Some properly critical subsets of Euclidean spaces

Cornel Pintea

Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. We first characterize the properly critical subsets of the real line and use this characterization to produce some properly critical subsets of the higher dimensional spaces, both for real valued functions and for some vector-valued maps.

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1 Introduction

If M is a boundaryless smooth manifold and $f : M^n \longrightarrow \mathbb{R}$ is a smooth function, then the critical set of f, denoted by $C(f)$ in this note, is obviously closed since $C(f)$ is the preimage of zero through the continuous function $M \longrightarrow \mathbb{R}$, $x \longmapsto ||(\nabla f)(x)||^2$, where the norm is considered with respect to some arbitrary Riemannian metric on M. Given a closed subset C of M, the inverse problem of deciding whether C is the critical set of some smooth map $f : M \longrightarrow \mathbb{R}$ has been intensively studied over the last decades, especially when M is either the two or the three dimensional Euclidean space. In higher dimension the closedness necessary condition for *criticality* is far from being sufficient, although for subsets of $\mathbb{R}$ it is also sufficient, as we will see in the next section. Indeed, a circle in $\mathbb{R}^2$ is not the critical set of any smooth function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, while the Antoine's necklace is, according to Grayson and Pugh [2], the critical set of a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ which is even proper. Recall that the Antoine's necklace is a wild Cantor set in $\mathbb{R}^3$. Consequently it is worthwhile to call *critical* a closed subset of M^n which is the critical set of some smooth real valued function on M^n. One can also consider the *criticality problem* for some subclasses of $C^\infty(M^n, \mathbb{R})$, some examples of closed sets which are not CS^∞-critical are presented in [4], [5] and [6]. On the other hand, we will call *properly critical* a closed subset of M^n which is the critical set of some proper real valued function on M^n. Recall that a function $f : M^n \longrightarrow \mathbb{R}$ is said to be *proper* if $f^{-1}(K)$ is compact for every compact subset K of $\mathbb{R}$ and observe that a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is proper if and only if $|f(x)| \longrightarrow \infty$ as $||x|| \longrightarrow \infty$. While the criticality and proper criticality are obviously equivalent if M is compact, this nontrivial equivalence is proved in [3] for compact subsets of $\mathbb{R}^2$, where a characterization of the (properly) critical compact

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subsets of $\mathbb{R}^2$ is also provided. We also mention [10] for some (non)critical subsets of certain compact surfaces.

As mentioned above, the closed subsets of $\mathbb{R}$ are all critical, but not all of them are properly critical. In this paper we first provide a characterization of properly critical subsets of the real line and use it to produce some properly critical subsets of higher dimensional Euclidean spaces. We particularly get the proper criticality of $C^n \subseteq \mathbb{R}^n$, where $C \subseteq [0,1]$ is the middle third Cantor set.

We end this section by proving the closedness of the union of some closed subsets in a metric space, which are *away from each other* in the sense specified by Definition 1.1. This type of sets appears several times in this paper, while their property is only used twice and we could not find an explicit reference for it.

Definition 1.1. We say that the subsets $\{A_i\}_{i\in I}$ of a metric space (X,d) are *away from each other* if there exists a constant $c > 0$ such that $d(A_i, A_j) \geq c$ for all $i,j \in I$, $i \neq j$, where $d(A,B),\ A,B \subseteq X$, stands for $\inf\{d(a,b) \,|\, a \in A, b \in B\}$. Such a family of sets are obviously pairwise disjoint.

Lemma 1.2.

1. If $\{A_i\}_{i\in I}$ is a family of closed subsets of a metric space (X,d) which are away from each other, then the following equality holds
$$\overline{\bigcup_{i\in I} A_i} = \bigcup_{i\in I} \bar{A}_i.$$
2. If $\{C_i\}_{i\in I}$ is a family of closed subsets of a metric space (X,d) which are away from each other, then their union $\bigcup_{i\in I} C_i$ is also closed.

