Multitime maximum principle for curvilinear integral cost

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Abstract. Recently we have created a multitime maximum principle gathering together some concepts in Mechanics, Field Theory, Differential Geometry, and Control Theory. The basic tools of our theory are variational PDE systems, adjoint PDE systems, Hamiltonian PDE systems, duality, multitime maximum principle, incavity on manifolds etc. Now we justify the multitime maximum principle for curvilinear integral cost using the m-needle variations. Section 1 recalls the multitime control theory and proves the equivalence between curvilinear integral costs and multiple integral costs. Section 2 formulates variants of multitime maximum principle using control Hamiltonian 1-forms produced by a curvilinear integral cost and a controlled m-flow evolution. Section 3 refers to original proofs of the multitime maximum principle using simple and multiple multitime m-needle control variations. The key is to use completely integrable first order PDEs (controlled evolution and variational PDEs) and their adjoint PDEs. Section 4 formulates and proves sufficient conditions that the multitime maximum principle be true.

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1 Multitime control theory

The *multitime control theory* is concerned with partial derivatives dynamical systems and their optimization over multitime [15]-[30]. Such problems are well-known also as the multidimensional control problems of Dieudonné-Rashevsky type [9], [10], [31], [32], but both techniques and results in these papers are different from ours. We confirm the expectations of Lev Pontryaguin, Lawrence Evans and Jacques-Louis Lions regarding the analogy between *optimal control of systems governed by first order PDEs* and *optimal control of systems containing first order ODEs.* The ideas we use were stimulated by the original point of view of Lawrence C. Evans [4] on

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(single-time) Pontryaguin maximum principle [11] and by the papers [1]-[3], [5]-[10], [31], [32].

The multidimensional optimal control arise in the description of torsion of prismatic bars in the elastic case as well as in the elastic-plastic case. Another instance are optimization problems for convex bodies under geometrical restrictions, e.g., maximization of the area surface for given width and diameter.

Let (T, h) be an m-dimensional C^∞ Riemannian manifold, (M, g) be an n-dimensional C^∞ Riemannian manifold and $J^1(T, M)$ the associated *jet bundle of first order*. Let (U, η, M) be a *control fiber bundle*. The manifold M is called *state manifold* and the components x^i, $i = 1, \dots, n$ of a point $x \in M$ are called *state variables*. Then (x^i, u^a), $i = 1, \dots, n$; $a = 1, \dots, k$ are adapted coordinates in U, and $(t^\alpha, x^i, x^i_\alpha = \dfrac{\partial x^i}{\partial t^\alpha})$, $i = 1, \dots, n$; $\alpha = 1, \dots, m$ are natural coordinates in $J^1(T, M)$. The components u^a of the point $u \in U_x = \eta^{-1}(x)$ are called *controls*.

Let $X_\alpha : U \to J^1(T, M)$, $X_\alpha = X^i_\alpha(x, u)\dfrac{\partial}{\partial x^i}$ be a C^∞ fibered mapping, over the identity in the state manifold M, which produces a continuous control PDEs system (*controlled m-flow*)

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(x(t), u(t)), \quad i = 1, \dots, n; \quad \alpha = 1, \dots, m,$$

where $t = (t^1, \dots, t^m)$ is the multi-parameter of evolution (multitime). This PDEs system has solutions if and only if the complete integrability conditions

$$(CIC) \qquad \frac{\partial X^i_\alpha}{\partial x^j} X^j_\beta + \frac{\partial X^i_\alpha}{\partial u^a}\frac{\partial u^a}{\partial t^\beta} = \frac{\partial X^i_\beta}{\partial x^j} X^j_\alpha + \frac{\partial X^i_\beta}{\partial u^a}\frac{\partial u^a}{\partial t^\alpha}$$

are satisfied. These determine the set of *admissible controls*

$$\mathcal{U} = \{u(\cdot) : R^m_+ \to U \mid u(\cdot) \text{ is measurable and satisfies CIC}\}.$$

