Splitting theorems for the double tangent bundles of Fréchet manifolds

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Abstract. In this paper for a vector bundle (v.b.) (E, p, M), we show that there are two splitting theorems for TE at the presence of a connection. In spite of natural difficulties with non-Banach modeled v.b.'s, we generalize these theorems for a wide class of Fréchet v.b.'s i.e. those which can be considered as projective limits of Banach v.b.'s. Afterward using the concept of parallelism we propose an alternative way of studying ordinary differential equations on Banach v.b.'s as well as a suitable basis for further steps. Notwithstanding of the lack of a general solvability-uniqueness theorem for differential equations on non-Banach modeled v.b.'s, we will prove an interesting result for the category of our discussing v.b.'s. For the case of the tangent bundle of a projective system of manifolds as a corollary we observe that according to [1], the connection may be replaced with an equivalence structure like a dissection, Christoffel structure, spray or a Hessian Structure. These splitting theorems have applications in the study of the geometry of bundle of accelerations.

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1 Introduction

The study of Fréchet manifolds was the subject of investigation for many authors due to interaction of this field with modern differential geometry and theoretical physics. (see for example [2], [3], [7], [12], [16], [17], and [20].) In the study of Mathematical Physics one often encounters with manifolds and spaces which are obtained as projective limits. Some of these fields are the loop quantization of Gauge theories as quantum gravity, the 2D Yang-Mills theory [4], [5], and string theory [21]. Moreover projective systems of manifolds arise in other areas of differential geometry like the study of group of diffeomorphisms and the geometry of infinite jets ([18], [23], [22], [25]).

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At the second section we introduce our notation for v.b.'s and manifolds. All the maps, for the sake of simplicity, are assumed to be smooth but less degrees of differentiability may be assumed. In section 3 we state two main theorems for a Banach v.b. $\pi : E \longrightarrow M$ which propose two ways of slitting of TE at the presence of a connection. We used the nominating of tensorial splitting and Dombrowski splitting due to the fact that in the case of $E = TM$ similar theorems are proved in [11] and [15]. Afterward the notion of parallelism on v.b.'s is introduced and the local equations are determined. In fact we propose an alternative way of studying ordinary differential equations on Banach v.b.'s which serves as a suitable basis for the further steps i.e. ordinary differential equations on non-Banach v.b.'s.

Section 4 is devoted to the study of non-Banach v.b.'s and manifolds. There are two main problems with these geometric objects. The first is the pathological structure of $GL(\mathbb{F})$ i.e. the general linear group of a Fréchet space. In fact if $\mathbb{F}$ is non-Banach, $GL(\mathbb{F})$ does not admit a reasonable topological group structure (see [12]. [14]). As we know when we are engaged with v.b.'s with fibres of type $\mathbb{F}$, the transition functions take their values in $GL(\mathbb{F})$ and we fail to have the important geometric objects like v.b's, tangent bundles and frame bundles [10]. The other obstacle is the lack of a general solvability-uniqueness theorem for ordinary differential equations on non-Banach manifolds and even spaces. As we will see, both of these overcome if we restrict ourselves to the category of those Fréchet v.b.'s which may be considered as projective limits of Banach v.b.'s.

As a corollary we prove that for the second order tangent bundle, TTM of a projective limit manifold M there are two different splittings at the presence of a linear connection. Afterward using [1] we prove that the linear connection can be substituted with a Christoffel structure, spray, dissection or a Hessian structure. (In [15] these splitting theorems are proved for Banach manifolds and at the presence of a spray.) Finally flat connection on the Banach and Fréchet model spaces, and the canonical direct connection on Banach and Fréchet Lie groups are used to construct our introduced splittings on their double tangent bundles.

2 Preliminaries

Let $p : E \longrightarrow M$ be a smooth Banach vector bundle with fibres of type $\mathbb{E}$ and $\mathbb{B}$ as the model space of M (both Banach spaces). According to [27] a connection on a smooth vector bundle $p : E \longrightarrow M$ is a smooth splitting of the short exact sequence of bundles on E;

$$0 \longrightarrow VE \xrightarrow{i} TE \xrightarrow{\tilde{p}} p^*TM \longrightarrow 0 \tag{2.1}$$

where VE is the vertical sub-bundle of E, i is the inclusion map, $p^*(TM) \subset E \times TM$ is the pullback bundle induced by the projection $p : E \longrightarrow M$ and $\tilde{p}$ is defined by the tangent map $Tp : TE \longrightarrow TM$.

Since VE is isomorphic to p^*E, there is a canonical isomorphism $r : VE \longrightarrow E$. The connection map $D : TE \longrightarrow E$ is defined to be $D = rv$ where v is the left splitting of (2.1).

The connection map D is fibre preserving for both vector bundle structures of TE, i.e. $\pi_E : TE \longrightarrow E$ and $TP : TE \longrightarrow TM$, and fibre linear for the first mentioned structure. If D is also fibre linear for the second indicated structure, then D is called a linear connection on E.

Let $\Phi = (\phi, \bar{\phi}) : E|_U \longrightarrow \phi(U) \times \mathbb{E}$ be a local trivialization where (U, Φ) is a chart of M and assume that $\Psi = ((\psi, \bar{\psi}), V)$ is another local trivialization with $U \cap V \neq \varnothing$. Then $\Psi \circ \Phi^{-1}(x, v) = ((\psi \circ \phi^{-1})(x), G_{\Psi\Phi}(x)v)$ where $G_{\Psi\Phi} : U \cap V \longrightarrow GL(\mathbb{E}, \mathbb{E})$ is smooth and for any $x \in U \cap V$, $G_{\Psi\Phi}(x)$ is a toplinear isomorphism. The canonical induced trivialization for TE takes the form

$$\begin{aligned} T(\Psi \circ \Phi^{-1})(x, \xi, y, \eta) &= \big((\psi \circ \phi^{-1})(x), G_{\Psi\Phi}(x)\xi, T(\psi \circ \phi^{-1})(x)y, \\ &\quad G_{\Psi\Phi}(x)\eta + TG_{\Psi\Phi}(x)(y, \xi)\big). \end{aligned} \tag{2.2}$$

for $(x, \xi, y, \eta) \in \phi(U) \times \mathbb{E} \times \mathbb{B} \times \mathbb{E}$. By these means on (Φ, U), D has the form

$$D_\Phi(x, \xi, y, \eta) = (x, \eta + \omega_\Phi(x, y)\xi)$$

where $D_\Phi = \Phi \circ D \circ T\Psi^{-1}$ and $\omega_\Phi : \phi(U) \times \mathbb{E} \longrightarrow L(\mathbb{B}, \mathbb{E})$ is smooth and is called the local form of the connection D.

