Connections on the generalized tangent bundle of a Riemannian manifold

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Abstract. Properties of covariant connections defined on the generalized tangent bundle of a Riemannian manifold are established and their invariance with respect to generalized complex structures induced by a B-field transformation is discussed. The Kähler case is detailed. An extension of the notion of statistical structure to generalized geometry will be defined and a particular example will be given.

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1 Introduction

Generalized complex geometry represents a larger framework, containing both symplectic and complex geometry. *Generalized complex structures* were defined by N. Hitchin [4] and M. Gualtieri who developed Hitchin's ideas in his Ph.D. thesis [3]. The idea is to pass from the tangent and cotangent bundles of a smooth manifold M to the generalized tangent bundle $TM \oplus T^*M$. M. Gualtieri proved that a symplectic or a complex structure on M induces a generalized complex structure, but not any generalized complex structure can be derived from a symplectic or a complex one. Precisely, if ω (respectively, J) is a symplectic (respectively, a complex) structure on M, then

$$\mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} [respectively,\ \mathcal{J}_J := \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}]$$

is a generalized complex structure, called of symplectic (respectively, of complex) type. Examples of generalized complex structures which don't derive from a symplectic or a complex one can be found in [3].

In what follows, we shall define two operators having properties of covariant connections (and shall call them also covariant connections) on the generalized tangent bundle and prove invariance properties of these connections depending on the additional structure of the manifold.

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The Courant bracket defined on smooth sections of $TM \oplus T^*M$ generalizes the Lie bracket on vector fields. A specific property of the Courant bracket is that it admits other symmetries besides the diffeomorphisms, namely, the B-field transformations. In the present paper, we shall use a different bracket [5] on smooth sections of the generalized tangent bundle and find conditions on the 2-form B such that B-field transformations to constitute symmetries for it (see Proposition 3.1).

For the case when the generalized complex structure is of complex type, $\mathcal{J}_J$, using the B-field transformation $e^B := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$ [where B is viewed as a map from $\Gamma(TM)$ to $\Gamma(T^*M)$], we shall prove that, under certain assumptions, the connections defined are invariant with respect to the new generalized complex structure $(\mathcal{J}_J)_B := e^B \mathcal{J}_J e^{-B}$ obtained from $\mathcal{J}_J$. The same for the case when the generalized complex structure is of symplectic type.

2 Invariant connections on $TM \oplus T^*M$

The notion of *dual connections* often appears in the context of statistical mathematics, giving rise to *dual statistical manifolds*.

Let ∇ and ∇' be dual connections on the Riemannian manifold (M, g) [that is, $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla'_X Z)$, for any X, Y, $Z \in \Gamma(TM)$] and consider their extensions:

$$\tilde{\nabla} : \Gamma(TM) \times \Gamma(T^*M) \to \Gamma(T^*M), \quad (\tilde{\nabla}_X \alpha)(Y) := X(\alpha(Y)) - \alpha(\nabla_X Y)$$

and

$$\bar{\nabla} : \Gamma(T^*M) \times \Gamma(T^*M) \to \Gamma(T^*M), \quad \bar{\nabla}^\alpha \beta := \tilde{\nabla}_{\sharp\alpha} \beta,$$

where $\sharp$ is the inverse of the isomorphism $\flat(X) := i_X g$, $X \in \Gamma(TM)$.

