac{2(cc_2r^2-c_1)}{r^2}\{c_1(n-2) - r^2[cc_2 - d_1(n-1)]\}g_{jk} \\ &+\{4c_1^2d_1 + r^2c_1[d_1^2(n+2) - c^2c_2^2(n-2)] + r^4d_1(d_1^2 - c^2c_2^2)n\}g_{0j}g_{0k}. \end{aligned} \tag{4.2}$$

By doing a detailed analysis of all the situations in which the difference expressed by (4.2) vanishes, we may prove the following theorem.

Theorem 4.1. *The tangent sphere bundle T_rM of an n-dimensional Riemannian (M, g) of constant sectional curvature c is Einstein with respect to the metric G induced from the natural diagonal lifted metric $\widetilde{G}$ defined on TM, i.e. it exists a real constant ρ such that*

$$Ric(X, Y) = \rho G(X, Y),\ \forall X, Y \in \mathcal{T}^1_0(T_rM),$$

if and only if

$$c_1 = \frac{r^2d_1n}{n-2},\ c_2 = \frac{d_1n}{c(n-2)},\ \rho = \frac{c(n-1)^2(n-2)}{r^2d_1n^2}.$$

Proof. The difference expressed by (4.2) vanishes if and only if the both coefficients are equal to zero. The vanishing condition for the coefficient of g_{jk} leads to two cases which must be studied:

$$I)\quad c_1 = cc_2r^2,\ II)\ c_1 = \frac{r^2[cc_2 - d_1(n-1)]}{n-2}.$$

In the case *I*, the coefficient of $g_{0j}g_{0k}$ from (4.2) becomes

$$-(cc_2 + d_1)^2[cc_2(n-2) - d_1n]r^4 = 0,$$

so we have two possible values for c_2:

$$c_2 = -\frac{d_1}{c},\ \text{or}\ c_2 = \frac{d_1n}{c(n-2)}. \tag{4.3}$$

Replacing the first one into the expression of c_1, we obtain that $c_1 + r^2d_1 = 0$, thus the constant ρ, some components of K as well as the Levi Civita connection are not defined. So this situation should be excluded.

When c_2 takes the second value from (4.3), c_1 and ρ have the forms presented in the case I from the theorem.

If c_1 has the expression from case II the coefficient of $g_{0j}g_{0k}$ given in (4.2) decomposes as

$$-\frac{r^4(cc_2-d_1)\{c^2c_2^2(n-2)^2+d_1[d_1n^2+2cc_2(n^2-4n+2)]\}}{(n-2)^2}=0. \tag{4.4}$$

The vanishing condition for the first factor and the value of c_1 from case II lead us again to the situation where $c_1+r^2d_1=0$, so it should be excluded.

The second factor which appears in (4.4) is a second degree function of c_2. The discriminant of the attached equation is

$$\Delta=16c^2d_1^2(1-n)(n^2-3n+1).$$

Since the dimension n of the base manifold is a natural number bigger then one, Δ is positive only for $n=2$, value which makes vanish the denominator of the expression (4.4), so the case must be treated separately, starting with the expressions of the differences $RicHH_{jk}-\rho G^{(1)}_{jk}$ and $RicTT_{jk}-\rho G^{(2)}_{jk}$, which become of the forms

$$\begin{aligned}RicHH_{jk}-\rho G^{(1)}_{jk} = {} & \tfrac{cc_2[2(c_1+r^2d_1)-cc_2r^2]-d_1(2c_1+r^2d_1)-2\rho c_1c_2(c_1+r^2d_1)}{2c_2(c_1+d_1r^2)}g_{jk}\\ & +\tfrac{d_1[4c_1^2+r^4(d_1^2-c^2c_2^2)]+4r^2c_1d_1^2-2\rho r^2c_1c_2d_1(c_1+r^2d_1)}{2r^2c_1c_2(c_1+d_1r^2)}g_{0j}g_{0k},\\ RicTT_{jk}-\rho G^{(2)}_{jk} = {} & \frac{r^4(d_1^2-c^2c_2^2)+2\rho r^2c_1c_2(c_1+r^2d_1)}{2r^2c_1(c_1+r^2d_1)}\left(\frac{1}{r^2}g_{0j}g_{0k}-g_{jk}\right).\end{aligned}$$

From these relations we get

$$\rho=\frac{r^2(c^2c_2^2-d_1^2)}{2c_1c_2(c_1+r^2d_1)},$$

and then

$$\begin{aligned}RicHH_{jk}-\rho G^{(1)}_{jk} = {} & 2(cc_2r^2-c_1)(d_1-cc_2)g_{jk}\\ & +2d_1[2c_1(c_1+r^2d_1)+r^4(d_1^2-c^2c_2^2)]g_{0j}g_{0k}.\end{aligned}$$

This last difference vanishes if and only if $c_1=r^2cc_2$ and $d_1=-cc_2$, or $d_1=cc_2$ and $c_1=-r^2cc_2$, but the both cases must be excluded, since they lead to $c_1+r^2d_1=0$. Thus the theorem is proved. □

Corollary 4.2. *The tangent sphere bundle T_rM of an n-dimensional Riemannian space form (M,g) may never be Einstein with respect to the metric induced by the Sasaki metric g^s from TM.*

In fact, in this case, the coefficients should be $c_1=c_2=1,\ d_1=0$, and they do not satisfy the condition from theorem 4.1.

