On complete hypersurfaces with two distinct principal curvatures in a hyperbolic space

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Abstract. We investigate complete hypersurfaces in a hyperbolic space with two distinct principal curvatures and constant m-th mean curvature. By using Otsuki's idea, We obtain some global classification results. As their applications, we obtain some global rigidity results for hyperbolic cylinders and obtain some non-existence results.

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1 Introduction

In 1970 Otsuki [3] studied the minimal hypersurfaces in a unit $(n+1)$-sphere $\mathbb{S}^{n+1}(1)$ $(n \geq 3)$ with two distinct principal curvatures and proved that if the multiplicities of the two principal curvatures are both greater than 1, then they are the Clifford minimal hypersurfaces. As for the case when the multiplicity of one of the two principal curvatures is $n-1$, it corresponds to an ordinary differential equation. Recently, there has been a surge of new interest in the theory of hypersurfaces in space forms based on Otsuki's work (see e.g., [1, 2, 4, 5, 6, 7, 8, 9]). The key of the study is to analyze the case when one of the two principal curvatures is simple.

In this paper we focus our interest on hypersurfaces in hyperbolic space. By using Otsuki's idea, we obtain some global classification results for immersed hypersurfaces in $\mathbb{H}^{n+1}(-1)$ of constant m-th mean curvature and two distinct principal curvatures of multiplicities $n-1, 1$. As their applications, we obtain some global rigidity results for hyperbolic cylinders and obtain some non-existence results.

2 The local construction of the isometric immersion

In this section, we shall provide the explicit construction of isometric immersion of hypersurface in $\mathbb{H}^{n+1}(-1)$ with constant m-th mean curvature H_m and two distinct principal curvatures with multiplicities $n-1, 1$. Since the argument is similar to that

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of [8], we shall only give the outline and omit the most of detailed discussions and computations.

Let M be an n-dimensional hypersurface in the hyperbolic space $\mathbb{H}^{n+1}(-1)$ of constant curvature -1. We choose a local orthonormal frame field $e_1, \cdots, e_n, e_{n+1}$ of $\mathbb{H}^{n+1}(-1)$ along M with coframe $\omega_1, \cdots, \omega_{n+1}$ such that, when restricted on M, $e_1, \cdots, e_n$ are tangent to M. It is well-known that there exist local functions h_{ij} such that $\omega_{n+1\ i} = \sum_j h_{ij}\omega_j, (h_{ij} = h_{ji})$ which determines the *second fundamental form* $h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j$ of M. We call the eigenvalues of matrix (h_{ij}) the *principal curvatures* of M. The *mean curvature* of M is given by $H = \frac{1}{n}\mathrm{tr}(h) = \frac{1}{n}\sum_i h_{ii}$. M is said to be of *constant mean curvature* if H is a constant. In particular, when $H = 0$, M is said to be *minimal.* We choose local frame field $e_1, \cdots, e_n$ such that $h_{ij} = \lambda_i\delta_{ij}$. For $1 \le m \le n$, the *m-th mean curvature* H_m of M is defined by

$$C_n^m H_m = \sum_{1 \le i_1 < \cdots < i_m \le n} \lambda_{i_1} \cdots \lambda_{i_m}, \quad C_n^m = \frac{n!}{m!(n-m)!}.$$

The important class of hypersurfaces in the hyperbolic space is the following.

Example 2.1. (The hyperbolic cylinders in $\mathbb{H}^{n+1}(-1)$) For $1 \le k \le n-1, \lambda > 1$, let $M_{k,n-k}(\lambda) = \mathbb{S}^k(\lambda^2 - 1) \times \mathbb{H}^{n-k}(\frac{1}{\lambda^2} - 1)$, where $\mathbb{H}^k(c)$ denotes the k-dimensional hyperbolic space of constant curvature c, while $\mathbb{S}^k(c')$ denotes the k-dimensional sphere of constant curvature c'. We view $x = (x_1, x_2) \in M_{k,n-k}(\lambda)$ as a vector in $\mathbb{R}_1^{n+2} = \mathbb{R}^{k+1} \times \mathbb{R}_1^{n+1-k}$, then $x \in \mathbb{H}^{n+1}(-1)$. This is the standard isometric embedding of $M_{k,n-k}(\lambda)$ into $\mathbb{H}^{n+1}(-1)$ as a hypersurface, and it has two distinct principal curvatures λ of multiplicity k and $\mu = \frac{1}{\lambda}$ of multiplicity $n-k$ (for suitably chosen e_{n+1}) , and clearly $M_{k,n-k}(\lambda)$ has constant m-th mean curvature for all $1 \le m \le n$. We shall refer $M_{k,n-k}(\lambda)$ as the *hyperbolic cylinders* in $\mathbb{H}^{n+1}(-1)$.

It is natural to ask that whether there are hypersurfaces in $\mathbb{H}^{n+1}(-1)$ with two distinct principal curvatures and constant m-th mean curvature other than the hyperbolic cylinders as described in Example 2.1. The answer is negative when the two principal curvatures are both non-simple and two principal curvatures are nonzero when $m \ge 2$. In fact one can prove the following proposition by the similar argument as in [3].

