Helmholtz conditions and their generalizations

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Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. In the paper, properties of morphisms in the variational sequence are investigated. The Euler–Lagrange morphism $\mathcal{E}_1$ is well-understood. It is also known that the kernel of the Helmholtz morphism $\mathcal{E}_2$ consists of locally variational dynamical forms, and is characterized by Helmholtz conditions. We study the image of $\mathcal{E}_2$ and the kernel of the next morphism $\mathcal{E}_3$, and solve the corresponding local and global inverse problem *when a three-form comes (via a variational map) from a dynamical form, i.e., corresponds to a system of differential equations.* We find identities, that are a generalization of the Helmholtz conditions to this situation, and show that the problem is closely related to the question on existence of a closed three-form. The obtained results extend known results on Lagrangians and locally variational dynamical forms to general dynamical forms, and open a new possibility to study non-variational equations by means of closed three-forms, as a parallel to extremal problems (variational equations) that are studied by means of closed two-forms (Cartan forms, symplectic geometry).

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1 Introduction: Helmholtz conditions and the inverse problem of the calculus of variations

Consider a system of second order ordinary differential equations

$$E_i(t, x^k, \dot{x}^k, \ddot{x}^k) = 0, \quad 1 \leq i \leq m, \tag{1.1}$$

for curves $c : I \to \mathbb{R}^m$, $c(t) = (x^k(t))$, where the functions E_i, $1 \leq i \leq m$, are defined on an open subset of $\mathbb{R} \times \mathbb{R}^{3m}$. The problem to decide when there exists a function $L(t, x^k, \dot{x}^k)$, such that for all i,

$$E_i = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}, \tag{1.2}$$

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is called the *(covariant) inverse problem of the calculus of variations.*

Above, d/dt is the total derivative operator defined by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\dot{x}^i + \frac{\partial f}{\partial \dot{x}^i}\ddot{x}^i, \tag{1.3}$$

where summation over repeated indices is understood.

Equations (1.1) satisfying (1.2) are called *variational*, and a corresponding function L is called a *Lagrangian.* Solutions of variational equations are *extremals* of a variational functional defined by the Lagrangian L.

It is well-known that equations (1.1) are variational if and only if the "left-hand-sides" E_i satisfy the following identities, called *Helmholtz conditions* [4, 15]:

$$\begin{aligned} &\frac{\partial E_i}{\partial \ddot{x}^k} - \frac{\partial E_k}{\partial \ddot{x}^i} = 0, \quad \frac{\partial E_i}{\partial \dot{x}^k} + \frac{\partial E_k}{\partial \dot{x}^i} - \frac{d}{dt}\Big(\frac{\partial E_i}{\partial \ddot{x}^k} + \frac{\partial E_k}{\partial \ddot{x}^i}\Big) = 0, \\ &\frac{\partial E_i}{\partial x^k} - \frac{\partial E_k}{\partial x^i} - \frac{1}{2}\frac{d}{dt}\Big(\frac{\partial E_i}{\partial \dot{x}^k} - \frac{\partial E_k}{\partial \dot{x}^i}\Big) = 0. \end{aligned} \tag{1.4}$$

Then, a Lagrangian for equations (1.1) can be constructed by the following formula

$$L = x^i \int_0^1 E_i(t, ux^k, u\dot{x}^k, u\ddot{x}^k)du \tag{1.5}$$

due to Tonti [16] and Vainberg [17] (see [13]).