Define now $\hat{\nabla}^*$ and $\check{\nabla}^*$ two connections on $\mathcal{S} := \{X + \alpha \in \Gamma(TM \oplus T^*M) : i_X g = \alpha\}$, respectively, by the relations

$$\hat{\nabla}^*_{X+\alpha} Y + \beta := \nabla_X Y + \bar{\nabla}'^{\alpha} \beta, \quad \check{\nabla}^*_{X+\alpha} Y + \beta := \nabla'_X Y + \bar{\nabla}^\alpha \beta.$$

An invariance property of these connections with respect to the generalized complex structures of symplectic and respectively, of complex type, is given by the following theorem:

Theorem 2.1. *1. If J is a complex structure on the Riemannian manifold (M, g) such that $\mathcal{J}_J(\mathcal{S}) \subset \mathcal{S}$ and ∇ and ∇' are J-invariant, then $\mathcal{J}_J \hat{\nabla}^* = \hat{\nabla}^* \mathcal{J}_J$ and $\mathcal{J}_J \check{\nabla}^* = \check{\nabla}^* \mathcal{J}_J$, where $\mathcal{J}_J := \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$ is the generalized complex structure induced by J;*

2. If ω is a symplectic form on the Riemannian manifold (M, g) such that $\mathcal{J}_\omega(\mathcal{S}) \subset \mathcal{S}$ and ω is ∇- and ∇'-parallel, then $\mathcal{J}_\omega \hat{\nabla}^* = \check{\nabla}^* \mathcal{J}_\omega$ and $\mathcal{J}_\omega \check{\nabla}^* = \hat{\nabla}^* \mathcal{J}_\omega$, where $\mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$ is the generalized complex structure induced by ω.

Proof. Let $X + \alpha$, $Y + \beta \in \mathcal{S}$. Then

1.
$$\begin{aligned} \mathcal{J}_J(\hat{\nabla}^*_{X+\alpha}Y+\beta) &:= \mathcal{J}_J(\nabla_X Y + \bar{\nabla}'^{\alpha}\beta) \\ &:= J(\nabla_X Y) - J^*(\bar{\nabla}'^{\alpha}\beta) \\ &= \nabla_X JY - \bar{\nabla}'^{\alpha} J^*\beta \\ &:= \hat{\nabla}^*_{X+\alpha}\mathcal{J}_J(Y+\beta). \end{aligned}$$

Similarly for $\check{\nabla}^*$;

2.
$$\begin{aligned} \mathcal{J}_\omega(\hat{\nabla}^*_{X+\alpha}Y+\beta) &:= \mathcal{J}_\omega(\nabla_X Y + \bar{\nabla}'^{\alpha}\beta) \\ &:= -\omega^{-1}(\bar{\nabla}'^{\alpha}\beta) + \omega(\nabla_X Y), \end{aligned}$$

$$\begin{aligned} \check{\nabla}^*_{X+\alpha}\mathcal{J}_\omega(Y+\beta) &:= \check{\nabla}^*_{X+\alpha}(-\omega^{-1}(\beta)+\omega(Y)) \\ &:= \nabla'_X(-\omega^{-1}(\beta)) + \bar{\nabla}^{\alpha}\omega(Y). \end{aligned}$$

Let $\omega^{-1}(\bar{\nabla}'^{\alpha}\beta) =: Z$. Then $\bar{\nabla}'^{\alpha}\beta = \omega(Z)$ and for any $W \in \Gamma(TM)$, $(\bar{\nabla}'^{\alpha}\beta)(W) = \omega(Z,W)$ equivalent $X(\beta(W)) - \beta(\nabla'_X W) = \omega(Z,W)$. But $\omega(\nabla'_X(\omega^{-1}(\beta)), W) := -(\nabla'\omega)(X,\omega^{-1}(\beta),W) + X(\omega(\omega^{-1}(\beta),W)) - \omega(\omega^{-1}(\beta),\nabla'_X W) = -(\nabla'\omega)(X,\omega^{-1}(\beta),W) + X(\beta(W)) - \beta(\nabla'_X W)$. For $\nabla'\omega = 0$ follows $Z = \nabla'_X(\omega^{-1}(\beta))$.