Proposition 2.1. *Let M be a (connected) hypersurface in $\mathbb{H}^{n+1}(-1)$ with two distinct principal curvatures of multiplicities $k, n-k$ and constant m-th mean curvature H_m. If $2 \le k \le n-2$, then M is either locally a hyperbolic cylinder $M_{k,n-k}(\lambda)$ described as in Example 2.1, or M has two distinct principal curvatures $\lambda_1 = \cdots = \lambda_k = 0, \lambda_{k+1} = \cdots = \lambda_n$ and $m > n-k$ (In this case $H_m = 0$).*

Thus, to consider the hypersurfaces with two distinct principal curvatures and constant m-th mean curvature, we need only to deal the case when one of the two principal curvatures is simple. Let M be a (connected) hypersurface in $\mathbb{H}^{n+1}(-1)$ with constant m-th mean curvature H_m and two distinct principal curvatures λ, μ with multiplicities $n-1, 1$. Since the multiplicities are constant, their eigenspaces are completely integrable, and we can show as in [8] that the integral curves corresponding to μ are geodesics, and they are orthogonal trajectories of the family of the integral submanifolds corresponding to λ. Let u be the parameter of arc length of the geodesics corresponding to μ, and we may put $\omega_n = du$. Then $\lambda = \lambda(u)$ is locally a function of u, and by a similar computation as in [8] we get

$$\left(\log|\lambda^m - H_m|^{\frac{1}{n}}\right)'' - \left(\left(\log|\lambda^m - H_m|^{\frac{1}{n}}\right)'\right)^2 - \frac{nH_m - (n-m)\lambda^m}{m\lambda^{m-2}} + 1 = 0. \tag{2.1}$$

By putting $w = |\lambda^m - H_m|^{-\frac{1}{n}}$, (2.1) is reduced to

$$\frac{d^2w}{du^2} = -w\left(\frac{nH_m - (n-m)\lambda^m}{m\lambda^{m-2}} - 1\right). \tag{2.2}$$

Note that

$$w = \begin{cases} (\lambda^m - H_m)^{-\frac{1}{n}}, & \text{for} \quad \lambda^m - H_m > 0, \\ (H_m - \lambda^m)^{-\frac{1}{n}}, & \text{for} \quad \lambda^m - H_m < 0, \end{cases}$$

(2.2) can be rewritten as

$$\frac{d^2w}{du^2} = -wf^+(w)$$

$$:= -w\left(-\frac{(n-m)}{m}(w^{-n} + H_m)^{\frac{2}{m}} + \frac{n}{m}H_m(w^{-n} + H_m)^{\frac{2}{m}-1} - 1\right), \tag{2.3}$$

for $\lambda^m - H_m > 0$, or

$$\frac{d^2w}{du^2} = -wf^-(w)$$

$$:= -w\left(-\frac{(n-m)}{m}(H_m - w^{-n})^{\frac{2}{m}} + \frac{n}{m}H_m(H_m - w^{-n})^{\frac{2}{m}-1} - 1\right), \tag{2.4}$$

for $\lambda^m - H_m < 0$. Integrating (2.3) or (2.4), we get

$$\left(\frac{dw}{du}\right)^2 = C - F^+(w) := C - w^2(w^{-n} + H_m)^{\frac{2}{m}} + w^2 \tag{2.5}$$

for $\lambda^m - H_m > 0$, or

$$\left(\frac{dw}{du}\right)^2 = C - F^-(w) := C - w^2(H_m - w^{-n})^{\frac{2}{m}} + w^2 \tag{2.6}$$

for $\lambda^m - H_m < 0$, where C is the integration constant. We have $\frac{d}{dw}F^+(w) = 2wf^+(w)$ and $\frac{d}{dw}F^-(w) = 2wf^-(w)$. We view $\mathbb{H}^{n+1}(-1)$ as a hypersurface in $\mathbb{R}^{n+2}_1$, then the local orthonormal frame $e_1, \cdots, e_{n+1}$ of $\mathbb{H}^{n+1}(-1)$ along M gives rise to a local frame $e_1, \cdots, e_{n+2}$ of $\mathbb{R}^{n+2}_1$ along M, where $e_{n+2} = x$ is the position vector of M in $\mathbb{R}^{n+2}_1$. By putting

$$W = e_1 \wedge \cdots \wedge e_{n-1} \wedge \left(\left(\log|\lambda^m - H_m|^{\frac{1}{n}}\right)' e_n + \lambda e_{n+1} + e_{n+2}\right),$$

we can get as in [8] that

$$dW = \left(\log|\lambda^m - H_m|^{\frac{1}{n}}\right)' W du. \tag{2.7}$$

(2.7) shows that the n-vector W in $\mathbb{R}^{n+2}_1$ is constant along integral submanifold $M^{n-1}(u)$. Hence there exists an n-dimensional linear subspace $\mathbb{E}^n(u)$ in $\mathbb{R}^{n+2}_1$ containing $M^{n-1}(u)$, and we can argument as in [8] that the curvature of $M^{n-1}(u)$ is

$$K = K(u) = \left(\left(\log|\lambda^m - H_m|^{\frac{1}{n}}\right)'\right)^2 + \lambda^2 - 1 = \frac{C}{w^2}. \tag{2.8}$$