If D is a linear connection on $\pi : E \longrightarrow M$, then for the local forms we have $\omega_\Phi : \phi(U) :\longrightarrow L(\mathbb{E}, L(\mathbb{B}; \mathbb{E}))$. Using (2.2) and the fact $D_\Psi \circ T(\Psi \circ \Phi^{-1}) = (\Psi \circ \Phi^{-1}) \circ D_\Phi$ we see that

$$\begin{aligned} D_\Psi \circ T(\Psi \circ \Phi^{-1})(x, \xi, y, \eta) &= \big((\psi \circ \phi^{-1})(x), G_{\Psi\Phi}(x)\eta + TG_{\Psi\Phi}(x)(y, \xi) \\ &+ \omega_\Psi((\psi \circ \phi^{-1})(x))[T(\psi \circ \phi^{-1})(x)y, G_{\Psi\Phi}(x)\xi]\big). \end{aligned}$$

On the other hand,

$$(\Psi \circ \Phi^{-1}) \circ D_\Phi(x, \xi, y, \eta) = \big((\psi \circ \phi^{-1})(x), G_{\Psi\Phi}(\eta + \omega_\Phi(x)[y, \xi])\big).$$

Comparing the second components of the above equalities for different charts of M yields the following well known local compatibility condition

$$G_{\Psi\Phi}(\omega_\Phi(x)[y, \xi]) = TG_{\Psi\Phi}(x)(y, \xi) + \omega_\Psi((\psi \circ \phi^{-1})(x))[T(\psi \circ \phi^{-1})(x)y, G_{\Psi\Phi}(x)\xi]. \tag{2.3}$$

For the manifold let $\pi_M : TM \longrightarrow M$ be its tangent bundle and $TM = \bigcup_{x \in M} T_xM$ where T_xM consists of all equivalent classes of the form $[c, x]$ such that

$$c \in C_x = \{c : (-\epsilon, \epsilon) \longrightarrow M;\ \epsilon > 0,\ c \text{ smooth and } c(0) = x\},$$

with the equivalence relation $c_1 \sim_x c_2$ iff $c_1'(0) = c_2'(0)$ for $c_1,\ c_2 \in C_x$. If $\mathcal{A} = \{(U_\alpha, \phi_\alpha);\ \alpha \in I\}$ is an atlas for M, then consider the atlases $\mathcal{B} = \{(\pi_M^{-1}(U_\alpha), \Phi_\alpha);\ \alpha \in I\}$ and $\mathcal{C} = \{(\pi_{TM}^{-1}(\pi^{-1}(U_\alpha)), \tilde{\Phi}_\alpha);\ \alpha \in I\}$ for TM and $T(TM)$ respectively.

A connection on a manifold M is a connection on its tangent bundle i.e. a v.b. morphism $D : T(TM) \longrightarrow TM$ with the local representation;

$$\begin{aligned} D_\alpha : \psi_\alpha(U_\alpha) \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} &\longrightarrow \psi_\alpha(U_\alpha) \times \mathbb{E} \\ (y, u, v, w) &\longmapsto (y, w + \omega_\alpha(y, u)v) \end{aligned}$$

where $D_\alpha = \varphi_\alpha \circ \tilde{\varphi}_\alpha^{-1}$ and $\omega_\alpha : \psi_\alpha(U_\alpha) \times \mathbb{E} \longrightarrow \mathcal{L}(\mathbb{E}, \mathbb{E})$, $\alpha \in I$, are the local forms of D. The connection D is linear if $\{\omega_\alpha\}_{\alpha \in I}$ are linear with respect to the second variable. Moreover we can fully determine a linear connection by the family of its Christoffel symbols defined by;

$$\Gamma_\alpha : \psi_\alpha(U_\alpha) \longrightarrow \mathcal{L}(\mathbb{E}, \mathcal{L}(\mathbb{E}, \mathbb{E}));\ \alpha \in I.$$

Using (2.3) we see that the compatibility condition for Christoffel symbols are

$$\sigma'_{\alpha\beta}(x)(\Gamma_\beta(x)[v,z]) = \Gamma_\alpha(\sigma_{\alpha\beta}(x))[\sigma'_{\alpha\beta}(x)(v), \sigma'_{\alpha\beta}(x)(z)] + \sigma''_{\alpha\beta}(x)(v,z), \tag{2.4}$$

where $(x, v, z) \in \psi_\alpha(U_\alpha) \times \mathbb{E} \times \mathbb{E}$.

3 Splitting theorems for Banach manifolds

Suppose that $p : E \longrightarrow M$ is a Banach vector bundle and (U, Φ) a local trivialization on it i.e. $E|_U \simeq \phi(U) \times \mathbb{E}$. Then there is a canonical diffeomorphism between $TE|U$ and $\phi(U) \times \mathbb{E} \times \mathbb{B} \times \mathbb{E}$. Now define the map $(\kappa_1, \kappa_2) : TE \longrightarrow p^*TM \oplus p^*E$ locally by

$$\begin{array}{rcl} \kappa_{1\Phi} : (\phi(U) \times \mathbb{E}) \times \mathbb{B} \times \mathbb{E} & \longrightarrow & (\phi(U) \times \mathbb{E}) \times \mathbb{B} \\ (x, \xi, y, \eta) & \longmapsto & (x, \xi, y) \end{array}$$

and

$$\begin{array}{rcl} \kappa_{2\Phi} : (\phi(U) \times \mathbb{E}) \times \mathbb{B} \times \mathbb{E} & \longrightarrow & (\phi(U) \times \mathbb{E}) \times \mathbb{E} \\ (x, \xi, y, \eta) & \longmapsto & (x, \xi, \eta + \omega_\Phi(x)[y, \xi]). \end{array}$$

Theorem 3.1. *Let $p : E \longrightarrow M$ be a Banach v.b. and D a linear connection on it. Then $(\kappa_1, \kappa_2) : TE \longrightarrow \pi^*TM \oplus \pi^*E$ (Whitney sum) defines a v.b.-isomorphism.*