Within the modern calculus of variations, variational objects and their properties can be effectively studied by methods of differential and algebraic geometry. In this paper we shall use the framework of the theory of variational sequences on fibred manifolds, introduced by Krupka [9, 8]. The variational sequence is a quotient sequence of the De Rham sequence, such that one of the morphisms is the *Euler–Lagrange mapping* $\mathcal{E}_1 : \lambda \to E_\lambda$, assigning to a Lagrangian (one-form $\lambda = Ldt$) its Euler-Lagrange form (two-form $E_\lambda = E_i(L)dx^i \wedge dt$, where $E_i(L)$ are the Euler–Lagrange expressions (1.2)). The next morphism, $\mathcal{E}_2 : E \to H_E$, assigns to a two-form $E = E_i dx^i \wedge dt$ a three-form H_E, called *Helmholtz form.* E represents a system of differential equations (1.1) that, of course, need not be variational. Due to exactness of the sequence, dynamical forms with identically zero Helmholtz forms are locally variational, i.e. represent *variational* equations. Components of a Helmholtz form H_E are "left-hand sides" of Helmholtz conditions, and nonzero Helmholtz forms correspond to *non-variational equations.* In other words, Helmholtz conditions describe the kernel of the morphism $\mathcal{E}_2$.

While the Euler–Lagrange mapping $\mathcal{E}_1$ is well-understood, yet almost nothing is known about the next variational morphisms $\mathcal{E}_2$, and especially $\mathcal{E}_3$. In the present paper we study the image of $\mathcal{E}_2$ and the kernel of the next mapping $\mathcal{E}_3$. This means to study the question *when a three-form comes (via a variational map) from a dynamical form*, i.e., corresponds to a system of differential equations. We show that the problem is closely related to the question on existence of a closed counterpart of a three-form (closed Lepage equivalent). We solve both the local and global version of the problem (Theorem 3.2, Theorem 3.3), and for the second order case compute identities that, in this sense, generalize the Helmholtz conditions (Theorem 3.4). We show that a Helmholtz form can be completed to a *closed* form in a *unique* way, and find the corresponding closed three-form explicitly (Theorem 3.5).

Our results extend known results on Lagrangians and locally variational dynamical forms to general dynamical forms, and open a new possibility to study *non-variational equations* by means of *closed three-forms*, as a parallel to extremal problems (variational equations) that are studied by means of *closed two-forms* (Cartan forms, symplectic and pre-symplectic geometry).

2 The variational sequence in fibred manifolds

Let $\pi : Y \to X$ be a smooth fibred manifold, $\dim X = 1$, $\dim Y = m + 1$, and $\pi_r : J^rY \to X$, $r \geq 1$, its jet prolongations. Denote by $\pi_{r,s} : J^rY \to J^sY$, $r > s \geq 0$, canonical jet projections. A differential q-form ($q > 1$) η on J^rY is called *contact* if $J^r\gamma^*\eta = 0$ for every section γ of π, *horizontal* or 0*-contact* if $i_\xi\eta = 0$ for every vertical vector field ξ on J^rY, and k*-contact*, $1 \leq k \leq q$, if for every vertical vector field ξ, $i_\xi\eta$ is $(k-1)$-contact. If lifted to $J^{r+1}Y$, every q-form η on J^rY can be canonically decomposed into a sum of k-contact components, η_k, where $k = 0, 1, \ldots, q$, We write $\eta_k = p_k\eta$, and $p_0 = h$, then

$$\pi^*_{r+1,r}\eta = h\eta + p_1\eta + \cdots + p_q\eta. \tag{2.1}$$

A contact q-form is called *strongly contact* if $\pi^*_{r+1,r}\eta = p_q\eta$.

In what follows, we denote Ω^r_q the sheaf of q-forms on J^rY, $\Omega^r_{0,c} = \{0\}$, $\Omega^r_{q,c}$ the sheaf of strongly contact q-forms on J^rY, $d\Omega^r_{q-1,c}$ the image sheaf of $\Omega^r_{q-1,c}$ by the exterior derivative d, and we put

$$\Theta^r_q = \Omega^r_{q,c} + d\Omega^r_{q-1,c}. \tag{2.2}$$

The De Rham sequence

$$0 \to \mathbb{R} \to \Omega^r_0 \to \Omega^r_1 \to \Omega^r_2 \to \Omega^r_3 \to \cdots \tag{2.3}$$