Now we consider the cases when $C < 0$ or $C > 0$. Let us first assume that $C < 0$. In this situation, $\lambda^2 = 1 - \left(\left(\log|\lambda^m - H_m|^{\frac{1}{n}}\right)'\right)^2 + K < 1$, and $\left(\log|\lambda^m - H_m|^{\frac{1}{n}}\right)' e_n + \lambda e_{n+1} + e_{n+2}$ is a timelike vecotr field, and $\mathbb{E}^n(u) \cong \mathbb{R}^n_1$, and thus $M^{n-1}(u) = \mathbb{E}^n(u) \cap \mathbb{H}^{n+1}(-1) \cong \mathbb{H}^{n-1}(K(u))$. The center of $\mathbb{H}^{n-1}(K(u))$ is given by

$$q = q(u) = x + \frac{\left(\log|\lambda^m - H_m|^{\frac{1}{n}}\right)' e_n + \lambda e_{n+1} + e_{n+2}}{K(u)}. \tag{2.9}$$

It is clear that the curve $q = q(u)$ lies in a fixed 2-plane $\mathbb{R}^2$ through the origin of $\mathbb{R}^{n+2}_1$ which is orthogonal to $\mathbb{E}^n(u)$. The tangent vector field of $q = q(u)$ is

$$q'(u) = \frac{(\lambda^2 - 1)e_n - \left(\log|\lambda^m - H_m|^{\frac{1}{n}}\right)' (\lambda e_{n+1} + e_{n+2})}{K(u)}. \tag{2.10}$$

Letting $\bar{e}_{n+1} = \sqrt{\frac{-K}{1+K}} q$, then we can show by using (2.9) and (2.10) that

$$\left\langle \frac{d\bar{e}_{n+1}}{du}, \frac{d\bar{e}_{n+1}}{du} \right\rangle = \frac{-\lambda^2 K}{(1+K)^2}.$$

Thus we can choose a new frame field of $\mathbb{R}^{n+2}_1$ along M as following:

$$\bar{e}_a = e_a, \quad \bar{e}_n = \frac{\left(\log|\lambda^m - H_m|^{\frac{1}{n}}\right)' e_n + \lambda e_{n+1} + e_{n+2}}{\sqrt{-K(u)}},$$

$$\bar{e}_{n+1} = \sqrt{\frac{-K}{1+K}} q, \quad \bar{e}_{n+2} = \frac{1+K}{\lambda\sqrt{-K}} \frac{d\bar{e}_{n+1}}{du}.$$

Then $\bar{e}_{n+1}, \bar{e}_{n+2}$ spans the fixed 2-plane $\mathbb{R}^2$. We can rewrite (2.9) as

$$x = q + \frac{1}{\sqrt{-K}} \bar{e}_n, \tag{2.11}$$

and the curve $q = q(u)$ in $\mathbb{R}^2$ can be expressed by

$$q = \sqrt{\frac{1+K}{-K}} \bar{e}_{n+1}. \tag{2.12}$$

We fix an orthonormal basis $\varepsilon_1, \varepsilon_2$ for $\mathbb{R}^2$ and write

$$\bar{e}_{n+1} = \cos\theta\varepsilon_1 + \sin\theta\varepsilon_2, \quad \bar{e}_{n+2} = -\sin\theta\varepsilon_1 + \cos\theta\varepsilon_2. \tag{2.13}$$

By the definition of $\bar{e}_{n+1}$ and $\bar{e}_{n+2}$ we have

$$\frac{d\theta}{du} = \frac{\lambda\sqrt{-K}}{1+K}. \tag{2.14}$$

Note that (2.11) define an isometric immersion $x : (a,b) \times \mathbb{H}^{n-1}(-1) \to \mathbb{H}^{n+1}(-1)$, here $(a,b) \times \mathbb{H}^{n-1}(-1)$ is endowed with a warped product metric as following:

$$ds^2 = du^2 + \frac{1}{-K} d\bar{s}^2, \tag{2.15}$$

here $d\bar{s}^2$ denotes the standard metric on hyperbolic $(n-1)$-space $\mathbb{H}^{n-1}(-1)$. As usual we write $(a,b) \times_\rho \mathbb{H}^{n-1}(-1)$ when endowed with the metric (2.15), here $\rho = \frac{1}{\sqrt{-K(u)}}$.

Conversely, assume that $w = w(u) : (a,b) \to \mathbb{R}$ be a positive solution of equation (2.5) or (2.6) for some constants H_m and $C < 0$, and we also assume that $\lambda = (w^{-n} + H_m)^{\frac{1}{m}}$ for (2.5) or $\lambda = (H_m - w^{-n})^{\frac{1}{m}}$ for (2.6) is well-defined, and define $K = K(u)$ by (2.8). We consider $\mathbb{H}^{n+1}(-1)$ as $\mathbb{H}^{n+1}(-1) \subset \mathbb{R}^{n+2}_1 = \mathbb{R}^n_1 \times \mathbb{R}^2$ and $\bar{e}_n$ denoting the position vector of $\mathbb{H}^{n-1}(-1)$ in $\mathbb{R}^n_1$, then (2.11) define an immersion $x : (a,b) \times_\rho \mathbb{H}^{n-1}(-1) \to \mathbb{H}^{n+1}(-1)$, where the curve $q = q(u)$ in $\mathbb{R}^2$ is determined by (2.12)-(2.14), and we can show as in [8] that $\lambda = \lambda(u)$ is the principal curvature with multiplicity $n-1$ of the immersion x and it has constant m-th mean curvature H_m.