Proof. (1) We only need to show the inclusion $\overline{\bigcup_{i\in I} A_i} \subseteq \bigcup_{i\in I} \bar{A}_i$, since the opposite inclusion is obvious. In this respect we consider an element

$$a \in \overline{\bigcup_{i\in I} A_i} \text{ and a sequence } (a_n) \text{ in } \bigcup_{i\in I} A_i$$

such that $a_n \longrightarrow a$ as $n \longrightarrow \infty$. If the terms of the sequence (a_n) are contained in some set A_{i_0}, $i_0 \in I$, for infinitely many indices $n \geq 1$, then the sequence (a_n) has a subsequence in that A_{i_0}. This shows that a, which is equally the limit of that subsequence, is actually contained in $\bar{A}_{i_0}$. Assume now that each A_i, $i \in I$ contains terms of the sequence (a_n) for only finitely many indices $n \geq 1$. In this case the sequence (a_n) has a subsequence, denoted in the same way, such that a_m and a_k, $m \neq k$ are in contained in different sets A_r and A_s respectively. This shows that $d(a_m, a_k) \geq c$ for $m \neq k$, which makes the convergence of the sequence (a_n) impossible, since convergent sequences are Cauchy sequences, namely their terms are arbitrarily close to each other for sufficiently large indices.

(2) Combining the closedness of the sets $\{C_i\}_{i\in I}$ with (1) one gets:

$$\overline{\bigcup_{i\in I} C_i} = \bigcup_{i\in I} \bar{C}_i = \bigcup_{i\in I} C_i,$$

which shows that the union $\bigcup_{i\in I} C_i$ is, indeed, closed. □

Remark 1.3. A family $\{A_i\}_{i\in I}$ of subsets of the real line are away from each other if and only if there exists a constant $c > 0$ such that for $i \neq j$, either $A_i \geq A_j + c$ or $A_j \geq A_i + c$, were the inequality $A \geq B$, for two subsets A, B of $\mathbb{R}$, is understood in the sense that $a \geq b$ for all $a \in A$ and all $b \in B$. Equivalently, $\inf(A_i) \geq \sup(A_j) + c$ or $\inf(A_j) \geq \sup(A_i) + c$.

2 Smooth functions on $\mathbb{R}$ with prescribed zero sets

As we have already mentioned few times before, the closedness necessary condition for criticality of subsets in $\mathbb{R}$ is also sufficient, namely, each closed subset of $\mathbb{R}$ is the critical set of a smooth function $F : \mathbb{R} \longrightarrow \mathbb{R}$. Indeed, a given closed subset C of $\mathbb{R}$ is the critical set of the antiderivative

$$F : \mathbb{R} \longrightarrow \mathbb{R},\ F(x) := \int_0^x f(t)dt,$$

of a positive function $f : \mathbb{R} \longrightarrow \mathbb{R}$ whose set of zeros is precisely C. The existence of f is ensured by the well known Whitney theorem [7, Théorème 1, p. 17]. For more details on the above mentioned construction we refere to [9] and the references therein. Let us recall that the Whitney's construction of f relies on the structure representation theorem of open subsets in Euclidean spaces, as countable unions of open balls. In the case of the real line, the representation of open sets as unions of countably many open intervals satisfies an additional condition which we will exploit in this section. Indeed, these intervals might be chosen to be pairwise disjoint.

Observe that, for $a, b \in \mathbb{R},\ a < b$, the smooth function

$$f_{a,b} : \mathbb{R} \longrightarrow \mathbb{R},\ f_{a,b}(x) = \frac{g(b-t)}{g(b-t) + g(t-a)},$$

where

$$g : \mathbb{R} \longrightarrow \mathbb{R},\ g(t) = \begin{cases} e^{-\frac{1}{t}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases},$$

has the following properties:

1. $0 \leq f_{a,b} \leq 1$;
2. $f_{a,b}\big|_{(-\infty,a]} = 1$;
3. $f_{a,b}\big|_{[b,\infty)} = 0$.

Moreover, the *bump function* $F_{x,r} : \mathbb{R} \longrightarrow \mathbb{R},\ F_{x,r}(t) = f_{r,2r}(|t-x|)$, has the following properties:

$$\begin{array}{lcl} 1.\ 0 \leq F_{x,r} \leq 1; & & 1.\ 0 \leq mF_{x,r} \leq m; \\ 2.\ F_{x,r}\big|_{(x-r,x+r)} = 1; & \text{Equivalently} & 2.\ mF_{x,r}\big|_{(x-r,x+r)} = m; \quad ,\ (\forall)m > 0. \\ 3.\ F_{x,r}\big|_{(x-2r,x+2r)^c} = 0. & & 3.\ mF_{x,r}\big|_{(x-2r,x+2r)^c} = 0 \end{array}$$

Theorem 2.1. *Every open subset D of the real line can be represented as a countable union of pairwise disjoint open intervals called* the components of D.