The evolution of the state manifold is totally characterized by the image set $S = Im(X_\alpha) \subset J^1(T, M)$ which is described by the control equations

$$x^i = x^i, \ x^i_\alpha = X^i_\alpha(x, u), \ i = 1, \dots, n \quad \alpha = 1, \dots, m.$$

Remark 1.1. *A problem of controlled evolution can be thought as a controlled immersion if $m < n$, controlled diffeomorphism if $m = n$ or controlled submersion if $m > n$. Particularly, if $m = n$, taking the trace after the indices i, α, we find a divergence type evolution (conservation laws).*

To simplify, we replace the manifolds M and T and the fiber U_x by their local representatives R^n, R^m, $U \subset R^k$ respectively. More precisely, for multitimes we use the orthant R^m_+. Having this in mind, for the multitimes $s = (s^1, ..., s^m)$ and $t = (t^1, ..., t^m)$, we denote $s \leq t$ if and only if $s^\alpha \leq t^\alpha$, $\alpha = 1, ..., m$ (*product order*). Then the parallelepiped Ω_{0t_0}, fixed by the diagonal opposite points $0(0, ..., 0)$ and $t_0 = (t^1_0, ..., t^m_0)$, is equivalent to the closed interval $0 \leq t \leq t_0$. Given $u(\cdot) \in \mathcal{U}$, the state $x(\cdot)$ is the solution of the evolution system

$$(PDE) \qquad \frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(x(t), u(t)), \ x(0) = x_0, \ t \in \Omega_{0t_0} \subset R^m_+.$$

This multitime evolution system is used as a constraint when we want to optimize a *multitime cost functional*. On the other hand, the cost functionals can be introduced at least in two equivalent ways:

- either using a curvilinear integral,

$$P(u(\cdot)) = \int_{\Gamma_{0t_0}} X_\beta(x(t), u(t))dt^\beta + g(x(t_0)),$$

where Γ_{0t_0} is an arbitrary C^1 curve joining the points 0 and t_0, the *running costs* $\omega = X_\beta(x(t), u(t))dt^\beta$ is a closed (completely integrable) 1-form (autonomous Lagrangian 1-form), and g is the *terminal cost*;

- or using the multiple integral,

$$Q(u(\cdot)) = \int_{\Omega_{0t_0}} X(x(t), u(t))dt^1...dt^m + g(x(t_0)),$$

where the *running costs* $X(x(t), u(t))$ is a continuous function (autonomous Lagrangian), and g is the *terminal cost.*

Theorem 1.2. *The controlled multiple integral*

$$I(t_0) = \int_{\Omega_{0t_0}} X^0(x(t), u(t))dt^1...dt^m,$$

with $X^0(x(t), u(t))$ *as continuous function, is equivalent to the controlled curvilinear integral*

$$J(t_0) = \int_{\Gamma_{0t_0}} X^0_\beta(x(t), u(t))dt^\beta,$$

where $\omega = X^0_\beta(x(t), u(t))dt^\beta$ *is a closed (completely integrable)* 1*-form and the functions* X^0_β *have partial derivatives of the form*

$$\frac{\partial}{\partial t^\alpha}, \ \frac{\partial}{\partial t^\alpha \partial t^\beta}\, (\alpha < \beta), \ \dots , \ \frac{\partial^{m-1}}{\partial t^1 ... \partial \hat{t}^\alpha ... \partial t^m}.$$

The *hat symbol* posed over ∂t^α designates that ∂t^α is omitted.