Also notice that for any $W \in \Gamma(TM)$,

$$\begin{aligned} (\omega(\nabla_X Y))(W) &= \omega(\nabla_X Y, W) \\ &:= -(\nabla\omega)(X,Y,W) + X(\omega(Y,W)) - \omega(Y,\nabla_X W) \\ &= -(\nabla\omega)(X,Y,W) + X((\omega(Y))(W)) - \omega(Y)(\nabla_X W) \\ &:= -(\nabla\omega)(X,Y,W) + (\bar{\nabla}^{\alpha}\omega(Y))(W). \end{aligned}$$

For $\nabla\omega = 0$ follows $\omega(\nabla_X Y) = \bar{\nabla}^{\alpha}\omega(Y)$.
Similarly for the other relation.

□

3 Invariance under a B-field transformation

Let B be a 2-form and ∇ a flat connection on M. Consider the bracket $[X+\alpha, Y+\beta]_\nabla := [X,Y] + \tilde{\nabla}_X\beta - \tilde{\nabla}_Y\alpha$ [5] and the B-field transformation $e^B := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$. Besides the diffeomorphisms, the bracket $[\cdot,\cdot]_\nabla$ has these B-field transformations as symmetries, if we require for B to satisfy a certain property, stated in the following proposition:

Proposition 3.1. *A necessary and sufficient condition for the B-field transformation to be a symmetry of $[\cdot,\cdot]_\nabla$ is to satisfy*

$$B(T_\nabla(X,Y),Z) = (\nabla B)(Y,X,Z) - (\nabla B)(X,Y,Z),$$

for any X, Y, $Z \in \Gamma(TM)$.

Proof. Let X, $Y \in \Gamma(TM)$ and α, $\beta \in \Gamma(T^*M)$. Then

$$\begin{aligned} e^B([X+\alpha, Y+\beta]_\nabla) &:= e^B([X,Y] + \tilde{\nabla}_X\beta - \tilde{\nabla}_Y\alpha) \\ &:= [X,Y] + B([X,Y]) + \tilde{\nabla}_X\beta - \tilde{\nabla}_Y\alpha \end{aligned}$$

and

$$\begin{aligned} [e^B(X+\alpha), e^B(Y+\beta)]_\nabla &:= [X + B(X) + \alpha, Y + B(Y) + \beta]_\nabla \\ &:= [X,Y] + \tilde{\nabla}_X(B(Y)+\beta) - \tilde{\nabla}_Y(B(X)+\alpha) \\ &= [X,Y] + \tilde{\nabla}_X(B(Y)) - \tilde{\nabla}_Y(B(X)) + \tilde{\nabla}_X\beta - \tilde{\nabla}_Y\alpha. \end{aligned}$$

But for any $Z \in \Gamma(TM)$

$$\begin{aligned} (\tilde{\nabla}_X(B(Y)))(Z) &- (\tilde{\nabla}_Y(B(X)))(Z) \\ &:= X(B(Y,Z)) - B(Y, \nabla_X Z) - Y(B(X,Z)) + B(X, \nabla_Y Z) \\ &:= (\nabla B)(X,Y,Z) + B(\nabla_X Y, Z) \\ &- (\nabla B)(Y,X,Z) - B(\nabla_Y X, Z) \\ &= (\nabla B)(X,Y,Z) - (\nabla B)(Y,X,Z) \\ &+ B(T_\nabla(X,Y),Z) + B([X,Y],Z) \end{aligned}$$

and therefore $(\nabla B)(X,Y,Z) - (\nabla B)(Y,X,Z) + B(T_\nabla(X,Y),Z) = 0$, for any X, Y, $Z \in \Gamma(TM)$. $\square$

Theorem 3.2. *If J is a complex structure on the Riemannian manifold (M,g) such that $(\mathcal{J}_J)_B(\mathcal{S}) \subset \mathcal{S}$, ∇ and ∇' are J-invariant and B satisfies $(\nabla' B)(X,JY,Z) = -(\nabla' B)(X,Y,JZ)$, for any X, Y, $Z \in \Gamma(TM)$ [respectively, $(\nabla B)(X,JY,Z) = -(\nabla B)(X,Y,JZ)$, for any X, Y, $Z \in \Gamma(TM)$], then $(\mathcal{J}_J)_B\hat{\nabla}^* = \hat{\nabla}^*(\mathcal{J}_J)_B$ [respectively, $(\mathcal{J}_J)_B\check{\nabla}^* = \check{\nabla}^*(\mathcal{J}_J)_B$], where $(\mathcal{J}_J)_B := e^B\mathcal{J}_J e^{-B}$, for $\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$ the generalized complex structure induced by J.*