If $I \subseteq \mathbb{R}$ is an open interval, we consider the function $F_I : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $F_I = F_{x_I, l(I)/2}$ if I is bounded, where $l(I)$ is the length of I and x_I is its midpoint, and $F_I = f_{s_I - 1, s_I}$, where $s_I = \sup(I)$, if I is unbounded below but bounded above. Finally, if I unbounded above but bounded below, then $F_I := F_{-I} \circ A$, where $A : \mathbb{R} \longrightarrow \mathbb{R}$, $A(x) = -x$. If $D \subseteq \mathbb{R}$ is an open subset with components (I_n), we next define the functions

$$F_D, G_D : \mathbb{R} \longrightarrow \mathbb{R} \text{ by } F_D := \sum_n \frac{1}{2^n c_n} F_{I_n} \text{ and } G_D = \sum_n m(I_n) F_{I_n}$$

respectively, where $m(I_n)$ is chosen in such a way that

$$m(I_n) \geq \frac{1}{\displaystyle\int_{\bar{I}_n} F_{I_n}(t)dt}, \text{ namely } \int_{\bar{I}_n} G_D(t)dt \geq 1$$

and $c_n = \max\left\{1, \sup(F_{I_n}), \sup(F_{I_n}), \ldots, \sup(F_{I_n}^{(n)})\right\}$. If D has infinitely many componnets, the representations

$$\sum_n \frac{1}{2^n c_n} F_{I_n} \text{ and } \sum_n m(I_n) F_{I_n}$$

for F_D and G_D respectively, relies on the basic fact that, for series with positive terms their order plays no role. Finally, the values $F_D(x)$ and $G_D(x)$ are well defined, for every $x \in \mathbb{R}$, because at most one of the functions F_{I_n} does not vanish at x.

Lemma 2.2. *Let $D \subseteq \mathbb{R}$ be an open subset with components (I_n).*

1. *F_D is a smooth function.*
2. *If the components of D are away form each other, then and G_D is smooth.*

Proof. (1) Indeed, if $k \geq 1$, then for $n \geq k$ one gets

$$\Big(\frac{1}{2^n c_n} F_{I_n}\Big)^{(k)} = \frac{1}{2^n c_n} F_{I_n}^{(k)} \leq \frac{1}{2^n}.$$

This shows, by using the classical theory of series with differentiable terms, that F_D is indeed smooth.

(2) If the components of D are away from each other and I is such a component, then obviously $G_D|_{(i_I - \frac{c}{2}, s_I + \frac{c}{2})} = G_I|_{(i_I - \frac{c}{2}, s_I + \frac{c}{2})}$, where, as usual, $i_I := \inf(I)$ and $s_I := \sup(I)$. This shows that G_D is smooth on the union

$$\bigcup_n \Big(i_{I_n} - \frac{c}{2}, s_{I_n} + \frac{c}{2}\Big), \text{ since } G_I|_{\left(i_{I_n} - \frac{c}{2}, s_{I_n} + \frac{c}{2}\right)} \text{ is smooth on } \Big(i_{I_n} - \frac{c}{2}, s_{I_n} + \frac{c}{2}\Big).$$

Therefore G_D is smooth on a neighborhood of $\bar{D}$, since, according to Lemma 1.2, the equality $\bar{D} = \overline{\bigcup_n I_n} = \bigcup_n \bar{I}_n$ holds. On $\mathbb{R} \setminus \bar{D} \subseteq \mathbb{R} \setminus D$, G_D is obviously smooth because $G_D|_{\mathbb{R} \setminus D} = 0$. □

Remark 2.3.

1. The function F_D, besides its smoothness, has the properties that $F_D \geq 0$ and $F_D^{-1}(0) = \mathbb{R} \setminus D$.
2. Although the function G_D may not be continuous, yet $G_D \geq 0$ and $G_D^{-1}(0) = \mathbb{R} \setminus D$.

3 The properly critical subsets of the real line

If $K \subseteq \mathbb{R}$ is a properly critical closed set, then its complement $K^c = \mathbb{R} \setminus K$ is unbounded both below and above. In order to justify this statement, consider a differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $C(f) = K$. The boundedness below of K^c shows that the interval $(-\infty, i_{K^c})$, where $i_{K^c} = \inf(K^c)$, is contained in K, namely $f'(x) = 0$ for all $x \in (-\infty, i_{K^c})$. Consequently f is constant on $(-\infty, i_{K^c})$ and $(-\infty, i_{K^c}) \subseteq f^{-1}(y)$, where y is the value of f taken all along the interval $(-\infty, i_{K^c})$, which shows that f is not proper. One can similarly show that f is not proper under the assumption K^c bounded above.