Proof. The multiple integral $I(t_0)$ suggests to introduce a new coordinate

$$x^0(t) = \int_{\Omega_{0,t}} X^0(x(t), u(t))dt^1...dt^m, \ t \in \Omega_{0t_0}, \ x^0(t_0) = I(t_0).$$

Taking

$$X^0_\alpha(x(t), u(t)) = \frac{\partial x^0}{\partial t^\alpha}(t),$$

we can write $x^0(t)$ as the curvilinear integral

$$x^0(t) = \int_{\gamma_{0t}} X^0_\alpha(x(s), u(s))ds^\alpha, \ x^0(t_0) = J(t_0),$$

where γ_{0t} is an arbitrary C^1 curve joining the points 0 and t in Ω_{0t_0}. Also

$$\frac{\partial^{m-1} X^0_\alpha}{\partial t^1 ... \partial \hat{t}^\alpha ... \partial t^m} = \frac{\partial^m x^0}{\partial t^1 ... \partial t^\alpha ... \partial t^m}.$$

Conversely, the curvilinear integral $J(t_0)$ suggests to define a new coordinate by

$$x^0(t) = \int_{\gamma_{0t}} X^0_\alpha(x(s), u(s)) ds^\alpha,\ x^0(t_0) = J(t_0),$$

where γ_{0t} is an arbitrary C^1 curve joining the points 0 and t in Ω_{0t_0}, and $\omega = X^0_\alpha(x(s), u(s))ds^\alpha$ is a completely integrable 1-form. Since $X^0_\alpha = \dfrac{\partial x^0}{\partial t^\alpha}$, we can define

$$X^0 = \frac{\partial^{m-1} X^0_\alpha}{\partial t^1 ... \partial \hat{t}^\alpha ... \partial t^m} = \frac{\partial^m x^0}{\partial t^1 ... \partial t^\alpha ... \partial t^m}.$$

Then the new coordinate can be written as

$$x^0(t) = \int_{\Omega_{0t}} X^0(x(t), u(t)) dt^1 ... dt^m,\ t \in \Omega_{0,t_0},\ x^0(t_0) = I(t_0).$$

□

In this paper, we shall develop the optimization problems using cost functionals as path independent curvilinear integrals and constraints as m-flows, where the complete integrability conditons are piecewise satisfied.

Remark 1.3. *New aspects of control theory are developed in the papers [9], [10], [31] and [32] using weak derivatives instead of usual partial derivatives.*

Remark 1.4. *We can extend the holonomic controlled evolution to a nonholonomic controlled evolution, using the Pfaff system $dx^i = X^i_\alpha(x, u)dt^\alpha$. In the noholonomic case, the dimension of evolution is smaller than m.*

2 Maximum principle for multitime control theory based on a curvilinear integral cost

A *curvilinear integral cost* and a *multitime flow* were introduced in the *optimal control theory* by our papers [12]-[30].

Multitime optimal control problem. Find

$$\max_{u(\cdot)} \quad P(u(\cdot)) = \int_{\Gamma_{0t_0}} X_\beta(x(t), u(t)) dt^\beta + g(x(t_0))$$

$$\text{subject to} \quad \frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(t, x(t), u(t)),\ i = 1, ..., n,\ \alpha = 1, ..., m,$$

$$u(t) \in \mathcal{U},\ t \in \Omega_{0t_0},\ x(0) = x_0,\ x(t_0) = x_{t_0}.$$

The *multitime maximum principle* (necessary condition) will assert the existence of a *costate vector function* $(p^*_0, p^*)(\cdot) = ({p^*}_0(\cdot), {p^*}_i(\cdot))$ which, together with the optimal

m-sheet $x^*(\cdot)$, satisfies an appropriate PDEs system and a maximum condition. All these conditions can be written using the appropriate *control Hamiltonian* 1*-form*

$$H_\alpha(x, p_0, p, u) = p_0 X^0_\alpha(x, u) + p_i X^i_\alpha(x, u).$$

Theorem 2.1. (multitime maximum principle) *Suppose $u^*(\cdot)$ is optimal for (P), (PDE) and that $x^*(\cdot)$ is the corresponding optimal m-sheet. Then there exists a function $(p^*_0, p^*) = (p^*{}_0, p^*{}_i) : \Omega_{0t_0} \to R^{n+1}$ such that*