(where morphisms are the exterior derivatives d) has a subsequence

$$0 \to \Theta^r_1 \to \Theta^r_2 \to \Theta^r_3 \to \cdots \tag{2.4}$$

which is an exact sequence of soft sheaves. The quotient sequence

$$0 \to \mathbb{R} \to \Omega^r_0 \to \Omega^r_1/\Theta^r_1 \to \Omega^r_2/\Theta^r_2 \to \Omega^r_3/\Theta^r_3 \to \Omega^r_4/\Theta^r_4 \to \cdots \tag{2.5}$$

is also exact, and is called *the variational sequence* of order r. As proved in [9], *the variational sequence is an acyclic resolution of the constant sheaf* $\mathbb{R}$ *over* Y. Hence, due to the abstract De Rham theorem, *the cohomology groups of the cochain complex of global sections of the variational sequence are identified with the De Rham cohomology groups* H^qY *of the manifold* Y.

By construction, *morphisms* in the variational sequence are quotients of the exterior derivative operator d. In turns out that

$$\mathcal{E}_1 : \ \Omega^r_1/\Theta^r_1 \to \Omega^r_2/\Theta^r_2 \tag{2.6}$$

is the well-known *Euler–Lagrange mapping* of the calculus of variations. The next morphism

$$\mathcal{E}_2 : \ \Omega^r_2/\Theta^r_2 \to \Omega^r_3/\Theta^r_3 \tag{2.7}$$

is called *Helmholtz mapping*. It should be stressed that this mapping, discovered within the variational sequence theory, has not been known earlier in the calculus of variations.

Objects in the variational sequence are elements of the quotient sheaves Ω^r_q/Θ^r_q, $q \geq 1$, i.e., they are *equivalence classes of local rth-order differential q-forms*. We denote by $[\rho] \in \Omega^r_q/\Theta^r_q$ the *class* of $\rho \in \Omega^r_q$. By definition, the kernel of the Euler–Lagrange mapping $\mathcal{E}_1$ consists of *null Lagrangians*, and the image are *locally variational forms* (Euler–Lagrange forms of locally defined Lagrangians). Due to the exactness of the variational sequence, if $[\alpha] \in \Omega^r_2/\Theta^r_2$ is such that

$$\mathcal{E}_2([\alpha]) = [d\alpha] = 0 \tag{2.8}$$

then there exists $[\rho] \in \Omega^r_1/\Theta^r_1$ such that $[\alpha] = [d\rho] = \mathcal{E}_1([\rho])$, i.e. $[\alpha]$ is the image by the Euler–Lagrange mapping of a class $[\rho]$. In other words, the class $[\alpha]$ is locally variational—comes from a class $[\rho]$ that has the meaning of a *local Lagrangian*. If moreover $H^2Y = \{0\}$ then a *global* Lagrangian exists. Condition (2.8) for "local variationality" then provides Helmholtz conditions (of order r).

Classes in the variational sequence can be represented by differential forms. We shall explore the representation by so-called *source forms*, $(q-1)$-contact q-forms belonging to the ideal generated by contact forms $\omega^i = dx^i - \dot{x}^i dt$, $1 \leq i \leq m$. A *canonical source forms representation* is obtained by means of the *interior Euler operator*, $\mathcal{I}$, introduced to the variational bicomplex theory by Anderson [1, 2], and adapted to the finite order situation of the variational sequence theory in [6, 10]. This operator reflects in an intrinsic way the procedure of getting a distinguished representative of a class $[\rho] \in \Omega^r_q/\Theta^r_q$ by applying to ρ the operator p_{q-1} and the factorization by Θ^r_q. $\mathcal{I}$ is an $\mathbb{R}$-linear mapping $\Omega^r_q \to \Omega^{2r+1}_q$ such that

(1) $\mathcal{I}\rho$ belongs to the same class as $\pi^*_{2r+1,r}\rho$,

(2) $\mathcal{I}^2 = \mathcal{I}$ (up to a canonical projection),

(3) the kernel of $\mathcal{I} : \Omega^r_q \to \Omega^{2r+1}_q$ is Θ^r_q.