Proof. We have $(\mathcal{J}_J)_B = \begin{pmatrix} J & 0 \\ BJ + J^*B & -J^* \end{pmatrix}$.

Let $X+\alpha$, $Y+\beta \in \mathcal{S}$. Then

$$\begin{aligned} (\mathcal{J}_J)_B(\hat{\nabla}^*_{X+\alpha}Y+\beta) &:= (\mathcal{J}_J)_B(\nabla_X Y + \bar{\nabla}'^{\alpha}\beta) \\ &:= J(\nabla_X Y) + [B(J(\nabla'_X Y)) + J^*(B(\nabla'_X Y)) - J^*(\bar{\nabla}'^{\alpha}\beta)] \\ &= \nabla_X JY + [B(\nabla'_X JY) + J^*(B(\nabla'_X Y)) - \bar{\nabla}'^{\alpha} J^*\beta] \end{aligned}$$

and respectively,

$$\begin{aligned} \hat{\nabla}^*_{X+\alpha}(\mathcal{J}_J)_B(Y+\beta) &:= \hat{\nabla}^*_{X+\alpha}(JY + B(JY) + J^*(B(Y)) - J^*\beta) \\ &:= \nabla_X JY + \bar{\nabla}'^{\alpha}[B(JY) + J^*(B(Y)) - J^*\beta]. \end{aligned}$$

But for any $Z \in \Gamma(TM)$

$$B(\nabla'_X JY, Z) + (J^*(B(\nabla'_X Y)))(Z) := B(\nabla'_X JY, Z) + B(\nabla'_X Y, JZ)$$

and

$$\begin{aligned}(\bar{\nabla}'^{\alpha}[B(JY)+J^*(B(Y))])(Z) &:= X(B(JY,Z)+B(Y,JZ))\\ &- B(JY,\nabla'_X Z)-B(Y,J(\nabla'_X Z))\\ &:= (\nabla' B)(X,JY,Z)+B(\nabla'_X JY,Z)\\ &+ (\nabla' B)(X,Y,JZ)+B(\nabla'_X Y,JZ)\end{aligned}$$

from where we get the required relation. □

The next theorem gives the condition which should be satisfied by the connection ∇ (if we take $\nabla'=\nabla$) and by the 2-form B such that the connection $\hat{\nabla}^*_{X+\alpha}Y+\beta := \nabla_X Y+\bar{\nabla}^{\alpha}\beta$ to be $(\mathcal{J}_\omega)_B$-invariant, where $(\mathcal{J}_\omega)_B := e^B\mathcal{J}_\omega e^{-B}$.

Theorem 3.3. *If ω is a symplectic form on the Riemannian manifold (M,g) such that $(\mathcal{J}_\omega)_B(\mathcal{S})\subset\mathcal{S}$ and ω and B are ∇-parallel, then $(\mathcal{J}_\omega)_B\hat{\nabla}^*=\hat{\nabla}^*(\mathcal{J}_\omega)_B$, where $(\mathcal{J}_\omega)_B := e^B\mathcal{J}_\omega e^{-B}$, for $\mathcal{J}_\omega=\begin{pmatrix}0 & -\omega^{-1}\\ \omega & 0\end{pmatrix}$ the generalized complex structure induced by ω.*

Proof. We have $(\mathcal{J}_\omega)_B=\begin{pmatrix}\omega^{-1}B & -\omega^{-1}\\ \omega+B\omega^{-1}B & -B\omega^{-1}\end{pmatrix}$.