Proof. Clearly κ is a vector bundle morphism. Moreover (κ_1, κ_2) is smooth and linear on fibres and locally is a bijection. Let (U, Φ) and (V, Ψ) be local trivializations of p with $U \cap V \neq \varnothing$. Then

$$\begin{array}{cl} & (\kappa_{1\Psi}, \kappa_{2\Psi}) \circ T(\Psi \circ \Phi^{-1})(x, \xi, y, \eta) \\ = & (\kappa_{1\Psi}, \kappa_{2\Psi})\Big((\psi \circ \phi^{-1})(x), G_{\Psi\Phi}(x)\xi, T(\psi \circ \phi^{-1})(x)y \\ & , G_{\Psi\Phi}(x)\eta + TG_{\Psi\Phi}(x)(y, \xi)\Big) \\ = & \Big((\psi \circ \phi^{-1})(x), G_{\Psi\Phi}(x)\xi, T(\psi \circ \phi^{-1})(x)y\Big) \oplus \Big((\psi \circ \phi^{-1})(x), G_{\Psi\Phi}(x)\xi \\ & , G_{\Psi\Phi}(x)\eta + TG_{\Psi\Phi}(x)(y, \xi) + \omega_\Psi((\psi \circ \phi^{-1})(x))[T(\psi \circ \phi^{-1})(x)y, G_{\Psi\Phi}(x)\xi]\Big) \\ \overset{*}{=} & \Big((\psi \circ \phi^{-1})(x), G_{\Psi\Phi}(x)\xi, T(\psi \circ \phi^{-1})(x)y\Big) \oplus \Big((\psi \circ \phi^{-1})(x), G_{\Psi\Phi}(x)\xi \\ & , G_{\Psi\Phi}(x)\eta + G_{\Psi\Phi}(\omega_\Phi(x)[y, \xi])\Big) \\ = & (\Psi \circ \Phi^{-1}) \circ (\kappa_{1\Phi}, \kappa_{2\Phi})(x, \xi, y, \eta) \end{array}$$

i.e. (κ_1, κ_2) defines a v.b.-isomorphism. (In * we used the equality 2.3) $\square$

The next useful theorem proposes another splitting for TE. Note that in the case of $E = TM$ this theorem splits TTM to three copies of TM. In fact this is a generalization of Dombrowski's splitting theorem [11] for Banach vector bundles. In [11] the theorem is proved for $E = TM$ with M finite dimensional and this is developed by Lang for Banach manifolds [15].

Theorem 3.2. *Let $p : E \longrightarrow M$ be a Banach vector bundle with a connection D on it. Then the map*

$$(\pi_E, Tp, \kappa_2) : TE \longrightarrow E \oplus TM \oplus E$$

locally defined by $(x, \xi, y, \eta) \longmapsto (x, \xi) \oplus (x, y) \oplus (x, \eta + \omega(x)(y, \xi))$, is a fibre bundle isomorphism.

Proof. It can be easily checked that the map is a fibre bundle morphism and a local bijection. Moreover the compatibility condition ,similar to the previous theorem, yields that these local isomorphisms are compatible and consequently the bundles are isomorphic. □

3.1 Parallelism and ordinary differential equations on Banach vector bundles

If A is a section of $\pi : E \longrightarrow M$ and X a vector field on M, then the covariant derivative $D_X A$ is defined to be $D \circ TA \circ X$. Let $\alpha : (-\epsilon, \epsilon) \longrightarrow M$. Denote the basic section of $T\mathbb{R}$ by $\partial : t \longmapsto (t, 1)$ and put $\alpha'(t) = T\alpha(\partial(t))$.

For a curve c in E we say that c is parallel along α if $\pi \circ c = \alpha$ and $D \circ c' = 0$. If (U, Φ) is a local trivialization for p then $D \circ c' = 0$ locally means that

$$\begin{aligned} &D_U\big((\phi\circ\alpha)(t), (\bar{\phi}\circ\mathbf{c})(t), (\phi\circ\alpha)'(t), (\bar{\phi}\circ\mathbf{c})'(t) = \\ &\big((\phi\circ\alpha)(t), (\bar{\phi}\circ\mathbf{c})'(t) + \omega_\phi((\phi\circ\alpha)(t))[(\phi\circ\alpha)'(t), (\bar{\phi}\circ\mathbf{c})(t)] = 0 \end{aligned} \tag{3.1}$$

where $c(t)$ is considered as $c(t) = (\alpha(t), \mathbf{c}(t))$.
In the case of $E = TM$ where we consider the canonical lift of the curve α, the above equation takes the familiar form

$$(\phi\circ\alpha)''(t) + \omega_\phi((\phi\circ\alpha)(t))[(\phi\circ\alpha)'(t), (\phi\circ\alpha)'(t)] = 0$$

which is the autoparallel (geodesic when D is a metric connection) equation on M.

Here we state the following theorem which is a direct consequence of the existence and uniqueness theorem for solution of ordinary differential equations on Banach manifolds.

Theorem 3.3. *Let $p : E \longrightarrow M$ be a Banach vector bundle and D be a connection on it. Then for a curve α in M and $\xi \in E_{\alpha(0)}$ there exists a unique parallel curve c (in E) along α with $c(0) = \xi$.*

4 The Fréchet case

In the sequel we introduce our notations about a special class of Fréchet manifolds (v.b.'s) which are obtained as projective limits of Banach manifolds (v.b.'s). Suppose that $\{(M^i, \varphi^{ji})\}_{i,j\in\mathbb{N}}$ is a projective system of Banach manifolds with the limit $M = \varprojlim M^i$ such that M^i is modeled on the Banach space $\mathbb{B}^i$ and $\{\mathbb{B}^i, \rho^{ji}\}_{i\in\mathbb{N}}$ also form a projective system of Banach spaces.

Furthermore let for each $x = (x)_{i\in\mathbb{N}} \in M$ there exists a projective system of local charts $\{(U^i, \psi^i)\}_{i\in\mathbb{N}}$ such that $x^i \in U^i$ and $U = \varprojlim U^i$ is open in M.

The vector bundle structure of TM for a Fréchet manifold M links to pathological structure of general linear group $GL(\mathbb{F})$ and this causes troubles. It is shown in [12] that by considering the generalized topological Lie group

$$\mathcal{H}_0(\mathbb{F}) = \{(l^i)_{i\in\mathbb{N}} \in \prod_{i\in I} GL(\mathbb{E}^i) \ : \ \varprojlim l^i \ exists\}$$

rather than $GL(\mathbb{F})$ this obstacle can be solved. Moreover as we will see in the rest of this paper, the problems related to the lack of a general solvability for differential equations on Fréchet manifolds will overcome with the introduced technique.