Source forms for classes $[\rho] \in \Omega^r_1/\Theta^r_1$ are *horizontal forms* $\lambda = Ldt$, called *Lagrangians*. Source forms for classes $[\alpha] \in \Omega^r_2/\Theta^r_2$ are two-forms $E = E_i\omega^i \wedge dt$, called *dynamical forms* (corresponding to *differential equations*). Note that in this representation, if $[\rho]$ is represented by λ then $[d\rho] = \mathcal{E}_1([\rho])$ is represented by the dynamical form E_λ, the Euler–Lagrange form of λ. If $[\alpha] \in \Omega^r_2/\Theta^r_2$ is represented by a dynamical form E then $[d\alpha] = \mathcal{E}_2([\alpha])$ is represented by a source three-form H_E, the Helmholtz-form of E. According to [10], for a general class $[\rho] \in \Omega^r_3/\Theta^r_3$, where

$$p_2\rho = \sum_{k,l=0}^{r} H^{kl}_{ij}\omega^i_k \wedge \omega^j_l \wedge dt, \tag{2.9}$$

we get the canonical source form

$$\mathcal{I}\rho = \frac{1}{2}\sum_{k,l=0}^{r}\sum_{p=0}^{k}(-1)^k\binom{k}{p}\frac{d^{k-p}}{dt^{k-p}}(H^{lk}_{ji} - H^{lk}_{ij})\omega^j_{p+l} \wedge \omega^i \wedge dt. \tag{2.10}$$

Above,

$$\omega^i_k = dx^i_k - x^i_{k+1}dt, \quad 1 \le i \le m, \quad 0 \le k \le r-1 \tag{2.11}$$

are basic contact forms of order r, $\omega^i_0 = \omega^i$.

3 The image of the Helmholtz mapping: generalization of Helmholtz conditions to 3-forms

The aim of this paper is to study the *image of the Helmholtz mapping*

$$\mathcal{E}_2 : \Omega^r_2/\Theta^r_2 \to \Omega^r_3/\Theta^r_3, \tag{3.1}$$

and the *kernel* of the next variational morphism

$$\mathcal{E}_3 : \Omega^r_3/\Theta^r_3 \to \Omega^r_4/\Theta^r_4. \tag{3.2}$$

First, we note that $\operatorname{Im}\mathcal{E}_2 = \operatorname{Ker}\mathcal{E}_3$. Indeed, if $[\beta] \in \operatorname{Ker}\mathcal{E}_3$, i.e., $[\beta] \in \Omega^r_3/\Theta^r_3$ is such that $\mathcal{E}_3([\beta]) = [d\beta] = 0$ then due to exactness of the variational sequence there exists $[\alpha] \in \Omega^r_2/\Theta^r_2$ such that $[\beta] = [d\alpha] = \mathcal{E}_2([\alpha])$, i.e. $[\beta]$ is the image by the Helmholtz mapping of a class $[\alpha]$ (that has the meaning of a *differential equation*, and can be represented by a local *dynamical form*). Conversely, if $[\beta] \in \operatorname{Im}\mathcal{E}_2$, i.e. $[\beta] = \mathcal{E}_2([\alpha])$ for a class $[\alpha] \in \Omega^r_2/\Theta^r_2$ then $[\beta] = [d\alpha]$, hence $\mathcal{E}_3([\beta]) = \mathcal{E}_3([d\alpha]) = [dd\alpha] = 0$ (the zero class in Ω^r_4/Θ^r_4), so that $[\beta] \in \operatorname{Ker}\mathcal{E}_3$.

In the representation by source forms, classes $[d\alpha] \in \Omega^r_3/\Theta^r_3$ are represented by 3-forms which arise from local dynamical forms as their Helmholtz forms.

Let us introduce the following definitions:

Definition 3.1. We shall call source forms representing elements in Ω^r_3/Θ^r_3 *Helmholtz-like forms.*

Let $U \subset J^sY$ be an open set. We say that a Helmholtz-like form H is *Helmholtz over* U if there exists a dynamical form E on U such that $H|_U = H_E$. If there is an open covering $\{W_\iota\}$ of J^sY such that H is Helmholtz over W_ι for every ι, we say that H is *locally Helmholtz.* We call H *globally Helmholtz* if there exists a dynamical form E on J^sY such that $H = H_E$.