In this section we show that the above mentioned necessary condition for proper criticality of closed subsets of the real line is also sufficient.

Theorem 3.1. *A closed subset K of $\mathbb{R}$ is properly critical if and only if its complement $K^c = \mathbb{R} \setminus K$ is unbounded both below and above.*

Proof. We only have to show that every closed subset of $\mathbb{R}$, say K, whose complement is unbounded both below and above is properly critical. Since K is closed, its complement K^c is open. From this point we divide the proof in two cases:

Case I. K is bounded. The required function is then

$$f_K : \mathbb{R} \longrightarrow \mathbb{R},\ f_K(x) = \int_0^x F_{K^c}(t)dt.$$

Indeed, f_K is differentiable since F_{K^c} is continuous, and $f'_K = F_{K^c}$, which shows that

$$C(f_K) = (f'_K)^{-1}(0) = (F_{K^c})^{-1}(0) = (K^c)^c = K.$$

In order to prove the properness of f_K we show that $\lim\limits_{x \to \pm\infty} f_K(x) = \pm\infty$. In this respect we observe that the intervals $(-\infty, i_K)$ and $(s_K, +\infty)$ are the unbounded components of K^c, where $i_K = \inf(K)$ and $s_K = \sup(K)$. Thus, the value of f_K at some $x \in (-\infty, i_K - 1)$ is

$$\begin{aligned} f_K(x) &= \int_0^x F_{K^c}(t)dt = \int_0^{i_K-1} F_{K^c}(t)dt + \int_{i_K-1}^x F_{K^c}(t)dt \\ &= \int_0^{i_K-1} F_{K^c}(t)dt + \int_{i_K-1}^x dt = x - i_K + 1 + \int_0^{i_K-1} F_{K^c}(t)dt. \end{aligned}$$

This shows that $\lim_{x\to-\infty} f_K(x) = -\infty$. Similarly, for $x \in (s_K + 1, +\infty)$ one has

$$f_K(x) = x - s_K - 1 + \int_0^{s_K+1} F_{K^c}(t)dt$$

and implicitly $\lim_{x\to+\infty} f_K(x) = +\infty$. For a slightly different proof of *Case I* we refer to [9].

Case II. K is unbounded. In this case K^c has infinitely many components since both K and its complement K^c are unbounded. This case is divided in three subcases:

1. *K is bounded below but unbounded above.* In this situation the complement K^c has a sequence of bounded components $(I_n)_{n\geq 0}$ such that $I_{n+1} > I_n + 1 \geq 1$ for all $n \geq 1$ whose union $\bigcup_{n\geq 1} I_n$ we denote by $\mathcal{I}$. Thus, the components (I_n) are away from each other, which shows, according to Lemma 2.2(2), that $G_{\mathcal{I}}$ is smooth. We also denote by $\mathcal{J}$ the union of the rest of components of K^c, namely those who are not terms of the sequence $(I_n)_{n\geq 1}$. For example, the unbounded component $(-\infty, i_K)$ of K^c, where $i_K = \inf(K)$, is contained in $\mathcal{J}$ as well as in $\mathbb{R} \setminus \mathcal{I}$. A proper function whose critical set is precisely K, in this situation, is

$$f_K : \mathbb{R} \longrightarrow \mathbb{R},\ f_K(x) = \int_0^x \big(F_{\mathcal{J}}(t) + G_{\mathcal{I}}(t)\big)dt,$$

which is smooth since $F_{\mathcal{J}}(t)$ and $G_{\mathcal{I}}$ are smooth. Further on, we have successively:

$$\begin{aligned} x \in (F_{\mathcal{J}} + G_{\mathcal{I}})^{-1}(0) &\iff F_{\mathcal{J}}(x) + G_{\mathcal{I}}(x) = 0 \\ &\iff F_{\mathcal{J}}(x) = G_{\mathcal{I}}(x) = 0 \\ &\iff x \in \big(F_{\mathcal{J}}\big)^{-1}(0) \text{ and } x \in \big(G_{\mathcal{I}}\big)^{-1}(0) \\ &\iff x \in \mathbb{R} \setminus \mathcal{J} \text{ and } x \in \mathbb{R} \setminus \mathcal{I} \\ &\iff x \in \big(\mathbb{R} \setminus \mathcal{J}\big) \cap \big(\mathbb{R} \setminus \mathcal{I}\big) \\ &\iff x \in \mathbb{R} \setminus (\mathcal{I} \cup \mathcal{J}) = K, \end{aligned}$$

which shows that $K = (F_{\mathcal{J}} + G_{\mathcal{I}})^{-1}(0) = (f'_K)^{-1}(0)$.