$$(PDE) \qquad \frac{\partial x^{*i}}{\partial t^\alpha}(t) = \frac{\partial H_\alpha}{\partial p_i}(x^*(t), p^*_0(t), p^*(t), u^*(t)),$$

$$(ADJ) \qquad \frac{\partial p^*{}_i}{\partial t^\alpha}(t) = -\frac{\partial H_\alpha}{\partial x^i}(x^*(t), p^*_0(t), p^*(t), u^*(t))$$

and

$$(M) \qquad H_\alpha(x^*(t), p^*_0(t), p^*(t), u^*(t)) = \max_{u \in U_x} H_\alpha(x^*(t), p^*_0(t), p^*(t), u),\ t \in \Omega_{0T^*}.$$

Also, the functions $t \to H_\alpha(x^*(t), p^*_0(t), p^*(t), u^*(t))$ are constants.

$$(t_0) \qquad p^*{}_0(t_0) = a_0,\ p^*{}_i(t_0) = \frac{\partial g}{\partial x^i}(x^*(t_0))$$

are satisfied.

We call $x^*(\cdot)$ the *state* of the optimally controlled system and $(p^*_0, p^*(\cdot))$ the *costate vector.*

Remark 2.2. (PDE) *means the identities*

$$\frac{\partial x^{*i}}{\partial t^\beta}(t) = X^i_\beta(x^*(t), u^*(t)), \quad \beta = 1, \dots, m; \quad i = 1, \dots, n,$$

(controlled evolution PDEs).

Remark 2.3. (ADJ) *means the identities*

$$\frac{\partial p^*{}_i}{\partial t^\beta}(t) = -\left(p^*{}_0(t) \frac{\partial X^0_\beta}{\partial x^i} + p^*{}_j(t) \frac{\partial X^j_\beta}{\partial x^i} \right)(x^*(t), u^*(t))$$

(adjoint PDEs).

Remark 2.4. *The relations (M) represent the maximization principle and the relation (t_0) means the terminal (transversality) condition.*

Remark 2.5. *The multitime maximum principle states necessary conditions that must hold on an optimal m-sheet of evolution.*

Example. Consider the problem of a mine owner who must decide at what rate to extract a complex ore from his mine. He owns rights to the ore from two-time date $0 = (0, 0)$ to two-time date $T = (T, T)$. A *two-time date* can be the pair (date, useful component frequence). At two-time date 0 there is $x_0 = (x_0^i)$ ore in the ground, and the instantaneous stock of ore $x(t) = (x^i(t))$ declines at the rate $u(t) = (u_\alpha^i(t))$ the mine owner extracts it. The mine owner extracts ore at cost $q_i \frac{u_\alpha^i(t)^2}{x^i(t)}$ and sells ore at a constant price $p = (p_i)$. He does not value the ore remaining in the ground at time T (there is no "scrap value"). He chooses the rate $u(t)$ of extraction in two-time to maximize profits over the period of ownership with no two-time discounting.

Solution (continuous two-time version). The manager want to maximizes the profit (curvilinear integral)

$$P(u(\cdot)) = \int_{\gamma_{0T}} \left(p_i u_\alpha^i(t) - q_i \frac{u_\alpha^i(t)^2}{x^i(t)} \right) dt^\alpha$$

subject to the law of evolution $\frac{\partial x}{\partial t^\gamma}(t) = -u_\gamma(t)$. Form the Hamiltonian 1-form

$$H_\alpha = p_i u_\alpha^i(t) - q_i \frac{u_\alpha^i(t)^2}{x^i(t)} - \lambda_i(t) u_\alpha^i(t),$$

differentiate and write the equations

$$\frac{\partial H_\alpha}{\partial u_\beta} = \left(p_i - 2q_i \frac{u_\alpha^i}{x^i} - \lambda_i \right) \delta_\alpha^\beta = 0, \text{ no sum after the index i;}$$

$$\frac{\partial \lambda_i}{\partial t^\alpha}(t) = -\frac{\partial H_\alpha}{\partial x^i} = -q_i \left(\frac{u_\alpha^i(t)}{x^i(t)} \right)^2, \text{ no sum after the index i.}$$