Note that relation between locally and globally Helmholtz forms is similar to that between locally and globally variational dynamical forms. A three-form H which is *globally Helmholtz* is a Helmholtz form of a (globally defined) dynamical form E. A form H which is *locally Helmholtz* comes from a family of local dynamical forms whose Helmholtz forms glue together to a global differential form; in this case a global dynamical form E such that $H = H_E$ need not exist.

Theorem 3.2. *Let H be a Helmholtz-like form. H is locally Helmholtz if and only if there exists a (possibly local) three-form β such that $p_2\beta = H$, and $d\beta = 0$.*

Proof. Let H be locally Helmholtz. Then (locally) $H = H_E$ for a dynamical form E. E is a source form for a class $[\alpha] \in \Omega^r_2/\Theta^r_2$ such that H is a source form for the class $[d\alpha] \in \Omega^r_3/\Theta^r_3$. By assumption, there is a representative $\bar\alpha$ of the class $[\alpha]$ such that $p_1\bar\alpha = E$. Putting $\bar\beta = d\bar\alpha$ we get a (possibly local) closed three-form such that

$p_2\bar{\beta} = p_2 d\bar{\alpha} \sim H_E$. This means that $H_E = p_2 d\bar{\alpha} + p_2 d\eta$ where η is 2-contact. Putting $\alpha_0 = \bar{\alpha} + \eta$ and $\beta = d\alpha_0$ we get a two-form equivalent with $\bar{\alpha}$ such that $p_1\alpha_0 = E$, and a 3-form such that $d\beta = 0$ and $p_2\beta = H_E$, as desired.

Conversely, let H be a source form such that $H = p_2\beta$ for a closed three-form β. Put $\alpha = A\beta$ where A is the contact homotopy operator [7]. Indeed, by definition of A, $\beta = dA\beta + Ad\beta = dA\beta = d\alpha$. Moreover, A is adapted to the decomposition to contact components (meaning that if η is k-contact then $A\eta$ is $(k-1)$-contact). Then

$$\tilde{E} = p_1\alpha = p_1 A\beta = p_1(Ap_2\beta) = p_1 AH = AH \tag{3.3}$$

is a (local) 1-contact two-form, representing the class $[\alpha]$. We shall show that $\tilde{E}$ is equivalent with a dynamical form E (that is, with a source form for the class $[\alpha]$).