For the properness of f_K we first observe that $f_K(x) = x - i_K + 1 + \int_0^{i_K-1} F_{\mathcal{J}}(t)dt$, for all $x \in (-\infty, i_K - 1)$, since $F_{\mathcal{J}}|_{(-\infty, i_K-1)} = 1$ and $G_{\mathcal{I}}|_{(-\infty, i_K-1)} = 0$. This shows that $\lim_{x\to-\infty} f_K(x) = -\infty$. In order to prove that

$$\lim_{x\to+\infty} f_K(x) = +\infty$$

it is enough to show that $\lim_{n\to\infty} f_K(s_{I_n}) = \infty$, taking into account that $f'_K = F_{\mathcal{J}} + G_{\mathcal{I}} \geq 0$, that is f_K is increasing. This is actually the case because s_{I_n} is increasing, $s_{I_n} \longrightarrow \infty$ as $n \longrightarrow \infty$ and

$$\begin{aligned} f_K(s_{I_n}) &= \int_0^{s_{I_n}} \big(F_{\mathcal{J}}(t) + G_{\mathcal{I}}(t)\big)dt \geq \int_{\bar{I}_1\cup\cdots\cup\bar{I}_n} \big(F_{\mathcal{J}}(t) + G_{\mathcal{I}}(t)\big)dt \\ &= \int_{\bar{I}_1\cup\cdots\cup\bar{I}_n} G_{\mathcal{I}}(t)dt = \int_{\bar{I}_1} G_{\mathcal{I}}(t)dt + \cdots + \int_{\bar{I}_n} G_{\mathcal{I}}(t)dt \\ &\geq 1 + \cdots + 1 = n. \end{aligned}$$

2. *K is bounded above but unbounded below.* The required function, in this situation, is $f_{-K} \circ A$, where f_{-K} is the function constructed in the previous subcase, which corresponds to the bounded below but unbounded above closed set $-K$.

3. *K is unbounded both below and above.* In this situation we may find a sequence $(I_n)_{n\in\mathbb{Z}}$ such that $I_{n-1} < I_n - 1$ and $I_n < 0$ for all $n < 0$ while $I_{n+1} > I_n + 1$ and $I_n > 0$ for all $n \geq 0$. Thus, the components of $\mathcal{I} = \bigcup\limits_{n\in\mathbb{Z}} I_n$ are away from each other, i.e. $G_{\mathcal{I}}$ is, according to Lemma 2.2, smooth.

A proper function whose critical set is precisely K, in this situation, is

$$f_K : \mathbb{R} \longrightarrow \mathbb{R},\ f_K(x) = \int_0^x \big(F_{\mathcal{J}}(t) + G_{\mathcal{I}}(t)\big)dt,$$

where $\mathcal{J}$ is the union of those components who are not terms of the sequence $(I_n)_{n\in\mathbb{Z}}$. The proof of the equality $K = (f_K')^{-1}(0)$ can be done in the same way as it was done in the first subcase. In order to prove the properness of f_K we just prove that $\lim\limits_{x\to\pm\infty} f_K(x) = \pm\infty$. In this respect we first observe that $f_K' = F_{\mathcal{J}} + G_{\mathcal{I}} \geq 0$, that is f_K is increasing and, just as above, $f_K(s_{I_n}) \geq n$ for all $n \geq 1$. On the other hand, for $n < 0$ one gets successively

$$\begin{aligned} f_K(i_{I_n}) &= \int_0^{i_{I_n}} \big(F_{\mathcal{J}}(t) + G_{\mathcal{I}}(t)\big)dt \leq -\int_{\bar{I}_{-1}\cup\cdots\cup\bar{I}_n} \big(F_{\mathcal{J}}(t) + G_{\mathcal{I}}(t)\big)dt \\ &= -\int_{\bar{I}_{-1}\cup\cdots\cup\bar{I}_n} G_{\mathcal{I}}(t)dt = -\int_{\bar{I}_{-1}} G_{\mathcal{I}}(t)dt - \cdots - \int_{\bar{I}_n} G_{\mathcal{I}}(t)dt \\ &\leq -1 - \cdots - 1 = n. \end{aligned}$$