Let $\bar{X}:=X+\alpha$, $\bar{Y}:=Y+\beta\in\mathcal{S}$. Then

$$\begin{aligned}(\mathcal{J}_\omega)_B(\hat{\nabla}^*_{\bar{X}}\bar{Y}) &:= (\mathcal{J}_\omega)_B(\nabla_X Y+\bar{\nabla}^{\alpha}\beta)\\ &:= \omega^{-1}(B((\nabla_X Y))-\bar{\nabla}^{\alpha}\beta)+\omega(\nabla_X Y)+B(\omega^{-1}(B(\nabla_X Y)-\bar{\nabla}^{\alpha}\beta))\end{aligned}$$

and respectively,

$$\begin{aligned}\hat{\nabla}^*_{\bar{X}}(\mathcal{J}_\omega)_B\bar{Y} &:= \hat{\nabla}^*_{X+\alpha}(\omega^{-1}(B(Y)-\beta)+\omega(Y)+B(\omega^{-1}(B(Y))-B(\omega^{-1}(\beta))))\\ &:= \nabla_X(\omega^{-1}(B(Y)-\beta))+\bar{\nabla}^{\alpha}[\omega(Y)+B(\omega^{-1}(B(Y)-\beta))].\end{aligned}$$

But for any $Z\in\Gamma(TM)$, according to the computations from Theorem 2.1

$$\begin{aligned}\omega(\nabla_X(\omega^{-1}(B(Y)-\beta)),Z) &= -(\nabla\omega)(X,\omega^{-1}(B(Y)-\beta),Z)\\ &+ X((B(Y)-\beta)(Z))-(B(Y)-\beta)(\nabla_X Z)\\ &= X((B(Y))-X(\beta(Z))-B(Y,\nabla_X Z)+\beta(\nabla_X Z)\\ &:= (\nabla B)(X,Y,Z)+B(\nabla_X Y,Z)-X(\beta(Z))+\beta(\nabla_X Z)\\ &= B(\nabla_X Y,Z)-X(\beta(Z))+\beta(\nabla_X Z).\end{aligned}$$

On the other hand,

$$B(\nabla_X Y,Z)-(\bar{\nabla}^{\alpha}\beta)(Z) := B(\nabla_X Y,Z)-X(\beta(Z))+\beta(\nabla_X Z).$$

Let us notice that for any $Z\in\Gamma(TM)$

$$\begin{aligned}(\bar{\nabla}^{\alpha}[\omega(Y) &+ B(\omega^{-1}(B(Y)-\beta))])(Z)\\ &:= (\nabla\omega)(X,Y,Z)+\omega(\nabla_X Y,Z)+X(B(\omega^{-1}(B(Y)-\beta),Z))\\ &- B(\omega^{-1}(B(Y)-\beta),\nabla_X Z)\\ &= \omega(\nabla_X Y,Z)+X(B(\omega^{-1}(B(Y)-\beta),Z))\\ &- B(\omega^{-1}(B(Y)-\beta),\nabla_X Z).\end{aligned}$$

But

$$\begin{aligned}
X(B(\omega^{-1}(B(Y)-\beta),Z)) &- B(\omega^{-1}(B(Y)-\beta),\nabla_X Z)\\
&:= (\nabla B)(X,\omega^{-1}(B(Y)-\beta),Z)\\
&+ B(\nabla_X(\omega^{-1}(B(Y)-\beta)),Z)\\
&= B(\nabla_X(\omega^{-1}(B(Y)-\beta)),Z)\\
&= B(\omega^{-1}(B(\nabla_X Y)-\bar{\nabla}^{\alpha}\beta),Z).
\end{aligned}$$