As the mine owner does not value the ore remaining at time T, we have $\lambda_i(T) = 0$. Using the above equations, it is easy to solve for the differential equations governing $u(t)$ and $\lambda_i(t)$:

$$2q_i \frac{u_\alpha^i(t)}{x^i(t)} = p_i - \lambda_i(t), \; \frac{\partial \lambda_i}{\partial t^\alpha}(t) = -q_i \left(\frac{u_\alpha^i(t)}{x^i(t)} \right)^2 \text{ no sum after i}$$

and using the initial and turn-T conditions, the equations can be solved numerically.

Free multitime, fixed endpoint problem. Given a control $u(\cdot) \in \mathcal{U}$, the state $x(\cdot)$ is the solution of the initial value problem

$$(PDE) \qquad \frac{\partial x^i}{\partial t^\alpha}(t) = X_\alpha^i(x(t), u(t)), \; x(0) = x_0, \; t \in \Omega_{0t_0} \subset R_+^m.$$

Suppose that a target point x_1 is given. Then we consider the cost functional (path independent curvilinear integral)

$$(P) \qquad P(u(\cdot)) = \int_{\gamma_{0\tau}} X_\beta^0(x(t), u(t)) dt^\beta,$$

where $\gamma_{0\tau}$ is an arbitrary C^1 curve joining the points 0 and τ in Ω_{0t_0}, for $\tau = \tau(u(\cdot)) \leq \infty$, $\tau = (\tau^1, ..., \tau^m)$ as the first multitime the solution of (PDE) hits the target point x^1. We ask to find an optimal control $u^*(\cdot)$ such that

$$P(u^*(\cdot)) = \max_{u(\cdot)\in\mathcal{U}} P(u(\cdot)).$$

The control Hamiltonian 1-form is

$$H_\alpha(x, p_0, p, u) = p_0 X^0_\alpha(x, u) + p_i X^i_\alpha(x, u).$$

Theorem 2.6. (multitime maximum principle) *Suppose $u^*(\cdot)$ is optimal for (P), (PDE) and that $x^*(\cdot)$ is the corresponding optimal m-sheet. Then there exists a function $(p^*_0, p^*) = ({p^*}_0, {p^*}_i) : \Omega_{0\tau^*} \to R^{n+1}$ such that*

$$(PDE) \qquad \frac{\partial {x^*}^i}{\partial t^\alpha}(t) = \frac{\partial H_\alpha}{\partial p_i}(x^*(t), p^*_0(t), p^*(t), u^*(t)),$$

$$(ADJ) \qquad \frac{\partial {p^*}_i}{\partial t^\alpha}(t) = -\frac{\partial H_\alpha}{\partial x^i}(x^*(t), p^*_0(t), p^*(t), u^*(t)),$$

and

$$(M) \qquad H_\alpha(x^*(t), p^*_0(t), p^*(t), u^*(t)) = \max_{u\in U_x} H_\alpha(x^*(t), p^*_0(t), p^*(t), u),\ t \in \Omega_{0\tau^*}.$$

Moreover,

$$H_\alpha(x^*(t), p^*_0(t), p^*(t), u^*(t))|_{\Omega_{0\tau^*}} = 0,$$

where τ^* denotes the first multitime the m-sheet $x^*(\cdot)$ hits the target point x_1.

We call $x^*(\cdot)$ the *state* of the optimally controlled system and $(p^*_0, p^*(\cdot))$ the *costate matrix*.