The inequalities $f_K(i_{I_n}) \leq n$ for all $n \leq -1$, combined with the increasing property of f_K and the fact that $m < n < 0 \Longrightarrow i_{I_m} < i_{I_n}$ as well as $i_{I_n} \longrightarrow -\infty$ as $n \longrightarrow -\infty$, show that $\lim\limits_{x\to-\infty} f_K(x) = -\infty$. □

Remark 3.2. 1. A closed subset K of the real line can be realized as the critical set of a smooth, positive and proper function. Indeed, K is the critical set of the function

$$\mathfrak{f}_K : \mathbb{R} \longrightarrow \mathbb{R},\ \mathfrak{f}_K(t) = \begin{cases} \left|\displaystyle\int_{x_0}^x F_{K^c}(t)dt\right| & \text{if } K \text{ is bounded} \\ \left|\displaystyle\int_{x_0}^x \left(F_{\mathcal{J}}(t) + G_{\mathcal{I}}(t)\right)dt\right| & \text{if } K \text{ is unbounded,} \end{cases}$$

which is obviously positive and proper. It is also smooth whenever x_0 belongs to K, as $(F_{K^c})^{(r)}(x_0) = (F_{\mathcal{J}} + G_{\mathcal{I}})^{(r)}(x_0) = 0$, for all $r \geq 0$.

2. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ associated to some closed subset of $\mathbb{R}$, which comes by means of Whitney's theorem, as well as the functions F_{K^c}, $F_{\mathcal{J}}$ are bounded while the function $G_{\mathcal{I}}$ might be unbounded. By choosing $m(I_n)$ such that

$m(I_n)^2 \int_{\bar{I}_n} (F_{I_n}(t))^2\, dt \geq 1$, one can see that the function

$$g_K : \mathbb{R} \longrightarrow \mathbb{R},\ g_K(x) = \begin{cases} \displaystyle\int_0^x (F_{K^c}(t))^2\, dt & \text{if } K \text{ is bounded} \\ \displaystyle\int_0^x (F_{\mathcal{J}}(t) + G_{\mathcal{I}}(t))^2\, dt & \text{if } K \text{ is unbounded} \end{cases}$$

is proper and $C(g_K) = K$. Observe that $g_K(x) = \int_0^x (f'_K(t))^2\, dt$.

4 Properly critical subsets of higher dimensional Euclidean spaces

The concept of *critical point* can be extended to vector valued functions. Indeed, if $F : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function, then $p \in \mathbb{R}^n$ is said to be a *critical point* of F if $rank_p F < \min\{n, m\}$, where $rank_p F := rank(JF)_p = \dim(Im(dF)_p)$ and $(JF)_p$ is the Jacobian matrix of F. The set $C(F)$ of critical points of F is still closed, while the set of its regular points $R(F) := \mathbb{R}^n \setminus C(F)$ is open.

Remark 4.1. 1. The critical set of a real-valued function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the same with the critical set of the vector-valued function $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by

$$G(x_1, \ldots, x_n) := \big(h(x_1, \ldots, x_n), x_2, \ldots, x_n\big),$$

where

$$h(x_1, \ldots, x_n) := \int_0^{x_1} ||(\nabla g)(t, x_2, \ldots, x =_n)||^2 dt.$$

Indeed, $\det(JG)_x = ||(\nabla g)(x)||^2$, which vanishes if and only if $x \in C(g)$. In other words $C(G) = \left(||(\nabla g)||^2\right)^{-1}(0) = C(g)$.

2. The role of the real valued function $\mathbb{R}^n \longrightarrow \mathbb{R}$, $x \longmapsto ||(\nabla g)(x)||^2$, in part (1), might be played by every real valued function. Indeed, every fiber $\varphi^{-1}(c)$ of a given smooth function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the critical set of the smooth vector valued function $\Phi_c : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by

$$\Phi_c(x_1, \ldots, x_n) = \Big(- cx_1 + \int_0^{x_1} \varphi(t, x_2, \ldots, x_n) = dt, x_2, \ldots, x_n \Big),$$

i.e. $C(\Phi_c) = \varphi^{-1}(c)$.

3. Every closed subset K of $\mathbb{R}^n$ is the critical set of a smooth vector-valued map $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Indeed, according to Whitney's theorem, there exists a smooth function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $K = \varphi^{-1}(0)$. Next we only need to apply part (2) in order to justify the statement.

Remark 4.1 was inspired by [8, Theorem 2.1], where the case $n = 2$ is treated. The question about the example presented within Remark 4.1(1) is whether the function

G is proper or not. Let us mention that the concept of *properness* can be extended to vector-valued maps. More precisely a vector-valued map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be *proper* if the preimage $f^{-1}(K)$ of every compact subset K of $\mathbb{R}^m$ is compact. In fact this concept can be extended to maps acting between topological spaces. Observe that a continuous map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is proper if and only if $||f(x)|| \longrightarrow \infty$ as $||x|| \longrightarrow \infty$.