□

4 Kähler case

We shall consider now the Kähler case. Recall that (M,J,g) is a *Hermitian manifold* if M is a smooth manifold, J a complex structure and g a Riemannian metric on M such that $g(JX,JY)=g(X,Y)$, for any X, $Y\in\Gamma(TM)$. We say that (M,J,g) is *Kähler manifold* if it is Hermitian manifold such that the Levi-Civita connection ∇ associated to g satisfies $\nabla J=0$ and the 2-form $\omega(X,Y):=g(X,JY)$ is closed. First, remark the next two results stated in the following lemma:

Lemma 4.1. *If ∇ and ∇' are dual connections on the Riemannian manifold (M,g), then:*

1. $g(T_{\nabla}(X,Y),Z)=g(T_{\nabla'}(X,Y),Z)+(\nabla' g)(X,Y,Z)-(\nabla' g)(Y,X,Z)$*, for any* X, Y, $Z\in\Gamma(TM)$*;*
2. $\nabla g=0$ *if and only if* $\nabla' g=0$.

Proof. From the compatibility condition of ∇ and ∇' follows

1.

$$\begin{aligned}
g(T_{\nabla}(X,Y),Z) &:= g(\nabla_X Y-\nabla_Y X-[X,Y],Z)\\
&= X(g(Y,Z))-g(Y,\nabla'_X Z)-Y(g(X,Z))+g(X,\nabla'_Y Z)\\
&+ g(T_{\nabla'}(X,Y)-\nabla'_X Y-\nabla'_Y X,Z)\\
&:= g(T_{\nabla'}(X,Y),Z)+(\nabla' g)(X,Y,Z)-(\nabla' g)(Y,X,Z),
\end{aligned}$$

for any X, Y, $Z\in\Gamma(TM)$;

2.

$$\begin{aligned}
(\nabla' g)(X,Y,Z) &:= X(g(Y,Z))-g(\nabla'_X Y,Z)-g(Y,\nabla'_X Z)\\
&:= X(g(Y,Z))-X(g(Z,Y))+g(\nabla_X Z,Y)\\
&- X(g(Y,Z))+g(\nabla_X Y,Z)\\
&:= -(\nabla g)(X,Y,Z),
\end{aligned}$$

for any X, Y, $Z\in\Gamma(TM)$.

□

If (M, J, g) is a Kähler manifold, and ∇' is a dual connection of the Levi-Civita connection ∇ associated to g, then from the previous lemma follows that $\nabla' \equiv \nabla$ and therefore $\hat{\nabla}^*_{X+\alpha} Y + \beta := \nabla_X Y + \bar{\nabla}^\alpha \beta$.

Corollary 4.2. *Assume that (M, J, g, ω) is Kähler manifold and ∇ is the Levi-Civita connection associated to g. Then:*

1. *(a) $\mathcal{J}_J \hat{\nabla}^* = \hat{\nabla}^* \mathcal{J}_J$, where $\mathcal{J}_J := \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$ is the generalized complex structure induced by J;*

   (b) $\mathcal{J}_\omega \hat{\nabla}^* = \hat{\nabla}^* \mathcal{J}_\omega$, where $\mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$ is the generalized complex structure induced by ω;

2. *for B a 2-form on M and $e^B := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$ the corresponding B-field transformation*

   (a) if B satisfies $(\nabla B)(X, JY, Z) = -(\nabla B)(X, Y, JZ)$, for any X, Y, $Z \in \Gamma(TM)$, then $(\mathcal{J}_J)_B \hat{\nabla}^* = \hat{\nabla}^* (\mathcal{J}_J)_B$, where $(\mathcal{J}_J)_B := e^B \mathcal{J}_J e^{-B}$;

   (b) if B is ∇-parallel, then $(\mathcal{J}_\omega)_B \hat{\nabla}^* = \hat{\nabla}^* (\mathcal{J}_\omega)_B$, where $(\mathcal{J}_\omega)_B := e^B \mathcal{J}_\omega e^{-B}$.