Remark 2.7. *A more careful statement of the multitime maximum principle is: there exist the constant ${p^*}_0$ and the function $({p^*}_i) : \Omega_{0t^*} \to R^n$ such that (PDE), (ADJ), and (M) hold. The vector function $p^*(\cdot)$ is a Lagrange multiplier, which appears owing to the constraint that the optimal m-sheet $x^*(\cdot)$ must satisfy (PDE).*

Remark 2.8. *If the number ${p^*}_0$ is 0, then the control Hamiltonian 1-form does not depend on the corresponding running costs X^0_α and in this case the maximum principle must be reformulated. Such a problem will be called abnormal problem.*

Remark 2.9. *The previous theory can be extended to the nonautonomous case*

$$(P) \qquad P(u(\cdot)) = \int_{\gamma_{0\tau}} X^0_\beta(t, x(t), u(t))dt^\beta$$

and

$$(PDE) \qquad \frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(t, x(t), u(t)),\ t \in R^m_+,$$

using the idea of reducing to the previous case by introducing new variables $x^\alpha = t^\alpha$, $\alpha = 1, ..., m$.

2.1 Maximum principle with transversality conditions

We look again at the dynamics

$$(PDE) \qquad \frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(x(t), u(t)),\ t \in R^m_+,$$

when the initial point x_0 belongs to the subset $M_0 \subset R^n$ and the terminal point x_1 is constrained to lie in the subset $M_1 \subset R^n$. In other word, we must choose the starting point $x_0 \in M_0$ in order to maximize

$$(P) \qquad P(u(\cdot)) = \int_{\gamma_{0\tau}} X^0_\beta(x(t), u(t))dt^\beta,$$

where $\gamma_{0\tau}$ is an arbitrary C^1 curve joining the points 0 and τ in Ω_{0t_0}, for $\tau = \tau(u(\cdot))$ as the first multitime we hit M_1.

Assumption. The subsets M_0 and M_1 are smooth submanifolds of R^n. In this context, we can use the tangent spaces $T_{x_0}M_0$ and $T_{x_1}M_1$.

Theorem 2.10. (more transversality conditions) *If the functions $u^*(\cdot)$ and $x^*(\cdot)$ solve the previous problem, with $x_0 = x^*(0),\ x_1 = x(\tau^*)$, then there exists the function $p^*(\cdot) : \Omega_{0\tau^*} \to R^n$ such that (PDE), (ADJ) and (M) hold for $t \in \Omega_{0\tau^*}$. Also,*

$$(t_0) \qquad \begin{array}{c} \text{the vector } p^*(\tau^*) \text{ is orthogonal to } T_{x_1}M_1, \\ \text{the vector } p^*(0) \text{ is orthogonal to } T_{x_0}M_0. \end{array}$$

Remark 2.11. *Let Γ_{0t_0} be an arbitrary C^1 curve joining the diagonal points 0 and t_0. According the terminal/transversality condition, for $t_0 > 0$ and*

$$P(u(\cdot)) = \int_{\Gamma_{0t_0}} X^0_\beta(x(t), u(t))dt^\beta + g(x(t_0)),$$

the condition (t_0) means $p^*{}_i(t_0) = \dfrac{\partial g}{\partial x^i}(x^*(t_0))$.

Remark 2.12. *Suppose $M_1 = \{x \,|\, f_k(x) = 0,\ k = 1, ..., \ell\}$. Since the normal space (orthogonal complement of the tangent space $T_{x_1}M_1$) is generated by the vectors $\dfrac{\partial f_k}{\partial x^i}(x_1),\ k = 1, ..., \ell$, we must have $p^*{}_i(\tau^*) = \lambda^k \dfrac{\partial f_k}{\partial x^i}(x_1)$, for some parameters (constants, Lagrange multipliers) $\lambda^1, ..., \lambda^\ell$.*

2.2 Maximum principle with state constraints

Let us return to the problem

$$(PDE) \qquad \frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(x(t), u(t)),\ x(0) = x_0,\ t \in \Omega_{0t_0} \subset R^m_+.$$

$$(P) \qquad P(u(\cdot)) = \int_{\gamma_{0\tau}} X^0_\beta(x(t), u(t))dt^\beta,$$

for $\tau = \tau(u(\cdot)) \leq \infty,\ \tau = (\tau^1, ..., \tau^m)$ as the first multitime with $x(\tau) = x^1$, and $\gamma_{0\tau}$ an arbitrary C^1 curve joining the points 0 and τ in Ω_{0t_0}. In this sense, we have a fixed endpoint problem.