In the following we assume that $C > 0$. Then $\left(\log|\lambda^m - H_m|^{\frac{1}{n}}\right)' e_n - \lambda e_{n+1} - e_{n+2}$ is a spacelike vecotr field, and $\mathbb{E}^n(u) \cong \mathbb{R}^n$, and thus $M^{n-1}(u) = \mathbb{E}^n(u) \cap \mathbb{H}^{n+1}(-1) \cong \mathbb{S}^{n-1}(K(u))$. The center of $\mathbb{S}^{n-1}(K(u))$ is again given by (2.9), and it lies in a fixed Lorentzian 2-plane $\mathbb{R}^2_1$. Now q is timelike, and we can choose the new frame field of $\mathbb{R}^{n+2}_1$ along M as following:

$$\bar{e}_a = e_a, \quad \bar{e}_n = \frac{-\left(\log|\lambda^m - H_m|^{\frac{1}{n}}\right)' e_n - \lambda e_{n+1} - e_{n+2}}{\sqrt{K(u)}},$$

$$\bar{e}_{n+1} = \sqrt{\frac{K}{1+K}} q, \quad \bar{e}_{n+2} = \frac{1+K}{\lambda\sqrt{K}} \frac{d\bar{e}_{n+1}}{du}.$$

Then $\bar{e}_{n+1}, \bar{e}_{n+2}$ spans the fixed Lorentzian 2-plane $\mathbb{R}^2_1$ with $\bar{e}_{n+1}$ timelike. Now the position vector of M in $\mathbb{R}^{n+2}_1$ can be written as

$$x = q + \frac{1}{\sqrt{K}} \bar{e}_n, \tag{2.16}$$

and the curve $q = q(u)$ in $\mathbb{R}^2_1$ can be expressed by

$$q = \sqrt{\frac{1+K}{K}} \bar{e}_{n+1}. \tag{2.17}$$

We fix an orthonormal basis $\varepsilon_1, \varepsilon_2$ for $\mathbb{R}^2_1$ with ε_1 timelike, and write

$$\bar{e}_{n+1} = \cosh\theta\varepsilon_1 + \sinh\theta\varepsilon_2, \quad \bar{e}_{n+2} = \sinh\theta\varepsilon_1 + \cosh\theta\varepsilon_2. \tag{2.18}$$

By the definition of $\bar{e}_{n+1}$ and $\bar{e}_{n+2}$ we have

$$\frac{d\theta}{du} = \frac{\lambda\sqrt{K}}{1+K}. \tag{2.19}$$

Now (2.16) define an isometric immersion $x : (a,b) \times_{\frac{1}{\sqrt{K}}} \mathbb{S}^{n-1}(1) \to \mathbb{H}^{n+1}(-1)$. Conversely, assume that $w = w(u) : (a,b) \to \mathbb{R}$ be a positive solution of equation (2.5) or (2.6) for some constants H_m and $C > 0$, we have the similar result as the case $C < 0$. In summary, we have the following

Theorem 2.1. *Let M be an n-dimensional spacelike hypersurface immersed into $\mathbb{H}^{n+1}(-1)$ for $n \geq 3$. Assume that M has constant m-th mean curvature H_m and that M has two distinct principal curvatures λ and μ with multiplicities $n-1$ and 1, respectively (when $m \geq 2$, we assume that $\lambda \neq 0$). Then $\lambda = \lambda(u)$ depends only on u, the arc parameter of the integral curves of μ, and $w = |\lambda^m - H_m|^{-\frac{1}{n}}$ satisfies the ordinary differential equation (2.5) or (2.6) for some constant C. Moreover,*
(1) if $C < 0$, then M is locally isometric to $(a,b) \times_\rho \mathbb{H}^{n-1}(-1)$ with $\rho = \frac{1}{\sqrt{-K(u)}}$, and the immersion x of M into $\mathbb{H}^{n+1}(-1)$ is given by (2.11)-(2.14), where $\bar{e}_n$ is the position vector of $\mathbb{H}^{n-1}(-1)$ in $\mathbb{R}^n_1$. Conversely, if $w = w(u) : (a,b) \to \mathbb{R}$ be a positive solution of equation (2.5) or (2.6) for some constants H_m and $C < 0$, and that $\lambda = (w^{-n} + H_m)^{\frac{1}{m}}$ for (2.5) or $\lambda = (H_m - w^{-n})^{\frac{1}{m}}$ for (2.6) is well-defined, and define $K = K(u)$ by (2.8). Then formulas (2.11)-(2.14) defines an isometric immersion $x : (a,b) \times_\rho \mathbb{H}^{n-1}(-1) \to \mathbb{H}^{n+1}(-1)$ which is a hypersurface with constant m-th mean curvature H_m and two distinct principal curvatures one of which is simple;
(2) if $C > 0$, then M is locally isometric to $(a,b) \times_\rho \mathbb{S}^{n-1}(1)$ with $\rho = \frac{1}{\sqrt{K(u)}}$, and the immersion x of M into $\mathbb{H}^{n+1}(-1)$ is given by (2.16)-(2.19). Conversely, if $w = w(u) : (a,b) \to \mathbb{R}$ be a positive solution of equation (2.5) or (2.6) for some constants H_m and $C > 0$, then (2.16)-(2.19) determines an isometric immersion of $(a,b) \times_\rho \mathbb{S}^{n-1}(1)$ into $\mathbb{H}^{n+1}(-1)$ with constant m-th mean curvature H_m and two distinct principal curvatures one of which is simple.