Suppose that for any $i \in \mathbb{N}$, (E^i, p^i, M^i) is a Banach v.b. on M^i with the fibres of type $\mathbb{E}^i$ where $\{\mathbb{E}^i, \lambda^{ji}\}_{i,j\in\mathbb{N}}$ forms also a projective system of Banach spaces. With these notations we state the following definition;

Definition 4.1. The system $\{(E^i, f^{ji})\}_{i\in\mathbb{N}}$ is called a strong projective system of Banach v.b.'s on $\{(M^i, \varphi^{ji})\}_{i\in\mathbb{N}}$ if for any $(x^i)_{i\in\mathbb{N}}$, there exists a projective system of trivializations (U^i, τ^i) (where $\tau^i : {p^i}^{-1}(U^i) \longrightarrow U^i \times \mathbb{E}^i$ are local diffeomorphisms) of (E^i, p^i, M^i), such that $x^i \in M^i$, $U = \varprojlim U^i$ is open in M and $(\varphi^{ji} \times \lambda^{ji}) \circ \tau^j = \tau^i \circ f^{ji}$ for all $i, j \in \mathbb{N}$ with $j \geq i$.

Remark 4.2. If $\{M^i, \varphi^{ji}\}_{i\in\mathbb{N}}$ is a projective system of Banach manifolds, then $\{TM^i,\}_{i\in\mathbb{N}}$, $\{TTM^i\}_{i\in\mathbb{N}}$ and $\{T^2M^i\}_{i\in\mathbb{N}}$ form strong systems of Banach v.b.'s. (see [9] and [12]). Hence we have concrete examples for our definition.

Remark 4.3. Galanis in [14] defined the strong system of Banach v.b.'s in a same way. The difference is in the base manifolds. More precisely in [14] the base manifold in all v.b.'s is a fixed Banach manifold M. Note that in the case that one needs to study the natural bundles over a projective system of manifolds, as we will see, the base manifold is not fixed and hence we stated our definition with a slight modification. A similar theorem is proved in [14] but with a fixed base manifold.

Proposition 4.4. *Let $\{(E^i, p^i, M^i)\}_{i\in\mathbb{N}}$ be a strong projective system of Banach vector bundles. Then $(\varprojlim E^i, \varprojlim p^i, \varprojlim M^i)$ is a Fréchet vector bundle.*

Proof. We claim that $E = \varprojlim E^i$ is a differentiable manifold modeled on $\varprojlim \mathbb{B}^i \times \varprojlim \mathbb{E}^i$ where $\{(\mathbb{B}^i, \rho^{ji})\}_{i\in\mathbb{N}}$ is the projective system of model spaces of the base manifolds. Let $e = (e^i)_{i\in\mathbb{N}} \in E$ then $x = (x^i = p^i(e^i))_{i\in\mathbb{N}}$ belongs to $M = \varprojlim M^i$ since for $j \geq i$;

$$\varphi^{ji}(x^j) = \varphi^{ji} \circ p^j(e^j) = p^i \circ f^{ji}(e^j) = p^i(e^i) = x^i.$$

Since the projective system is strong then for $x \in M$ there exists a projective system of trivializations $\{(U^i, \tau^i)\}_{i\in\mathbb{N}}$ such that $x^i \in U^i$ and $\varphi^{ji} \times \lambda^{ji} \circ \tau^j = \tau^i \circ f^{ji}$ for $j \geq i$. Define the projective system of charts $\{(V^i, \Phi^i)\}_{i\in\mathbb{N}}$ for $\{E^i\}_{i\in\mathbb{N}}$ to be $V^i = {p^i}^{-1}(U^i)$ and $\Phi^i = ((\phi^i \times id_{\mathbb{E}^i}) \circ \tau^i$ for any $i \in \mathbb{N}$.

It may be checked that these charts form a projective system with the limit ($V = \varprojlim V^i, \Phi = \varprojlim \Phi^i$).

The equality $\varphi^{ji} \circ p^j = p^i \circ f^{ji}$ yields that $p = \varprojlim p^i$ can be defined and p locally is the projection on the first component i.e. $pr^1 \circ \Phi = p$. Also the last equality yields that for any $x \in M$, $\Phi|_{p^{-1}(x)} = \varprojlim \Phi|_{p^{i-1}(x^i)}$ exists and is a linear isomorphism. Due to the pathological structure of $GL(\mathbb{E})$ we restrict ourselves to an appropriate subset of $GL(\mathbb{E})$ say it $\mathcal{H}^0(\mathbb{E})$. More precisely $\mathcal{H}^0(\mathbb{E})$ is the projective limit of Banach Lie groups $\mathcal{H}_i^0(\mathbb{E}^i)$ where

$$\mathcal{H}_i^0(\mathbb{E}^i) = \{(f_1, ..., f_i) \in \prod_{j=1}^{i} \mathbb{E}^j; \quad \lambda^{jk} \circ f_j = f_k \circ \lambda^{jk} \ for k \leq j \leq i\}$$

and consequently $\mathcal{H}^0(\mathbb{E})$ becomes a generalized Fréchet Lie group [13].

To see the differentiability of transition functions consider the projective limit trivializations (U, τ) and $(V, \bar{\tau})$ of $p : E \longrightarrow E$ with $x \in U \cap V$. Then we see that $\lambda^{ji} \circ \bar{\tau}^j \circ {\tau^j}^{-1}|_{x^j} = \bar{\tau}^i \circ {\tau^i}^{-1}|_{x^j} \circ \lambda^{ji}$ and consequently $(\bar{\tau}^1 \circ {\tau^1}^{-1}|_{x^1}, ..., \bar{\tau}^i \circ {\tau^i}^{-1}|_{x^j})$ belongs to $\mathcal{H}_i^0(\mathbb{E}^i)$ for any $i \in \mathbb{N}$. By using the canonical linear map

$$\begin{array}{rcl} \epsilon : \mathcal{H}^0(\mathbb{E}) & \longrightarrow & L(E) \\ (f_i)_{i\in\mathbb{N}} & \longmapsto & \varprojlim f_i \end{array}$$

we conclude that $T_{U\cap V} : U \cap V \longrightarrow GL(\mathbb{E}) : \ (x^i)_{i\in\mathbb{N}} \longmapsto \epsilon \circ (\tau^i|_{x^i})$ is smooth in the sense of [16] and [17]. □

Now we state a slight modified version of Proposition 2.1 of [14] to introduce projective limit of connections.