Example 4.2. According to Remark 4.1(1), the critical set of the real-valued function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the critical set of the vector-valued map

$$G : \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ G(x_1, \ldots, x_n) := \Big(\int_0^{x_1} ||(\nabla g)(t, x_2, \ldots, x_n)||^2 dt, x_2, \ldots, x_n\Big),$$

but the properness of g does not ensures the properness of G. Indeed, the real-valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $f(x) = \ln(1+||x||^2)$ is proper, since $f(x) \longrightarrow \infty$ as $||x|| \longrightarrow \infty$, while its associated vector-valued map

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ F(x_1, \ldots, x_n) := \Big(\int_0^{x_1} ||(\nabla f)(t, x_2, \ldots, x_n)||^2 dt, x_2, \ldots, x_n\Big)$$

is not proper. In order to justify that, let us observe that the first component of F, $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$$\begin{aligned}\varphi(x_1, \ldots, x_n) &= \int_0^{x_1} ||(\nabla f)(t, x_2, \ldots, x_n)||^2 dt = 4\int_0^{x_1} \frac{||(t, x_2, \ldots, x_n)||^2}{(1 + ||(t, x_2, \ldots, x_n)||^2)^2} dt \\ &= 2\frac{\big(2||x||^2 - 2x_1^2 + 1\big)}{\big(||x||^2 - x_1^2 + 1\big)^{3/2}} \arctan \frac{x_1}{\big(||x||^2 - x_1^2 + 1\big)^{1/2}} - \frac{2x_1}{(||x||^2 - x_1^2 + 1)(1 + ||x||^2)}\end{aligned}$$

is bounded. Therefore the preimage through F of the compact subset $[0, M] \times \{0\} \times \cdots \times \{0\}$, where M is an upper-bound of φ, contains $\mathbb{R} \times \{0\} \times \cdots \times \{0\}$, which is an unbounded subset of $\mathbb{R}^n$. This shows that F is not proper.

However the properness of some real valued functions is transferred to their associated vector-valued functions, especially when the variables can be separated and the critical sets are certain products of closed sets from the real line. More precisely we have the following:

Theorem 4.3. *If $K_1, \ldots, K_n \subseteq \mathbb{R}$ are closed subsets whose complements $K_1^c, \ldots, K_n^c$ have unbounded components, both below and above, then the following statements hold:*

1. *$K_1 \times \cdots \times K_n \subseteq \mathbb{R}^n$ is a properly critical subset of $\mathbb{R}^n$.*
2. *$K_1 \times \cdots \times K_n \subseteq \mathbb{R}^n$ is the critical set of a proper vector-valued map $F : \mathbb{R}^n \to \mathbb{R}^n$.*

Proof. (1) Indeed, if $F_i : \mathbb{R} \longrightarrow \mathbb{R}$ is a proper positive smooth function such that $C(F_i) = K_i$, $i = 1, \ldots, n$, then $C(F) = K_1 \times \cdots \times K_n$ (see Remark 3.2(1)), where $F : \mathbb{R}^n \longrightarrow \mathbb{R}$, $F(x_1, \ldots, x_n) = F_1(x_1) + \cdots + F_n(x_n)$. Indeed one successively has:

$$\begin{aligned}&(x_1, \ldots, x_n) \in C(F) \\ \Leftrightarrow\quad & \frac{\partial F}{\partial x_1}(x_1, \ldots, x_n) = \frac{\partial F}{\partial x_2}(x_1, \ldots, x_n) = \cdots = \frac{\partial F}{\partial x_n}(x_1, \ldots, x_n) = 0 \\ \Leftrightarrow\quad & F_1'(x_1) = F_2'(x_2) = \cdots = F_n'(x_n) = 0 \\ \Leftrightarrow\quad & (x_1, \ldots, x_n) \in C(F_1) \times C(F_2) \times \cdots \times C(F_n) = K_1 \times K_2 \times \cdots \times K_n.\end{aligned}$$

In order to show the properness of F it is enough to observe that $|x_i| \longrightarrow \infty$ for at least one $i \in \{1, 2, \dots, n\}$ as $||x|| \longrightarrow \infty$. This shows that $F(x) \longrightarrow \infty$ as $||x|| \longrightarrow \infty$ simply because $F_i(x_i) \longrightarrow \infty$ as $|x_i| \longrightarrow \infty$.