3 Global classification results: $m = 1$

In the following we shall consider the global results, namely, the complete hypersurface in $\mathbb{H}^{n+1}(-1)$ of constant m-th mean curvature and two distinct principal curvatures with one of which is simple. Clearly, it is related to the complete solution of ordinary differential equation (2.5) or (2.6), here we call a solution $w = w(u)$ of (2.5) or (2.6) to be *complete* if it is defined on $\mathbb{R}$. Let us first consider the case when $m = 1$ in this section. In this situation, replace e_{n+1} by $-e_{n+1}$ if necessary, we can always assume that $\lambda - H > 0$, here $H = H_1$ is the mean curvature. That is to say, we need only to consider complete hypersurface M in $\mathbb{H}^{n+1}(-1)$ which satisfies the following

Condition (*): The hypersurface (or the immersion) is of constant mean curvature H and two distinct principal curvatures $\lambda > H, \mu$ with multiplicities $n-1, 1$.

Notice that $\frac{d}{dw}F^+(w) = 2wf^+(w) = 2w(-(n-1)\lambda^2 + nH\lambda - 1)$. Denoting

$$w_* = \left(\frac{-(n-2)H + \sqrt{n^2H^2 - 4(n-1)}}{2(n-1)}\right)^{-1/n},$$

we have following tables:

Table 1: $m = 1, \lambda > H > 1$

w	0	$(0, w_0)$	$w_0 = w_*$	$(w_0, +\infty)$	$+\infty$
$f^+ = f^+(w)$	$-\infty$	< 0	0	> 0	$H^2 - 1$
$F^+ = F^+(w)$	$+\infty$	$\searrow$	$\Lambda = \Lambda(H) > 0$	$\nearrow$	$+\infty$

Table 2: $m = 1, \lambda > H, -\frac{2\sqrt{n-1}}{n} < H \le 1$

w	0	$(0, +\infty)$	$+\infty$
$f^+ = f^+(w)$	$-\infty$	< 0	$H^2 - 1 \le 0$
$F^+ = F^+(w)$	$+\infty$	$\searrow$	$0(H = 1)$ $-\infty(H \ne 1)$

Table 3: $m = 1, \lambda > H = -\frac{2\sqrt{n-1}}{n}$

w	0	$(0, w_0)$	$w_0 = \left(\frac{n-2}{n\sqrt{n-1}}\right)^{-\frac{1}{n}}$	$(w_0, +\infty)$	$+\infty$
$f^+ = f^+(w)$	$-\infty$	< 0	0	< 0	< 0
$F^+ = F^+(w)$	$+\infty$	$\searrow$	$\Lambda = \Lambda(H) < 0$	$\searrow$	$-\infty$

The following theorem can be shown by use of Tables 1-5. Since the proof is similar to that of [8, 9], we omit it.

Theorem 3.1. *Suppose that $n \ge 3$.*
(1) Let $H > 1$. Then for any $C < \Lambda$, there exists no complete positive solution for (2.5). On the other hand, for each $C > \Lambda$, there exists a unique complete positive solution $w = w(u) : \mathbb{R} \to (0, +\infty)$ up to a parameter translation. Each solution is periodical, and it determines an isometric immersion of $\mathbb{R} \times_{\frac{1}{\sqrt{K}}} \mathbb{S}^{n-1}(1)$ into $\mathbb{H}^{n+1}(-1)$ satisfying the condition (*), and the immersion is given by (2.16)-(2.19). There is only a constant solution $w = w_0$ for (2.5) with $C = \Lambda$ which is corresponding to the hyperbolic cylinder $\mathbb{S}^{n-1}(\lambda_0^2 - 1) \times \mathbb{H}^1(\frac{1}{\lambda_0^2} - 1)$, here $\lambda_0 = \frac{nH + \sqrt{n^2H^2 - 4(n-1)}}{2(n-1)}$.
(2) Let $H = 1$. Then for any $C \le 0$, there exists no complete positive solution for (2.5). On the other hand, for each $C > 0$, there exists a unique complete positive solution $w = w(u) : \mathbb{R} \to (0, +\infty)$ up to a parameter translation. The solution $w = w(u)$ can be chosen in a way that it is an even function which is strictly increasing and unbounded on $(0, +\infty)$. Each solution determines an isometric immersion of $\mathbb{R} \times_{\frac{1}{\sqrt{K}}} \mathbb{S}^{n-1}(1)$ into $\mathbb{H}^{n+1}(-1)$ satisfying the condition (*) and the immersion is

Table 4: $m = 1, \lambda > H, -1 < H < -\frac{2\sqrt{n-1}}{n}$

w	0	$(0, w_1)$	w_1	(w_1, w_2)	w_2	$(w_2, +\infty)$	$+\infty$
$f^+ = f^+(w)$	$-\infty$	< 0	0	> 0	0	< 0	$H^2 - 1 < 0$
$F^+ = F^+(w)$	$+\infty$	$\searrow$	$\Lambda_1 < 0$	$\nearrow$	$\Lambda_2 < 0$	$\searrow$	$-\infty$

$$w_1 = \left(\frac{-(n-2)H + \sqrt{n^2H^2 - 4(n-1)}}{2(n-1)} \right)^{-\frac{1}{n}} \qquad w_2 = \left(\frac{-(n-2)H - \sqrt{n^2H^2 - 4(n-1)}}{2(n-1)} \right)^{-\frac{1}{n}}$$