Proposition 4.5. *Let ($\varprojlim E^i = E, \varprojlim p^i = p, \varprojlim M^i = M$) be an strong P.L.B. v.b.'s. If $\{D^i\}_{i\in\mathbb{N}}$ is a projective system of connections (possibly nonlinear) on this system, then $\varprojlim D^i$ is a connection on $p : E \longrightarrow M$ with the local component $\{\omega_U\}$ given by $\omega_U(x) = \varprojlim \omega^i_{U^i}(x^i)$ where $x = (x^i)_{i\in\mathbb{N}} \in U = \varprojlim U^i$ and $\omega^i_{U^i}$ is the corresponding local components of D^i for any $i \in \mathbb{N}$.*

Proof. The proof follows from the previous theorem and section 2 of [14]. □

Lemma 4.6. *Let $\{(E^i, p^i, M^i)\}_{i\in\mathbb{N}}$ be a strong system of Banach v.b.'s, then $\{{p^i}^*TM^i \oplus {p^i}^*E^i\}$ also forms a strong system of Banach v.b.'s with the limit isomorphic to $p^*TM \oplus p^*E$.*

Proof. First we show that $\{{p^i}^*E^i\}_{i\in\mathbb{N}}$ is a P.L.B. v.b.'s with the limit isomorphic to P^*E. For $j \geq i$ define

$$\begin{array}{rcl} g^{ji} : {p^j}^*E^j & \longrightarrow & {p^i}^*E^i \\ (x^j, e^j, d^j) & \longmapsto & (f^{ji}(x^j, e^j), f^{ji}(x^j, d^j)) \end{array}$$

where $(x^j, e^j, d^j) := ((x^j, e^j), (x^j, d^j)) \in {p^j}^*E^j$. (We will use these notations during the proof alternatively.) Clearly $g^{ik} \circ g^{ji} = g^{jk}$ for $j \geq i \geq k$, and this ensures us that $\{{p^i}^*E^i, g^{ji}\}_{i,j\in\mathbb{N}}$ forms a projective system. Let $\{(U^i, \tau^i)\}_{i\in\mathbb{N}}$ be a system of trivializations for the system $\{(E^i, p^i, M^i)\}_{i\in\mathbb{N}}$. Then the corresponding systems for $\{({p^i}^*E^i, {p^i}^*(p^i) := {p^i}^*, E^i)\}_{i\in\mathbb{N}}$ is $\{({p^i}^{-1}(U^i), {\tau^i}^*)\}_{i\in\mathbb{N}}$ where

$$ {\tau^i}^* : {p^i}^{*-1}({p^i}^{-1}(U^i)) \longrightarrow {p^i}^{-1}(U^i) \times \mathbb{E}^i; \quad (e^i, \bar{e}^i) \longmapsto (\tau^i(e^i), \tau^i(\bar{e}^i)) $$

(If we intend to work with ${p^i}^*TM^i$, like the second part of the proof, we can choose the trivializations on $p^i : E^i \longrightarrow M^i$ with domains contained in domains of charts of M^i and the corresponding natural induced maps.)
Let $e \in E$. As it is shown in prop. 4.4, $(x^i) = (p^i(e^i)) \in M$ and consequently there exists a projective system of trivializations, say $\{(U^i, \tau^i)\}_{i\in\mathbb{N}}$, such that $(\varphi^{ji} \times \lambda^{ji}) \circ \tau^j = \tau^i \circ f^{ji}$ for $j \geq i$. Therefore

$$ \begin{aligned} &(f^{ji} \times \lambda^{ji}) \circ {\tau^j}^*(x^j, e^j, d^j) = \big((\varphi^{ji} \times \lambda^{ji}) \circ \tau^j(x^j, e^j), (\varphi^{ji} \times \lambda^{ji}) \circ \tau^j(x^j, d^j)\big) \\ &= \big(\tau^i \circ f^{ji}(x^j, e^j), \tau^i \circ f^{ji}(x^j, d^j)\big) = {\tau^i}^* \circ g^{ji}(x^j, e^j, d^j). \end{aligned} $$

Hence $\{({p^i}^*E^i, {p^i}^*, E^i)\}_{i\in\mathbb{N}}$ is a strong system of Banach v.b.'s and prop. 3.4. guarantees that $(p^*E = \varprojlim {p^i}^*E^i, p^* = \varprojlim {p^i}^*, E = \varprojlim E^i)$ is a Fréchet v.b.

As it is shown in [12] for the projective system $\{M^i\}_{i\in\mathbb{N}}$, $\{TM^i\}_{i\in\mathbb{N}}$ forms a (strong) system of v.b.'s with the limit TM isomorphic to $\varprojlim TM^i$. Using the first part of this lemma with a modification (as stated above), $\{({p^i}^*TM^i, {p^i}^*, E^i)\}_{i\in\mathbb{N}}$ is also a strong system of v.b.'s with the limit $(p^*TM, p^*(\pi_M), E)$.

After a direct calculation and just writing the steps of the proof for p^*E and p^*TM together, we see that $\{{p^i}^*E^i \oplus {p^i}^*TM^i\}_{i\in\mathbb{N}}$ also satisfies the conditions of prop 4.4 and hence the proof is complete. □

To be more familiar with projective limit manifolds techniques, we state a direct proof for the case of $E = TM$ in appendix.