(2) Consider the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $f(x_1, \dots, x_n) = g_{K_1}(x_1) + g_{K_2}(x_2) + \cdots + g_{K_n}(x_n)$, i.e.

$$f(x_1, \dots, x_n) = \int_0^{x_1} \left(f'_{K_1}(t)\right)^2 dt + \int_0^{x_2} \left(f'_{K_2}(t)\right)^2 dt + \cdots + \int_0^{x_n} \left(f'_{K_n}(t)\right)^2 dt.$$

Recall that $C(g_{K_i}) = C(f_{K_i}) = K_i$ and g_{K_i} is proper $i = 1, \dots, n$ (see Theorem 3.1 and Remark 3.2). The required function is $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by

$$\begin{aligned} F(x_1, \dots, x_n) &= \Big(\int_0^{x_1} ||(\nabla f)(t, x_2, \dots, x_n)||^2 dt, x_2, \dots, x_n \Big) \\ &= \left(\int_0^{x_1} \left(f'_{K_1}(t)\right)^2 dt + x_1 \sum_{i=2}^{n} \left(f'_{K_i}(x_i)\right)^2, x_2, \dots, x_n \right), \end{aligned}$$

which is proper since g_{K_1} is proper and

$$\begin{aligned} ||F(x_1, \dots, x_n)||^2 &= \left(\int_0^{x_1} \left(f'_{K_1}(t)\right)^2 dt \right)^2 + 2x_1 \sum_{i=2}^{n} \left(f'_{K_i}(x_i)\right)^2 \int_0^{x_1} \left(f_{K_1}(t)\right)^2 dt \\ &\quad + x_1^2 \left[\left(f'_{K_2}(x_2)\right)^2 + \cdots + \left(f'_{K_n}(x_n)\right)^2 \right]^2 + x_2^2 + \cdots + x_n^2 \\ &\geq \left(\int_0^{x_1} \left(f'_{K_1}(t)\right)^2 dt \right)^2 + \sum_{i=2}^{n} x_i^2 = \left(g_{K_1}(x_1)\right)^2 + \sum_{i=2}^{n} x_i^2. \end{aligned}$$

From the obvious equality $\det (JF)(x) = ||(\nabla f)(x)||^2$, it follows that $C(F) = C(f)$, which combined with the equalities $C(f) = C(g_{K_1}) \times \cdots \times C(g_{K_n}) = K_1 \times \cdots \times K_n$, ensured by the proof of part (1), shows that $K_1 \times \cdots \times K_n$ is the critical set of the proper vector-valued map F. □

Corollary 4.4. *If $K_1, \dots, K_n$ are compact subsets of $\mathbb{R}$, then $K_1 \times \cdots \times K_n$ is a properly critical subset of $\mathbb{R}^n$ as well as the critical set of a proper vector-valued map $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. If $C \subseteq \mathbb{R}^n$ is the middle third Cantor set, then one particularly gets that C^n is a properly critical subset of $\mathbb{R}^n$ as well as the critical set of a proper vector-valued map $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.*

Recall that the middle third Cantor set C is constructed by removing the open middle third interval I_1 from the interval $I_0 := [0, 1]$, removing the middle third of each of the two remaining intervals (I_2), and continuing this procedure ad infinitum. The choice of the initial interval (I_0) to be $[0, 1]$ is not really important and we can actually construct in a similar way a set, say $C_{a,b}$, for every interval $[a, b]$, $a < b$, which has all properties of $C = C_{0,1}$. In fact $C_{a,b}$ is homeomorphic to C for all $a, b \in \mathbb{R}$, $a < b$. In particular $C_{a,b}$ is compact for all $a, b \in \mathbb{R}$, $a < b$, which shows, by Lemma 1.2, that the union $\mathcal{C} := \bigcup_{n \in \mathbb{Z}} C_{2n,2n+1}$ is a closed subset of $\mathbb{R}$. Its complement $\mathcal{C}^c$ is obviously unbounded both below and above.

Corollary 4.5. *$\mathcal{C}^n$ is a properly critical subset of $\mathbb{R}^n$ as well as the critical set of some proper vector-valued map $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.*

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Author's address:

Cornel Pintea
"Babeş-Bolyai" University,
Faculty of Mathematics and Computer Sciences,
M. Kogălniceanu 1, 400084 Cluj-Napoca, Romania.
E-mail: cpintea@math.ubbcluj.ro