References

[1] M. T. K. Abbassi and O. Kowalski, *On g-natural metrics with constant scalar curvature on unit tangent sphere bundles*, Topics in Almost Hermitian Geometry and Related Fields, World Scientific 2005, 1-29.

[2] C. L. Bejan, V. Oproiu, *Tangent bundles of quasi-constant holomorphic sectional curvatures*, Balkan J. Geom. Appl., 11, 1 (2006), 11-22.

[3] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[4] E. Boeckx, *When are the tangent sphere bundles of a Riemannian manifold reductible?*, Trans. Amer. Math. Soc., 355 (2003), 2885-2903.

[5] E. Boeckx, L. Vanhecke, *Unit tangent sphere bundles with constant scalar curvature*, Czechoslovak. Math. J., 51 (2001), 523-544.

[6] G. Calvaruso, D. Perrone, *H-contact unit tangent sphere bundles*, Rocky Mountain J. Math., 37 (2000), 207-219.

[7] C. Eni, *A pseudo-Riemannian metric on the tangent bundle of a Riemannian manifold*, Balkan J. Geom. Appl., 13, 2 (2008), 35-42.

[8] J. Janyška, *Natural vector fields and 2-vector fields on the tangent bundle of a pseudo-Riemannan manifold*, Archivum Mathematicum (Brno), 37 (2001), 143-160.

[9] I. Kolář, P. Michor, J. Slovak, *Natural Operations in Differential Geometry*, Springer Verlag, Berlin, 1993, 1999.

[10] O. Kowalski and M. Sekizawa, *Natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles - a classification,* Bull. Tokyo Gakugei Univ. (4), 40 (1988), 1-29.

[11] O. Kowalski and M. Sekizawa, *On tangent sphere bundles with small and large constant radius,* Ann. Global. Anal. Geom. 18 (2000), 207-219.

[12] O. Kowalski and M. Sekizawa, *On the scalar curvature of tangent sphere bundles with arbitrary constant radius,* Bull. Greek Math. Soc. 44 (2000), 17-30.

[13] D. Krupka, J. Janyška, *Lectures on Differential Invariants,* Folia Fac. Sci. Nat. Univ. Purkinyanae Brunensis, 1990.

[14] M. L. Labbi, *On two natural Riemannian metrics on a tube,* Balkan J. Geom. Appl. 12, 2 (2007), 81-86.

[15] K. P. Mok, E. M. Patterson and Y.C. Wong, *Structure of symmetric tensors of type (0,2) and tensors of type (1,1) on the tangent bundle*, Trans. Amer. Math. Soc. 234 (1977), 253-278.

[16] M. I. Munteanu, *New CR-structures on the unit tangent bundle,* An. Univ. Timisoara Ser. Mat.-Inform. 38 (2000), no. 1, 99–110.

[17] M. I. Munteanu, *CR-structures on the translated sphere bundle,* Proceedings of the Third International Workshop on Differential Geometry and its Applications and the First German-Romanian Seminar on Geometry (Sibiu, 1997). Gen. Math. 5 (1997), 277–280.

[18] M. I. Munteanu, *Some aspects on the geometry of the tangent bundles and tangent sphere bundles of a Riemannian manifold*, Mediterranean Journal of Mathematics, 5 (2008), 1, 43-59.

[19] P. T. Nagy, *On the Tangent Sphere Bundle of Riemannian 2-Manifold*, Tôhoku Math. J., 29 (1977), 203-208.

[20] P. T. Nagy, *Geodesics on the Tangent Sphere Bundle of a Riemannian Manifold*, Geom. Dedicata, 7/2 (1978), 233-243.

[21] V. Oproiu, *A Generalization of natural almost Hermitian structures on the tangent bundles*, Math. J. Toyama Univ. 22 (1999), 1-14.

[22] V. Oproiu, *General natural almost Hermitian and anti-Hermitian structures on the tangent bundles*, Bull. Math. Soc. Sci. Math. Roum., 43 (91) (2000), 325-340.

[23] V. Oproiu, *On the differential geometry of the tangent bundle,* Rev. Roumaine Math. Pures Appl. 13 (1968), 847-855.

[24] V. Oproiu, *Some new geometric structures on the tangent bundles*, Publ. Math. Debrecen, 55/3-4 (1999), 261-281.

[25] J. H. Park, K. Sekigawa, *When are the tangent sphere bundles of a Riemannian manifold η-Einstein?*, Ann. Global An. Geom, 36 (3) (2009), 275-284.

[26] S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds,* Tôhoku Math. J., 10 (1958), 338-354.

[27] S. Tanno, *The standard CR-structure on the unit tangent bundle,* Tôhoku Math. J., 44 (1992), 535-543.

[28] Y. Tashiro, *On contact structures of tangent sphere bundles,* Tôhoku Math. J., 21 (1969), 117-143.

[29] C. Udrişte, *Diagonal lifts from a manifold to its tangent bundle,* Rendiconti di Matematica (Roma) (4), 9, 6 (1976), 539-550.

[30] C. Udrişte, *On the metric II+III in the tangent bundle of a Riemannian manifold,* Rev. Roumaine Math. Pures Appl., 22, 9 (1977), 1309-1320.

[31] C. Udrişte, C. Dicu, *Properties of the pseudo-Riemannian manifold $(TM, G = g^C - g^D - g^V)$ constructed over the Riemannian manifold (M, g),* Mathematica (Cluj) 21 (44) (1979), 33-41.

[32] K. Yano and S. Ishihara, *Tangent and Cotangent Bundles*, M. Dekker Inc. 1973.

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