Theorem 4.7. *Under the assumptions as in the previous lemma, let $\{D^i\}_{i\in\mathbb{N}}$ be a projective system of linear connections with the limit $D = \varprojlim D^i$. Then the map $(\kappa_1, \kappa_2) = \varprojlim(\kappa_1^i, \kappa_2^i) : TE \longrightarrow p^*TM \oplus p^*E$ is a v.b diffeomorphism.*

Proof. For any $i \in \mathbb{N}$, D^i is a linear connection on the Banach v.b. $p^i : E^i \longrightarrow M^i$. Then by theorem 3.1, (κ_1^i, κ_2^i) is a vector bundle isomorphism . On the other hand for any $j \geq i$,

$$ \begin{aligned} &(\varphi^{ji} \times \lambda^{ji} \times \lambda^{ji}) \circ \kappa_2^j(x^j, \xi^j, y^j, \eta^j) = (\varphi^{ji}(x), \lambda^{ji}(\eta^j) + \lambda^{ji}\omega^j(x^j)[y^j, \xi^j]) \\ &= (x^i, \eta^i + \omega^i(x^i)[y^i, \xi^i]) = \kappa_2^i \circ (\varphi^{ji} \times \lambda^{ji} \times \rho^{ji} \times \lambda^{ji})(x^j, \xi^j, y^j, \eta^j) \end{aligned} $$

where $\{\lambda^{ji}\}_{i,j\in\mathbb{N}}$ are the connecting morphisms of $\{\mathbb{E}^i\}_{i\in\mathbb{N}}$. In a similar way $(\varphi^{ji} \times \lambda^{ji} \times \rho^{ji}) \circ \kappa_1^j = \kappa_1^i \circ (\varphi^{ji} \times \lambda^{ji} \times \rho^{ji} \times \lambda^{ji})$, that is$\{(\kappa_1^i, \kappa_2^i)\}_{i\in\mathbb{N}}$ forms a projective system of maps with the limit $(\kappa_1, \kappa_2) = (\varprojlim \kappa_1^i, \varprojlim \kappa_2^i)$. This map is locally given by

$$ (\kappa_1, \kappa_2)(x^i, \xi^i, y^i, \eta^i)_{i\in\mathbb{N}} \longmapsto \big((x^i, \xi^i, y^i) \oplus (x^i, \xi^i, \eta^i + \omega^i(x^i)[y^i, \xi^i])\big)_{i\in\mathbb{N}}. $$

Since (κ_1, κ_2) is the projective limit of v.b.-diffeomorphisms, it is a generalized v.b.-diffeomorphism. □

One can consider the previous assumptions and prove the following theorem for the case of projective limit v.b.'s.

Theorem 4.8. *The projective limit map* $(\pi_E, Tp, \kappa_2) : TE \longrightarrow E \oplus TM \oplus E$ *exists and is a fibre bundle diffeomorphism.*

4.1 Double tangent bundles

Let $\{M^i\}_{i\in\mathbb{N}}$ be a projective system of manifolds with the limit $M = \varprojlim M^i$ and $D = \varprojlim D^i$ be a linear connection on it. Then we have the following generalizations of tensorial and Dombrowski's splitting theorems for Fréchet manifolds.

Theorem 4.9. *Two morphisms* $(\kappa_1, \kappa_2) : TTM \longrightarrow \pi_M^* TM \oplus \pi_M^* TM$ *and* $(\pi_{TM}, T\pi_M, \kappa_2) : TTM \longrightarrow TM \oplus TM \oplus TM$ *exist and are v.b. and fibre bundle diffeomorphisms respectively.*

Remark 4.10. According to [1], the linear connection can be replaced with a Christoffel structure, spray, Hessian structure or a dissection. Note that these structures must be locally associated with bilinear symmetric maps. (For a detailed study in the case of second order structures on Fréchet manifolds see [1]).

4.2 Parallelism and ordinary differential equations on Fréchet vector bundles

Let $\alpha = \varprojlim \alpha^i$ be a curve in M such that for any $i \in \mathbb{N}$, $\alpha^i : (-\epsilon, \epsilon) :\longrightarrow M^i$ is a smooth curve. In spite of the lack of a general solvability-uniqueness theorem for non-Banach manifolds and even spaces we have the following Theorem.

Theorem 4.11. *Let* $(E^i, p^i, M^i)_{i\in\mathbb{N}}$ *be a strong system of Banach v.b.'s and* $D = \varprojlim D^i$ *be a projective limit connection on it. Then for the curve* α *in* M *and* $\xi = (\xi^i)_{i\in\mathbb{N}} \in E_{\alpha(0)}$ *there exists a unique curve* c *along* α *in* E *which is parallel with respect to* D *and* $c(0) = \xi$.

Proof. For any $i \in \mathbb{N}$, D^i is a connection on $p^i : E^i \longrightarrow M^i$ and α^i is a curve in M^i. According to 3.3, for $\xi^i \in E_{\alpha^i(0)}$ there exists a unique curve c^i in E^i with $c^i(0) = \xi^i$ and

$$(\bar{\phi}^i \circ \mathbf{c}^i)'(t) + \omega^i_{U^i}((\phi^i \circ \alpha^i(t)))[(\phi^i \circ \alpha^i)'(t), (\bar{\phi}^i \circ \mathbf{c}^i)(t)] = 0 \tag{4.1}$$

for any local trivialization (U^i, Φ^i). For $j \geq i$, we claim that $f^{ji} \circ c^j = c^i$ and consequently $c = \varprojlim c^i$ exists as a curve in E.

$$\begin{aligned}
&(\bar{\phi}^i \circ f^{ji} \circ \mathbf{c}^j)'(t) + \omega^i_{U^i}((\phi^i \circ \alpha^i)(t))[(\phi^i \circ \alpha^i)'(t), (\bar{\phi}^i \circ f^{ji} \circ \mathbf{c}^j)(t)] \\
&= (\lambda^{ji} \circ \bar{\phi}^j \circ \mathbf{c}^j)'(t) + \omega^i_{U^i}((\phi^i \circ \alpha^i)(t))[(\phi^i \circ \alpha^i)'(t), (\lambda^{ji} \circ \bar{\phi}^j \circ \mathbf{c}^j)(t)] \\
&= \lambda^{ji} \circ (\bar{\phi}^j \circ \mathbf{c}^j)'(t) + \omega^i_{U^i}((\phi^i \circ \alpha^i)(t))[(\phi^i \circ \alpha^i)'(t), \lambda^{ji} \circ (\bar{\phi}^j \circ \mathbf{c}^j)(t)] \\
&= \lambda^{ji} \circ \big((\bar{\phi}^j \circ \mathbf{c}^j)'(t) + \omega^j_{U^j}((\phi^j \circ \alpha^j)(t))[(\phi^j \circ \alpha^j)'(t), (\bar{\phi}^j \circ \mathbf{c}^j)(t)]\big) = 0.
\end{aligned}$$