Proof. Follows from Theorems 3.3 and 3.2. □

For (M, J, g, ω) a Kähler manifold, the two generalized complex structures $(\mathcal{J}_J, \mathcal{J}_\omega)$ form a *generalized Kähler structure* (i.e. a pair (J_1, J_2) of commuting generalized complex structures such that $G := -J_1 J_2$ is a positive defined metric on $TM \oplus T^*M$, called *generalized Kähler metric* [3]). In our case, $G = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$ and moreover, $((\mathcal{J}_J)_B, (\mathcal{J}_\omega)_B)$ is also generalized Kähler structure with the corresponding generalized Kähler metric $G_B := -(\mathcal{J}_J)_B (\mathcal{J}_\omega)_B = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}$.

It was proved [3] that any generalized Kähler metric is uniquely determined by a Riemannian metric g and a 2-form B and its torsion is the 3-form $h = dB$.

For the case when a pair (J_1, J_2) of almost complex structures anticommutes, V. Oproiu constructed a family of almost hyper-complex structures on TM and proved [6] that if (M, J, g) is a Kähler manifold and the almost hyper-complex structure defined by (J_1, J_2) is integrable, then M has constant sectional curvature (see Proposition 4 from [6]). He also proved that if the natural Riemannian metric on TM induced by g is almost Hermitian with respect to the almost complex structures J_1 and J_2, then the induced hyper-complex structure is hyper-Kähler if and only if J_1 and J_2 are integrable (see Theorem 5 from [6]). A similar condition for a natural lifted Hermitian structure on T^*M to be Kähler was given by S.-L. Druţă [2].

5 A generalized statistical structure

Statistical manifolds are pseudo-Riemannian manifolds (M, g) with an affine symmetric connection ∇ such that the tensor ∇g is symmetric. Statistical structures play an important role in statistical physics, in neural networks etc. A class of statistical manifolds is made of *Hessian manifolds* studied by H. Shima [7], [8], C. Udrişte and G. Bercu [9] etc., which are statistical manifolds having constant curvature 0.

Extending the notion of statistical manifold in the context of the generalized geometry, we shall call (∇^*, g^*) *generalized statistical structure* if ∇^* is a torsion free connection such that the metric g^* on $TM \oplus T^*M$ is ∇^*-parallel, where $g^*(X+\alpha, Y+\beta) := \frac{1}{2}(\alpha(Y)+\beta(X))$, for $X+\alpha,\ Y+\beta \in \Gamma(TM \oplus T^*M)$. An example of generalized statistical structure will be given in Theorem 5.3.

The next propositions relates the two affine connections $\hat{\nabla}^*$ and $\check{\nabla}^*$ to the metric g^*.

Proposition 5.1. *The connections $\hat{\nabla}^*$ and $\check{\nabla}^*$ are compatible with the metric g^*, where $g^*(X+\alpha, Y+\beta) := \frac{1}{2}(\alpha(Y)+\beta(X))$.*

Proof. Let $X+\alpha,\ Y+\beta,\ Z+\gamma \in \mathcal{S}$. Then

$$\begin{aligned}
g^*(\hat{\nabla}^*_{X+\alpha}Y+\beta, Z+\gamma) &+ g^*(Y+\beta, \check{\nabla}^*_{X+\alpha}Z+\gamma)\\
&:= g^*(\nabla_X Y + \bar{\nabla}'^{\alpha}\beta, Z+\gamma) + g^*(Y+\beta, \nabla'_X Z + \bar{\nabla}^{\alpha}\gamma)\\
&:= \frac{1}{2}[(\bar{\nabla}'^{\alpha}\beta)(Z) + \gamma(\nabla_X Y)] + \frac{1}{2}[\beta(\nabla'_X Z) + (\bar{\nabla}^{\alpha}\gamma)(Y)]\\
&:= \frac{1}{2}[X(\beta(Z)) - \beta(\nabla'_X Z) + \gamma(\nabla_X Y)\\
&+ \beta(\nabla'_X Z) + X(\gamma(Y)) - \gamma(\nabla_X Y)]\\
&:= X(g^*(Y+\beta, Z+\gamma)).
\end{aligned}$$