Table 5: $m = 1, \lambda > H, H \leq -1$

w	0	$(0, w_0)$	$w_0 = w_*$	$(w_0, +\infty)$	$+\infty$
$f^+ = f^+(w)$	$-\infty$	< 0	0	> 0	$H^2 - 1$
$F^+ = F^+(w)$	$+\infty$	$\searrow$	$\Lambda = \Lambda(H) < 0$	$\nearrow$	$0 (H = -1)$ $+\infty (H < -1)$

given by (2.16)-(2.19).
(3) Let $-\frac{2\sqrt{n-1}}{n} < H < 1$*. Then for any constant* C *there exists a unique complete positive solution for (2.5) up to a parameter translation with the same property as in (2). If* $C \neq 0$*, it determines an isometric immersion of either* $\mathbb{R} \times_{\frac{1}{\sqrt{K}}} \mathbb{S}^{n-1}(1)$ *into* $\mathbb{H}^{n+1}(-1)$ *by (2.16)-(2.19) when* $C > 0$*, or* $\mathbb{R} \times_{\frac{1}{\sqrt{-K}}} \mathbb{H}^{n-1}(-1)$ *into* $\mathbb{H}^{n+1}(-1)$ *by (2.11)-(2.14) when* $C < 0$*, of condition (*).*
(4) Let $H = -\frac{2\sqrt{n-1}}{n}$*. Then for any* $C \neq \Lambda$*, we have the same conclusion as in (3); There is only a constant solution* $w = w_0$ *for (2.5) with* $C = \Lambda$ *which is corresponding to the hyperbolic cylinder* $\mathbb{H}^{n-1}(-\frac{n-2}{n-1}) \times \mathbb{S}^1(n-2)$*.*
(5) Let $-1 < H < -\frac{2\sqrt{n-1}}{n}$*. Then for any* $C \in \mathbb{R} \backslash (\Lambda_1, \Lambda_2)$*, there is a unique unbounded solution for (2.5) with the same conclusion as in (3); furthermore, for* $C \in (\Lambda_1, \Lambda_2)$*, apartment from the unbounded solution, there exists a unique complete positive periodical solution for (2.5) up to a parameter translation. When* $C \neq 0$*, each complete solution determines an isometric immersion as in (3). The constant solutions* $w = w_1$ *and* $w = w_2$ *correspond to* $C = \Lambda_1$ *and* $C = \Lambda_2$*, and they correspond to the hyperbolic cylinders* $\mathbb{H}^{n-1}(\lambda_i^2 - 1) \times \mathbb{S}^1(\frac{1}{\lambda_i^2} - 1)$ *with* $\lambda_i = w_i^{-n} + H, i = 1, 2$*.*
(6) Let $H = -1$*. Then for any* $C < \Lambda(-1)$*, there exists no complete positive solution for (2.5). On the other hand, for any* $C > 0$*, we have the same conclusion as in (3); and for any* $C \in (\Lambda(-1), 0)$*, there exists a unique complete positive periodical solution for (2.5) which determines an isometric immersion as in (3). There is only a constant solution* $w = w_0$ *for (2.5) with* $C = \Lambda(-1)$ *which is corresponding to the hyperbolic cylinder* $\mathbb{H}^{n-1}(\lambda_0^2 - 1) \times \mathbb{S}^1(\frac{1}{\lambda_0^2} - 1)$*, here* $\lambda_0 = w_0^{-n} + H$*.*
(7) Let $H < -1$*. Then for any* $C < \Lambda(H)$*, there exists no complete positive solution for (2.5). On the other hand, for each* $C > \Lambda(H)$*, there exists a unique complete positive periodical solution up to a parameter translation. If* $C \neq 0$*, each solution*

determines an isometric immersion of either $\mathbb{R}\times_{\frac{1}{\sqrt{K}}}\mathbb{S}^{n-1}(1)$ *into* $\mathbb{H}^{n+1}(-1)$ *for* $C>0$ *by (2.16)-(2.19), or* $\mathbb{R}\times_{\frac{1}{\sqrt{-K}}}\mathbb{H}^{n-1}(-1)$ *into* $\mathbb{H}^{n+1}(-1)$ *for* $C<0$ *by (2.11)-(2.14), of condition (*). There is only a constant solution* $w=w_0$ *for (2.5) with* $C=\Lambda(H)$ *which is corresponding to the hyperbolic cylinder described as in (6).*

4 Global classification results: $m\geq 2$

In this section we shall consider the case when $m\geq 2$. In this situation λ never vanishes unless it equals to zero identically, and in the following we always assume that λ never vanishes, and replace e_{n+1} by $-e_{n+1}$ if necessary, we can always assume that $\lambda>0$. Hence we need only to deal with the following three cases: Case A: $\lambda^m>H_m\geq 0$; Case B: $H_m>\lambda^m>0$ and Case C: $\lambda^m>0>H_m$. For simplicity, We will say that the hypersurface M in $\mathbb{H}^{n+1}(-1)$ or the corresponding immersion *is of property A* (resp. *property B, property C*) If M has constant m-th mean curvature H_m and two distinct principal curvatures λ,μ of multiplicities $n-1,1$ with $\lambda^m>H_m\geq 0$ (resp. $H_m>\lambda^m>0,\lambda^m>0>H_m$). We have the following tables:

Table 6: $m\geq 2,\lambda^m>H_m>1$

w	0	$(0,w_0)$	w_0	$(w_0,+\infty)$	$+\infty$
$f^+=f^+(w)$	$-\infty(m<n)$ $-1(m=n)$	<0	0	>0	$H_m^{\frac{2}{m}}-1$
$F^+=F^+(w)$	$+\infty(m<n)$ $1(m=n)$	$\searrow$	$\Lambda_m>0$	$\nearrow$	$+\infty$

Table 7: $m\geq 2,\lambda^m>H_m,1\geq H_m\geq 0$

w	0	$(0,+\infty)$	$+\infty$
$f^+=f^+(w)$	$-\infty(m<n)$ $-1(m=n)$	<0	$H_m^{\frac{2}{m}}-1$
$F^+=F^+(w)$	$b=\begin{cases}+\infty(m<n)\\1(m=n)\end{cases}$	$\searrow$	$a=\begin{cases}0(H_m=1)\\-\infty(H_m<1)\end{cases}$

Theorem 4.1. *Suppose that* $n\geq 3,m\geq 2$.
(1) For $0\leq H_m\leq 1$*, let* a,b *be given by Table 7. Then for any* $C\in\mathbb{R}\backslash(a,b)$*, there exists no complete positive solution of (2.5), and for any* $C\in(a,b)$*, there exists a unique unbounded complete positive solution of (2.5) up to a parameter translation with the same property as in part (2) of Theorem 3.1, and when* $C\neq 0$*, it determines an isometric immersion of property A described as in part (3) of Theorem 3.1;*
(2) Let $H_m>1$*, and* $2\leq m<n$*. Then for any* $C<\Lambda_m$*, there exists no complete positive solution of (2.5); On the other hand, for each* $C>\Lambda_m$*, there exists a unique*

Table 8: $m \geq 2, H_m > \lambda^m > 0, H_m \geq 1$

w	$H_m^{-\frac{1}{n}}$	$(H_m^{-\frac{1}{n}}, +\infty)$	$+\infty$
$f^- = f^-(w)$	$+\infty(m > 2)$ $\frac{n}{2}H_2 - 1(m = 2)$	> 0	$H_m^{\frac{2}{m}} - 1$
$F^- = F^-(w)$	$-H_m^{-\frac{2}{n}}$	$\nearrow$	$\Lambda = +\infty(H_m > 1)$ $\Lambda = 0(H_m = 1)$

Table 9: $m > 2, H_m > \lambda^m > 0, H_m < 1$ or $1 > H_2 > \frac{2}{n}$ when $m = 2$

w	$H_m^{-\frac{1}{n}}$	$(H_m^{-\frac{1}{n}}, w_0)$	w_0	$(w_0, +\infty)$	$+\infty$
$f^- = f^-(w)$	$+\infty(m > 2)$ $\frac{n}{2}H_2 - 1(m = 2)$	$\searrow$	0	$\searrow$	$H_m^{\frac{2}{m}} - 1$
$F^- = F^-(w)$	$-H_m^{-\frac{2}{n}}$	$\nearrow$	$\Lambda_m < 0$	$\searrow$	$-\infty$

complete periodical positive solution with the same property as in the part (1) of Theorem 3.1. Each solution determines an isometric immersion of $\mathbb{R} \times_{\frac{1}{\sqrt{K}}} \mathbb{S}^{n-1}(1)$ *into* $\mathbb{H}^{n+1}(-1)$ *of Property A which is given by (2.16)-(2.19). There is only a constant solution* $w = w_0$ *for (2.5) with* $C = \Lambda_m$ *which is corresponding to the hyperbolic cylinder* $\mathbb{S}^{n-1}(\lambda_0^2 - 1) \times \mathbb{H}^1(\frac{1}{\lambda_0^2} - 1)$*, here* $\lambda_0 = (w_0^{-n} + H_m)^{\frac{1}{m}}$*;*
(3) Let $H_m > 1$*, and* $m = n$*. Then for any* $C < \Lambda_n$ *or* $C \geq 1$*, there is no complete positive solution of (2.5); and for* $C \in [\Lambda_m, 1)$ *there is a unique complete solution with the same conclusions as in (2).*

Theorem 4.2. *Suppose that* $n \geq 3, m \geq 2$*.*
(1) If $H_m \geq 1$*, then for any constant* C*, there exists no complete positive solution of (2.6); Consequently, there is no complete hypersurface of property B in this case.*
(2) If $0 < H_m < 1$*, then for* $C > \Lambda_m$*, there exists no complete solution of (2.6) with* $w > H_m^{-\frac{1}{n}}$*; on the other hand, for each* $C < \Lambda_m$*, there exists a unique complete positive solution* $w = w(u) : \mathbb{R} \to (H_m^{-\frac{1}{n}}, +\infty)$ *up to a parameter translation. The solution can be chosen so that it is a even function which is strictly increasing and unbounded on* $(0, +\infty)$*, and it determines an isometric immersion of* $\mathbb{R} \times_{\frac{1}{\sqrt{-K}}} \mathbb{H}^{n-1}(-1)$ *into* $\mathbb{H}^{n+1}(-1)$ *of property B. The immersion is given by (2.11)-(2.14). When* $m > 2$ *or* $m = 2$ *and* $H_2 > \frac{2}{n}$*, the constant solution* $w = w_0$ *corresponds to* $C = \Lambda_m$*, and it corresponds to the hyperbolic cylinder* $\mathbb{H}^{n-1}(\lambda_0^2 - 1) \times \mathbb{S}^1(\frac{1}{\lambda_0^2} - 1)$ *with* $\lambda_0 = (H_m - w_0^{-n})^{\frac{1}{m}}$*.*