Furthermore $f^{ji} \circ c^j(0) = f^{ji}(\xi^j) = \xi^i$. Using 3.3 and the fact that for α^i there exists a unique curve c^i with $c^i(0) = \xi^i$, we conclude that $f^{ji} \circ c^j = c^i$. Hence $\{c^i\}_{i\in\mathbb{N}}$ forms

a projective system of curves and $c = \varprojlim c^i$ is a curve in $E = \varprojlim E^i$. On the other hand $p \circ c = (p^i \circ c^i)_{i\in\mathbb{N}} = (\alpha^i)_{i\in\mathbb{N}} = \alpha$ i.e. c is parallel along α.
Let c_1 be another curve in E along α with $D \circ c_1' = 0$ and $c_1(0) = \xi$. For any $i \in \mathbb{N}$, $c_1^i = \lambda^i \circ c_1$ satisfies the equation (4.1) and $c_1^i(0) = \xi^i.(\lambda^i : \varprojlim E^i \longrightarrow E^i$ is the canonical projection.) 3.3 yields that $c_1^i = c_1$ and consequently $c_1 = \varprojlim c_1^i = \varprojlim c^i = c$. Note that the life-time of the solution maybe trivial i.e. just the origin. To avoid this we have to assume some Lipschitz condition for the local components as it is stated in appendix of [1]. □

Example 4.12. Splitting of the second order tangent bundle of Lie groups using the direct connection. Suppose that G is a Banach Lie group with the model space $\mathbb{G}$. Let $\mu : G \times \eth \longrightarrow TG$ be given by $\mu(m, v) = T_e\lambda_m(v)$, where λ_m is the left translation on G and $\eth$ is the Lie algebra of G. According to [26], there exists a unique connection D^G on G which is $(\mu, id_G)-$related to the canonical flat connection on the trivial bundle $E = (G \times \eth, pr_1, G)$. Locally the Christoffel symbols Γ^G of D^G are given by

$$\Gamma^G_\phi(x)(y, \xi) = Df_\phi(x)(y, f_\phi^{-1}(m)(\xi));\ x \in \phi(U),\ y, \xi \in \mathbb{G}$$

where f_ϕ is the local expression of the isomorphism $T_e\lambda_x : T_eG \longrightarrow T_xG$ and (U, ϕ) chart of G. If $G = \varprojlim G_i$ is obtained as projective limit of Banach Lie groups and D^{G_i} is the direct connection on $E^i = (G_i \times \eth_i, pr_1, G_i)$, then $D^G = \varprojlim D^{G_i}$ is exactly the direct connection on $E = (\varprojlim G_i \times \varprojlim \eth_i, pr_1, \varprojlim G_i)$ [13]. Theorems 3.7 and 3.8 show that D^G determines the following diffeomorphisms $(\kappa_1, \kappa_1) : TTG \longrightarrow \pi_G^*TG \oplus \pi_G^*TG$ and $(\pi_{TG}, T\pi_G, \kappa_1) : TTG \longrightarrow TG \oplus TG \oplus TG$ which locally on the chart (U, ϕ) have the form

$$(\kappa_1, \kappa_1)(x, \xi, y.\eta) = \big((x, \xi, y) \oplus (x, \xi, \eta + Df_\phi(x)(y, f_\phi^{-1}(m)(\xi)))\big)$$

and

$$(\pi_{TG}, T\pi_G, \kappa_1)(x, \xi, y, \eta) = (x, \xi) \oplus (x, y) \oplus (x, \eta + Df_\phi(x)(y, f_\phi^{-1}(m)(\xi))).$$

Example 4.13. Flat connections. Let $M = \mathbb{E}$ with the global chart $(\mathbb{E}, id_{\mathbb{E}})$. The canonical flat connection D^C on the trivial bundle $E = (M \times \mathbb{E}, pr_1, M)$ is locally given by the global Christoffel symbol $\{\Gamma^C\}$, where $\Gamma^C(x)(\xi) = 0$, for any $(x, \xi) \in \mathbb{E} \times \mathbb{E}$. Let $M = \mathbb{F} = \varprojlim \mathbb{E}_i$ and consider it with the global chart $(\mathbb{F}, id_{\mathbb{F}}) = \varprojlim(\mathbb{E}_i, id_{\mathbb{E}_i})$. Moreover consider the canonical flat connection $D^C = \varprojlim D_i^C$ given by the form $\Gamma^C = \varprojlim \Gamma_i^C$ on $(M \times \mathbb{F}, pr_1, M)$. Then the diffeomorphisms of theorems 3.7 and 3.8 are given by

$$\begin{aligned}(\kappa_1, \kappa_1)(x, \xi, y.\eta) &= \big((x, \xi, y) \oplus (x, \xi, \eta)\big),\\ (\pi_{TG}, T\pi_G, \kappa_1)(x, \xi, y, \eta) &= (x, \xi) \oplus (x, y) \oplus (x, \eta).\end{aligned}$$

5 Appendix

Lemma 5.1. *Let $\{M^i\}_{in\in\mathbb{N}}$ be a projective system of Banach manifolds and for any $i \in \mathbb{N}$, $\pi^i : TM^i \longrightarrow M^i$ its tangent bundle. Then $\{\pi_i^*TM^i\}_{i\in\mathbb{N}}$ also forms a projective system with the limit isomorphic to $\pi^*TM \equiv \varprojlim \pi_i^*TM^i$ (set theoretically).*

Proof. For i, $j \in \mathbb{N}$ with $j \geq i$ define

$$\begin{array}{rcl} p^{ji}: \pi_j^* TM^j & \longrightarrow & \pi_i * TM^i \\ ([\alpha, x]^j, [\beta, x]^j) & \longmapsto & ([\varphi^{ji} \circ \alpha, \varphi^{ji}(x)]^i, [\varphi^{ji} \circ \beta, \varphi^{ji}(x)]^i) \end{array}$$

where α and β are smooth curves in M^j with $\alpha(0) = \beta(0)$. Clearly p^{ji} are well defined and $p^{ik} \circ p^{ji} = p^{jk}$ for $j \geq i \geq k$. Consequently $\{\pi_i^* TM^i, p^{ji}\}_{i,j \in \mathbb{N}}$ forms a projective system. Let $\varphi^i : M \longrightarrow M^i$ be the canonical projection of M. Define

$$\begin{array}{rcl} p^i : \pi^* TM & \longrightarrow & \pi_i^* TM^i \\ ([\alpha, x], [\beta, x]) & \longmapsto & ([\varphi^i \circ \alpha, \varphi^i(x)]^i, [\varphi^i \circ \beta, \varphi^i(x)]^i). \end{array}$$