□

Proposition 5.2. *For any $X+\alpha,\ Y+\beta,\ Z+\gamma \in \mathcal{S}$,*

$$(\hat{\nabla}^* g^*)(X+\alpha, Y+\beta, Z+\gamma) = -(\check{\nabla}^* g^*)(X+\alpha, Y+\beta, Z+\gamma) =$$

$$= \frac{1}{2}[(\nabla g)(X,Y,Z) - (\nabla' g)(X,Y,Z)].$$

In particular, if $\nabla = \nabla'$, then g^* is $\hat{\nabla}^*$-parallel.

Proof. Let $X+\alpha$, $Y+\beta$, $Z+\gamma \in \mathcal{S}$. Then

$$\begin{array}{rcl}(\hat{\nabla}^* g^*)(X+\alpha, Y+\beta, Z+\gamma) & := & X(g^*(Y+\beta, Z+\gamma)) - g^*(\hat{\nabla}^*_{X+\alpha}Y+\beta, Z+\gamma) \\ & - & g^*(Y+\beta, \hat{\nabla}^*_{X+\alpha}Z+\gamma) \\ & := & \frac{1}{2}[X(\beta(Z)+\gamma(Y))] - \frac{1}{2}[(\bar{\nabla}'^{\alpha}\beta)(Z) + \gamma(\nabla_X Y) \\ & + & \beta(\nabla_X Z) + (\bar{\nabla}'^{\alpha}\gamma)(Y)] \\ & := & \frac{1}{2}[X(\beta(Z)) + X(\gamma(Y)) - X(\beta(Z)) + \beta(\nabla'_X Z) \\ & - & \gamma(\nabla_X Y) - \beta(\nabla_X Z) - X(\gamma(Y)) + \gamma(\nabla'_X Y)] \\ & = & \frac{1}{2}[g(Y, \nabla'_X Z - \nabla_X Z) + g(Z, \nabla'_X Y - \nabla_X Y)] \\ & := & \frac{1}{2}[X(g(Y,Z)) - (\nabla' g)(X,Y,Z) - X(g(Y,Z)) \\ & + & (\nabla g)(X,Y,Z)] \\ & = & \frac{1}{2}[(\nabla g)(X,Y,Z) - (\nabla' g)(X,Y,Z)].\end{array}$$

Similarly for $\check{\nabla}^* g^*$. □

For ∇' flat connection, consider the bracket [5]

$$[X+\alpha, Y+\beta]_{\nabla'} := [X,Y] + \tilde{\nabla}'_X\beta - \tilde{\nabla}'_Y\alpha.$$

According to Lemma 2 from [1], we have $T_{\hat{\nabla}^*}(X+\alpha, Y+\beta) = T_{\nabla}(X,Y)$, for any $X+\alpha$, $Y+\beta \in \mathcal{S}$. Similarly, for ∇ flat connection, considering the bracket $[X+\alpha, Y+\beta]_{\nabla} := [X,Y] + \tilde{\nabla}_X\beta - \tilde{\nabla}_Y\alpha$, we obtain $T_{\check{\nabla}^*}(X+\alpha, Y+\beta) = T_{\nabla'}(X,Y)$, for any $X+\alpha$, $Y+\beta \in \mathcal{S}$.

Take ∇ the Levi-Civita connection on the Riemannian manifold (M,g) and $\hat{\nabla}^*_{X+\alpha}Y + \beta := \nabla_X Y + \bar{\nabla}^{\alpha}\beta$.

Theorem 5.3. *Assume that the Levi-Civita connection ∇ associated to the Riemannian metric g on M is flat. Then, considering the bracket $[\cdot,\cdot]_{\nabla}$ defined above, $(\hat{\nabla}^*, g^*)$ is a generalized statistical structure.*

Proof. Follows from Proposition 5.2 and the above considerations. □

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