State constraints. Suppose our multitime dynamics remains in the submanifold $N = \{x \in R^n \,|\, f(x) \leq 0\}$, where $f : R^n \to R$ is a differentiable function. The functions f and X^i_α define new functions

$$c_\alpha(x, u) = \frac{\partial f}{\partial x^i}(x) X^i_\alpha(x, u).$$

If $x(t) \in \partial N$ for $t \in \Omega_{s_0 s_1}$, then $c_\alpha(x(t), u(t)) = 0$.

Theorem 2.13. (maximum principle for state constraints) *Suppose* $u^*(\cdot),\ x^*(\cdot)$ *solve the previous control theory problem, and that* $x^*(t) \in \partial N$ *for* $t \in \Omega_{s_0 s_1}$. *Then there exist the costate vector function* $p^*(\cdot) : \Omega_{s_0 s_1} \to R^n$ *and there exist* ${\lambda^*}^\gamma_\beta(\cdot) : \Omega_{s_0 s_1} \to R$ *such that*

$$(PDE) \qquad \frac{\partial {x^*}^i}{\partial t^\alpha}(t) = \frac{\partial H_\alpha}{\partial p_i}(x^*(t), p^*_0(t), p^*(t), u^*(t)),$$

$$(ADJ') \qquad \frac{\partial {p^*}_i}{\partial t^\beta}(t) = -\frac{\partial H_\beta}{\partial x^i}(x^*(t), p^*(t), u^*(t)) + {\lambda^*}^\gamma_\beta(t) \frac{\partial c_\gamma}{\partial x^i}(x^*(t), u^*(t))$$

$$(M') \qquad H_\beta(x^*(t), p^*(t), u^*(t)) = \max_{u \in U_x} \{H_\beta(x^*(t), p^*(t), u) \,|\, c_\alpha(x^*(t), u) = 0\}$$

hold, for multitimes $t \in \Omega_{s_0 s_1}$.

Remark 2.14. *Let* $q_1, ..., q_s$ *be differentiable functions on* U *which determine the subset*

$$A = \{u \in U \,|\, q_1(u) \leq 0, ..., q_s(u) \leq 0\}$$

in the control set (the condition $m \leq s$ *is necessary). In this case, instead of the relation* (M') *appear* (M''):

$$\frac{\partial H_\beta}{\partial u}(x^*(t), p^*(t), u^*(t)) = {\lambda^*}^\gamma_\beta(t) \frac{\partial c_\gamma}{\partial u}(x^*(t), u^*(t)) + {\mu^*}^r_\beta(t) \frac{\partial q_r}{\partial u}(u^*(t)).$$

The functions ${\lambda^*}^\gamma_\beta(\cdot)$ *are those appearing in* (ADJ'). *If* $x^*(t)$ *lies in the interior of* N, *for say the multitimes* $t \in \Omega_{0s_0}$, *then the ordinary multitime maximum principle holds.*

Remark 2.15. (Jump conditions) *Let* s_0 *be a multitime that* p^* *hits the boundary* ∂N. *Then* $p^*(s_0 - 0) = p^*(s_0 + 0)$. *This means that there is no jump in* p^* *when we hit* ∂N. *However,*

$$p^*(s_1 + 0) = p^*(s_1 - 0) - {\lambda^*}^\beta_\beta(s_1) \frac{\partial f}{\partial x}(x^*(s_1)),$$

i.e., there is possibly a jump in p^* *when we leave* ∂N. *Of course, these statements are true when the gluing (contact) sheets, i.e., the unconstrained evolution sheet and the evolution boundary* ∂N *have the same dimension.*

3 Proofs of the multitime maximum principle

The proofs of multitime maximum principle relies either on control variations similar to those in variational calculus (for interior solutions, see also [15]-[30]) or on control *m-needle variations*. These give rise to variations of a reference m-sheet. In spite of the fact that these variations are not equivalent, they assert similar statements.