Since $p^{ji} \circ p^j = p^i$ then we obtain the mapping

$$\begin{array}{rcl} P : \pi^* TM & \longrightarrow & \varprojlim \pi_i^* TM^i \\ ([\alpha, x], [\beta, x]) & \longmapsto & ([\varphi^i \circ \alpha, \varphi^i(x)]^i, [\varphi^i \circ \beta, \varphi^i(x)]^i)_{i \in \mathbb{N}}. \end{array}$$

This is an injection since $P([\alpha, x], [\beta, x]) = P([\bar{\alpha}, x], [\bar{\beta}, x])$ gives

$$d\varphi^i(\alpha(0))(\alpha'(0)) = (\varphi^i \circ \alpha)'(0) = (\varphi^i \circ \bar{\alpha})'(0) = d\varphi^i(\bar{\alpha}(0))(\bar{\alpha}'(0)).$$

Hence $\alpha'(0) = \bar{\alpha}'(0)$ and in a similar way $\beta'(0) = \bar{\beta}'(0)$. Note that $TM \equiv \varprojlim TM^i$ and $TTM \equiv \varprojlim TTM^i$ via the canonical system of projections $\{d\varphi^i\}_{i \in \mathbb{N}}$ and $\{dd\varphi^i\}_{i \in \mathbb{N}}$ respectively. On the other hand P is also surjective since for $,j \geq i$,

$$([\alpha, x], [\beta, x]) = ([\alpha^i, x^i]^i, [\beta^i, x^i]^i)_{i \in \mathbb{N}} \in \varprojlim \pi_i^* TM^i.$$

Thus we obtain

$$[\varphi^{ji} \circ \alpha^j, \varphi^{ji}(x^j)]^i = [\alpha^i, x^i]^i \; and \; [\varphi^{ji} \circ \beta^j, \varphi^{ji}(x^j)]^i = [\beta^i, x^i]^i.$$

Consequently $x = (x^i)_{i \in \mathbb{N}} \in M = \varprojlim M^i$ and if $(U = \varprojlim U^i, \Psi = \varprojlim \psi^i)$ is a projective limit chart of M through x and $(\pi^{-1}(U) = \varprojlim \pi^{-1}(U^i), T\Psi = \varprojlim T\psi^i)$ is the corresponding chart of TM, then $\rho^{ji}(\psi^j \circ \alpha^j)(0) = (\psi^i \circ \varphi^{ji} \circ \alpha^j)(0) = (\psi^i \circ \alpha^i)(0)$ and

$$\begin{array}{rcl} \rho^{ji}(\psi^j \circ \alpha^j)'(0) & = & \rho^{ji} lim_{t \longrightarrow 0} \dfrac{(\psi^j \circ \alpha^j)(t) - (\psi^j \circ \alpha^j)(0)}{t} \\ & = & lim_{t \longrightarrow 0} \dfrac{\rho^{ji}(\psi^j \circ \alpha^j)(t) - \rho^{ji}(\psi^j \circ \alpha^j)(0)}{t} = (\psi^i \circ \alpha^i)'(0) \end{array}$$

and also $\rho^{ji}(\psi^j \circ \beta)'(0) = (\psi^i \circ \beta^i)'(0)$.
Consequently $u = ((\psi^i \circ \alpha^i)(0))_{i \in \mathbb{N}}$, $v = ((\psi^i \circ \alpha^i)'(0))_{i \in \mathbb{N}}$ and $w = ((\psi^i \circ \beta^i)'(0))_{i \in \mathbb{N}}$ are elements of $\mathbb{E} = \varprojlim \mathbb{E}^i$. Consider the curves $h_1(t) = u + tv$ and $h_2(t) = u + tw$ in $\mathbb{E}$, the corresponding curves to α and β with respect to the chart $(U = \varprojlim U^i, \Psi = \varprojlim \psi^i)$. (in the other words locally define $\alpha(t) = \Psi^{-1} \circ h_1(t)$ and $\beta = \Psi^{-1} \circ h_2(t)$)

Note that $(\varphi^i \circ \alpha)(0) = (\varphi^i \circ \beta)(0) = x^i = \alpha^i(0) = \beta^i(0)$ and

$$\begin{array}{rcl} (\varphi^i \circ \alpha)'(0) & = & ({\psi^i}^{-1} \circ \rho^i \circ h_1)'(0) = T{\psi^i}^{-1}((\rho^i \circ h_1)'(0)) \\ & = & T{\psi^i}^{-1}(\rho^i(v)) = T{\psi^i}^{-1}((\psi^i \circ \alpha^i)'(0) = (\alpha^i)'(0) \end{array}$$

and similarly $(\varphi^i \circ \beta)'(0) = (\beta^i)'(0)$. Hence $\varphi^i \circ \alpha$, $\varphi^i \circ \beta$, α^i and β^i, respectively, are equivalent on M^i for any $i \in \mathbb{N}$ i.e. $P([\alpha, x], [\beta, x]) = ([\alpha^i, x^i], [\beta^i, x^i])_{i\in\mathbb{N}}$. These help us to conclude that P is the desired bijection. □

Here we state a theorem from [12] without proof.

Theorem 5.2. *TM admits an Eréchet type vector bundle structure over M with the structure group $\mathcal{H}^0(\mathbb{E})$ where*

$$\mathcal{H}^0(\mathbb{E}) = \{(f_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} GL(\mathbb{E}^i);\ \ \varprojlim f^i exists\}.$$

More precisely $\mathcal{H}^0(\mathbb{E}) = \varprojlim \mathcal{H}^0_i(\mathbb{E})$ where for any $i \in \mathbb{N}$

$$\mathcal{H}^0_i(\mathbb{E}) = \{(f_1, ..., f_i) \in \prod_{k=1}^{i} GL(\mathbb{E}^i);\ \rho^{jk} \circ f_j = f_k \circ \rho^{jk}\ \ for\ \ \ k \le j \le i\}$$

is a Banach lie group. In an analogous manner $\{TTM^i\}_{i\in\mathbb{N}}$ forms a projective system of manifolds with the limit $TTM = \varprojlim TTM^i$ and the structure group $\mathcal{H}^0(\mathbb{E}\times\mathbb{E})$ and fibres of type $\mathbb{E} \times \mathbb{E}$.

Using the above theorem it can be checked that π^*TM is a Fréchet vector bundle on TM with the structure group $\mathcal{H}^0(\mathbb{E})$.

Study of other types of the connections like [6] may be the subject of interest in this category with generalized Finsler structures. Also in other branches like [8] and [19] we may find applications for our theory.

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