In fibred coordinates,

$$\begin{aligned}
\tilde{E} &= \sum_{j=0}^{r} \tilde{E}_i^j \omega_j^i \wedge dt = \tilde{E}_i \omega^i \wedge dt - \sum_{j=1}^{r} \tilde{E}_i^j d\omega_{j-1}^i \sim \tilde{E}_i \omega^i \wedge dt + \sum_{j=1}^{r} d\tilde{E}_i^j \wedge \omega_{j-1}^i \\
&\sim \Big(\tilde{E}_i - \frac{d\tilde{E}_i^1}{dt}\Big)\omega^i \wedge dt - \sum_{j=2}^{r} \frac{d\tilde{E}_i^j}{dt}\omega_{j-1}^i \wedge dt \\
&= \Big(\tilde{E}_i - \frac{d\tilde{E}_i^1}{dt}\Big)\omega^i \wedge dt + \frac{d\tilde{E}_i^2}{dt} d\omega^i + \sum_{j=3}^{r} \frac{d\tilde{E}_i^j}{dt} d\omega_{j-2}^i \\
&\sim \Big(\tilde{E}_i - \frac{d\tilde{E}_i^1}{dt} + \frac{d^2\tilde{E}_i^2}{dt^2}\Big)\omega^i \wedge dt - \sum_{j=3}^{r} d\Big(\frac{d\tilde{E}_i^j}{dt}\Big) \wedge \omega_{j-2}^i \\
&\sim \Big(\tilde{E}_i - \frac{d\tilde{E}_i^1}{dt} + \frac{d^2\tilde{E}_i^2}{dt^2}\Big)\omega^i \wedge dt + \frac{d^2\tilde{E}_i^3}{dt^2}\dot{\omega}^i \wedge dt + \sum_{j=4}^{r} \frac{d^2\tilde{E}_i^j}{dt^2}\omega_{j-2}^i \wedge dt \\
&= \Big(\tilde{E}_i - \frac{d\tilde{E}_i^1}{dt} + \frac{d^2\tilde{E}_i^2}{dt^2}\Big)\omega^i \wedge dt - \frac{d^2\tilde{E}_i^3}{dt^2} d\omega^i - \sum_{j=4}^{r} \frac{d^2\tilde{E}_i^j}{dt^2} d\omega_{j-3}^i \\
&\sim \Big(\tilde{E}_i - \frac{d\tilde{E}_i^1}{dt} + \frac{d^2\tilde{E}_i^2}{dt^2} - \frac{d^3\tilde{E}_i^3}{dt^3}\Big)\omega^i \wedge dt - \sum_{j=4}^{r} \frac{d^3\tilde{E}_i^j}{dt^3}\omega_{j-3}^i \wedge dt \sim \cdots \\
&\sim \Big(\tilde{E}_i + \sum_{k=1}^{r} (-1)^k \frac{d^k\tilde{E}_i^k}{dt^k}\Big)\omega^i \wedge dt.
\end{aligned} \tag{3.4}$$

□

Theorem 3.3. *If H is locally Helmholtz and the cohomology group H^3Y is trivial then H is equivalent with a globally Helmholtz form.*

Proof. We have seen above that if H is locally Helmholtz then H is a source form for a class $[d\alpha]$. If, moreover, the group $H^3Y = \{0\}$ then the class $[\alpha]$ has a global representative $\bar{\alpha}$. Putting $E = \mathcal{I}\bar{\alpha}$ where $\mathcal{I}$ is the interior Euler operator, we get a global dynamical form. The global form $H_E = \mathcal{I}d\bar{\alpha}$ is then the Helmholtz form of E, and, by construction, it is equivalent with H. □

With help of Theorem 3.2 we can compute explicit conditions for a three-form be a locally Helmholtz form (a generalization of Helmholtz conditions to three-forms), and obtain a corresponding dynamical form E, and a closed three-form (closed counterpart of H_E), which, as we shall show, is *unique*. In what follows, we shall be interested in second-order Helmholtz-like forms; a generalization to the 3rd order can be found in [14].

Theorem 3.4. *A second-order source three-form (Helmholtz-like form) H is locally Helmholtz if and only if components of H, given by*

$$H = H^0_{ij}\,\omega^i \wedge \omega^j \wedge dt + H^1_{ij}\,\omega^i \wedge \dot\omega^j \wedge dt + H^2_{ij}\,\omega^i \wedge \ddot\omega^j \wedge dt, \quad H^0_{ij} = -H^0_{ji}, \tag{3.5}$$

satisfy the following identities:

$$\begin{gathered}
\Big(\frac{\partial H^2_{ij}}{\partial \ddot x^k}\Big)_{[jk]} = 0, \quad \Big(\frac{\partial H^1_{ij}}{\partial \ddot x^k} - \frac{\partial H^2_{ik}}{\partial \dot x^j}\Big)_{[ijk]} = 0, \quad \Big(\frac{\partial H^1_{ij}}{\partial \ddot x^k} - \frac{\partial H^2_{ik}}{\partial \dot x^j}\Big)_{(jk)} = 0,\\
\Big(\frac{\partial H^1_{ij}}{\partial \dot x^k} - \frac{1}{2}\frac{d}{dt}\Big(\frac{\partial H^1_{ij}}{\partial \ddot x^k} - \frac{\partial H^2_{ik}}{\partial \dot x^j}\Big)\Big)_{(ij),[jk]} = 0,\\
\Big(\frac{\partial H^0_{ij}}{\partial \ddot x^k} - \frac{1}{2}\frac{\partial H^1_{ij}}{\partial \dot x^k} - \frac{\partial H^2_{ik}}{\partial x^j} + \frac{1}{4}\frac{d}{dt}\Big(\frac{\partial H^1_{ij}}{\partial \ddot x^k} - \frac{\partial H^2_{ik}}{\partial \dot x^j}\Big)\Big)_{[ijk]} = 0,\\
\Big(\frac{\partial H^0_{ij}}{\partial \dot x^k} - \frac{\partial H^1_{ik}}{\partial x^j} - \frac{d}{dt}\Big(\frac{\partial H^0_{ij}}{\partial \ddot x^k} - \frac{\partial H^2_{ik}}{\partial x^j}\Big)\Big)_{[ij],(jk)} = 0,\\
\Big(\frac{\partial H^0_{ij}}{\partial x^k} - \frac{1}{3}\frac{d}{dt}\Big(\frac{\partial H^0_{ij}}{\partial \dot x^k} - \frac{\partial H^1_{ik}}{\partial x^j}\Big) + \frac{1}{3}\frac{d^2}{dt^2}\Big(\frac{\partial H^0_{ij}}{\partial \ddot x^k} - \frac{\partial H^2_{ik}}{\partial x^j}\Big)\Big)_{[ijk]} = 0,
\end{gathered} \tag{3.6}$$

where [ ] and ( ) denotes skew-symmetrization and symmetrization in the indicated indices, respectively.

Proof. By Theorem 3.2 we have to search for a 3-contact three-form

$$G = \sum_{p\le q\le r=0}^{2} G^{pqr}_{ijk}\omega^i_p \wedge \omega^j_q \wedge \omega^k_r \tag{3.7}$$

of order 2 such that $d(H+G) = 0$. We may assume that the components of G are skew-symmetric in the upper indices whenever at least two of the indices take the same value. Denote $\beta = H + G$. Condition $d\beta = 0$ means that $p_2 d\beta = 0$ and $p_3 d\beta = 0$. Computing the former we get that $p_2 d\beta = 0$ if and only if (3.6) are satisfied, proving that (3.6) are necessary for H be locally Helmholtz. However, (3.6) are also sufficient, since, by a straightforward computation, $p_3 d\beta = 0$ is a consequence of $p_2 d\beta = 0$. $\square$

Theorem 3.5. *Let H be a second-order source three-form (Helmholtz-like form) (3.5).*

(1) *Assume that there exists a second-order 3-contact form G such that $\beta = H+G$ is closed. Then H is locally Helmholtz, G is unique and takes the coordinate form*

$$\begin{aligned}
G = {}& \frac{1}{3}\Big(\frac{\partial H^0_{ij}}{\partial \dot x^k} - \frac{\partial H^1_{ik}}{\partial x^j} - \frac{d}{dt}\Big(\frac{\partial H^0_{ij}}{\partial \ddot x^k} - \frac{\partial H^2_{ik}}{\partial x^j}\Big)\Big)\omega^i \wedge \omega^j \wedge \omega^k\\
&+ \Big(\frac{\partial H^0_{ij}}{\partial \ddot x^k} - \frac{\partial H^2_{ik}}{\partial x^j}\Big)\omega^i \wedge \omega^j \wedge \dot\omega^k + \frac{1}{2}\Big(\frac{\partial H^1_{ij}}{\partial \ddot x^k} - \frac{\partial H^2_{ik}}{\partial \dot x^j}\Big)\omega^i \wedge \dot\omega^j \wedge \dot\omega^k.
\end{aligned} \tag{3.8}$$