Theorem 4.3. *Let* $H_m < 0, n \geq 3$*. Then there exists no complete solution of (2.5) with* $0 < w < (-H_m)^{-\frac{1}{n}}$ *for any* C*. Consequently, there exists no complete hypersurface in* $\mathbb{H}^{n+1}(-1)$ *of property C in this case.*

Table 10: $m = 2, \frac{2}{n} \geq H_2 > \lambda^2 > 0$

w	$H_m^{-\frac{1}{n}}$	$(H_m^{-\frac{1}{n}}, +\infty)$	$+\infty$
$f^- = f^-(w)$	$\frac{n}{2}H_2 - 1$	$\searrow$	$H_2 - 1$
$F^- = F^-(w)$	$\Lambda_2 = -H_2^{-\frac{2}{n}}$	$\searrow$	$-\infty$

5 Applications: the characterizations for hyperbolic cylinders and some non-existence results

In this last section we shall use the global classification results to give some characterizations for hyperbolic cylinders in $\mathbb{H}^{n+1}(-1)$ and obtain some non-existence results. We only state the results and omit the proofs.

Theorem 5.1. *Let H be a number with $|H| \leq \frac{2\sqrt{n-1}}{n}$, then the hyperbolic cylinder $\mathbb{H}^{n-1}(-\frac{n-2}{n-1}) \times \mathbb{S}^1(n-2)$ is the only complete hypersurface in $\mathbb{H}^{n+1}(-1)(n \geq 3)$ of constant mean curvature H with two distinct principal curvatures λ, μ satisfying $\inf(\lambda-\mu)^2 > 0$. Consequently, there is no complete hypersurface in $\mathbb{H}^{n+1}(-1)(n \geq 3)$ of constant mean curvature $|H| < \frac{2\sqrt{n-1}}{n}$ with two distinct principal curvatures λ, μ satisfying $\inf(\lambda-\mu)^2 > 0$.*

Theorem 5.2. *Let H be a constant with $|H| > 1$, and M a complete hypersurface in $\mathbb{H}^{n+1}(-1)(n \geq 3)$ of constant mean curvature H and two distinct principal curvatures with multiplicities $n-1, 1$. Set*

$$S_{\pm} = -n + \frac{n^3H^2}{2(n-1)} \mp \frac{n(n-2)}{2(n-1)}|H|\sqrt{n^2H^2 - 4(n-1)}.$$

(1) If the square length of the second fundamental form satisfies $S \leq S_+$ or $S \geq S_-$, then $S = S_+$ or $S = S_-$, and M is isometric to hyperbolic cylinder $\mathbb{S}^{n-1}(\lambda_+^2 - 1) \times \mathbb{H}^1(\frac{1}{\lambda_+^2} - 1)$ or $\mathbb{H}^{n-1}(\lambda_-^2 - 1) \times \mathbb{S}^1(\frac{1}{\lambda_-^2} - 1)$, here

$$\lambda_{\pm} = \frac{n|H| \pm \sqrt{n^2H^2 - 4(n-1)}}{2(n-1)}.$$

(2) If the square length of the second fundamental form is constant, then $S = S_+$ or $S = S_-$, and M is isometric to hyperbolic cylinder $\mathbb{S}^{n-1}(\lambda_+^2 - 1) \times \mathbb{H}^1(\frac{1}{\lambda_+^2} - 1)$ or $\mathbb{H}^{n-1}(\lambda_-^2 - 1) \times \mathbb{S}^1(\frac{1}{\lambda_-^2} - 1)$.

Let w_0 be given by Table 6, and put

$$\lambda_0 = \left(w_0^{-n} + H_m\right)^{\frac{1}{m}}, \tag{5.1}$$

$$S_0 = (n-1)\lambda_0^2 + \left(\frac{nH_m - (n-m)\lambda_0^m}{m\lambda_0^{m-1}}\right)^2. \tag{5.2}$$

Theorem 5.3. *(1) Let $n \geq 3, m \geq 2$, H_m be a number with $|H_m| > 1$, and λ_0, S_0 be given by (5.1) and (5.2). Let M be a complete hypersurface in $\mathbb{H}^{n+1}(-1)$ with constant m-th mean curvature H_m and two distinct principal curvatures one of which is simple. If the square length of the second fundamental form satisfies $S \leq S_0$ or $S \geq S_0$, then $S = S_0$, and M is isometric to the hyperbolic cylinder $\mathbb{S}^{n-1}(\lambda_0^2 - 1) \times \mathbb{H}^1(\frac{1}{\lambda_0^2} - 1)$.*
(2) Let $n \geq 3$ and m be even. Then there exists no complete hypersurface in $\mathbb{H}^{n+1}(-1)$ of constant negative m-th mean curvature and two distinct principal curvatures.

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