A corresponding dynamical form E for H is given by the formula

$$E = \mathcal{I}AH. \tag{3.9}$$

In coordinates, $E = E_i\omega^i \wedge dt$ where

$$E_i = \tilde{E}_i - \frac{d\tilde{E}_i^1}{dt} + \frac{d^2\tilde{E}_i^2}{dt^2}, \tag{3.10}$$

$$\begin{aligned}
\tilde{E}_i &= 2x^j \int_0^1 H_{ji}^0(t, ux^p, u\dot{x}^p, u\ddot{x}^p)u\,du - \dot{x}^j \int_0^1 H_{ij}^1(t, ux^p, u\dot{x}^p, u\ddot{x}^p)u\,du \\
&\quad - \ddot{x}^j \int_0^1 H_{ij}^2(t, ux^p, u\dot{x}^p, u\ddot{x}^p)u\,du, \\
\tilde{E}_i^1 &= x^j \int_0^1 H_{ji}^1(t, ux^p, u\dot{x}^p, u\ddot{x}^p)u\,du, \\
\tilde{E}_i^2 &= x^j \int_0^1 H_{ji}^2(t, ux^p, u\dot{x}^p, u\ddot{x}^p)u\,du.
\end{aligned} \tag{3.11}$$

(2) *Conversely, if H is locally Helmholtz then there exists a unique (global) 3-contact form G on J^2Y such that $\beta = H + G$ is closed. G is given by formula (3.8).*

Proof. (1) Denote H as above, then G should take the form (3.8), where G_{ijk}^{000} is completely skew-symmetric in ijk, $G_{ijk}^{001} = -G_{jik}^{001}$, $G_{ijk}^{011} = -G_{ikj}^{011}$, etc. From $p_2 d\beta = 0$ we get the following identities:

$$\begin{aligned}
&\Big(\frac{\partial H_{ij}^0}{\partial x^k} - \frac{dG_{ijk}^{000}}{dt}\Big)_{[ijk]} = 0, \quad \Big(\frac{\partial H_{ij}^0}{\partial \dot{x}^k} - \frac{\partial H_{ik}^1}{\partial x^j} - \frac{dG_{ijk}^{001}}{dt} - 3G_{ijk}^{000}\Big)_{[ij]} = 0, \\
&\Big(\frac{\partial H_{ij}^0}{\partial \ddot{x}^k} - \frac{\partial H_{ik}^2}{\partial x^j} - G_{ijk}^{001}\Big)_{[ij]} = 0, \quad \Big(\frac{\partial H_{ij}^1}{\partial \dot{x}^k} - \frac{dG_{ijk}^{011}}{dt} - 2G_{ijk}^{001}\Big)_{[jk]} = 0, \\
&\frac{\partial H_{ij}^1}{\partial \ddot{x}^k} - \frac{\partial H_{ik}^2}{\partial \dot{x}^j} - 2G_{ijk}^{011} = 0, \quad \Big(\frac{\partial H_{ij}^2}{\partial \ddot{x}^k}\Big)_{[jk]} = 0, \quad G_{[ijk]}^{011} = 0, \\
&G_{ijk}^{111} = G_{ijk}^{002} = G_{ijk}^{012} = G_{ijk}^{112} = G_{ijk}^{122} = G_{ijk}^{222} = 0.
\end{aligned} \tag{3.12}$$

From (3.12) we get formulas for the components of G and conditions (3.6), hence H is locally Helmholtz. Since $d\beta = 0$ we have locally $\beta = d\alpha$, where $\alpha = A\beta$, and a corresponding local dynamical form E for H is a source form for α, i.e., $E = \mathcal{I}\alpha = \mathcal{I}p_1\alpha = \mathcal{I}AH$.

(2) By Theorem 3.2, if H is locally Helmholtz then G exists locally. However, G is unique, hence global (defined on J^2Y). □

Note that we have proved also the following result which can be viewed as a geometric version of the "generalized Helmholtz conditions" (3.6) above, and is an extension to three-forms of a well-known result in the calculus of variations on manifolds (see [3, 5, 11, 12]).

Corollary 3.6. *Let β be a three-form on J^2Y such that $p_2\beta = H$ is a source form on J^2Y. The following conditions are equivalent:*

(1) *H is locally Helmholtz (comes from a possibly local dynamical form as its Helmholtz form).*

(2) $p_2 d\beta = 0$.

(3) $d\beta = 0$.

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