Here we use the m-needle variations. To explain their meaning, we consider a candidate optimal control $u^*(\cdot)$, the corresponding optimal m-sheet $x^*(\cdot)$, and a multitime point s of *approximate continuity* for the functions $X_\alpha(x^*(\cdot), u^*(\cdot))$ and $u(\cdot) \in \mathcal{U}$. An *m-needle variation* is a family of controls $u_\epsilon(\cdot)$ obtained replacing $u^*(\cdot)$ with $u(\cdot)$ on the set $\Omega_{0s} \setminus \Omega_{0s-\varepsilon}$. Any m-needle variation gives rise to a *gradient variation* y_α of an m-sheet $x(t)$ satisfying the *variational PDE* $\dfrac{\partial y^i_\alpha}{\partial t^\beta}(t) = y^j_\alpha(t)\dfrac{\partial X^i_\beta}{\partial x^j}(x(t), u(t))$, in the classical sense, only after the multitime s. Of course, the last PDEs satisfies the complete integrability conditions.

3.1 Simple controls variations

The response $x(\cdot)$ to a given control $u(\cdot)$ is the unique solution of the completely integrable PDEs system

$$(PDE) \qquad \frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(x(t), u(t)),\ x(0) = x_0,\ t \in \Omega_{0t_0} \subset R^m_+.$$

Let us find how the m-needle changes in the control affect the response. In this sense, we fix the multitime $s = (s^1, ..., s^m), s^\alpha > 0,\ \alpha = 1, ..., m,$ and a control $v(\cdot) \in \mathcal{U}$. We select $\epsilon = (\epsilon^1, ..., \epsilon^m), \epsilon^\alpha > 0$ with the property $0 < s^\alpha - \epsilon^\alpha < s^\alpha$ and define the modified control

$$u_\epsilon(t) = \begin{cases} v(t) & \text{if } t \in \Omega_{0s} \setminus \Omega_{0s-\epsilon} \\ u^*(t) & \text{otherwise,} \end{cases}$$

which is called a *simple m-needle variation* of $u^*(\cdot)$. We denote by $x_\epsilon(\cdot)$ the corresponding response of our system

$$(22) \qquad \frac{\partial x^i_\epsilon}{\partial t^\alpha}(t) = X^i_\alpha(x_\epsilon(t), u_\epsilon(t)),\ x_\epsilon(0) = x_0,\ t \in \Omega_{0t_0} \subset R^m_+.$$

Let us try to understand how the choices of s and $v(\cdot)$ cause $x_\epsilon(\cdot)$ to differ from $x(\cdot)$, for small $||\epsilon|| > 0$.

Lemma 3.1. (changing initial conditions) *If $x_\epsilon(\cdot)$ is a solution of the initial value problem*

$$\frac{\partial x^i_\epsilon}{\partial t^\alpha}(t) = X^i_\alpha(x_\epsilon(t), u(t)),\ x_\epsilon(0) = x_0 + \epsilon^\alpha y_{\alpha 0} + o(\epsilon),\ t \in R^m_+,$$

then $x_\epsilon(t) = x(t) + \epsilon^\alpha y_\alpha(t) + o(\epsilon)$ as $\epsilon \to 0$, uniformly for t in compact subsets of R^m_+, where $y_\alpha = (y^i_\alpha) = \left(\dfrac{\partial x^i_\epsilon}{\partial \epsilon^\alpha}|_{\epsilon=0}\right)$ is the solution of the initial value variational problem

$$\frac{\partial y^i_\alpha}{\partial t^\beta}(t) = y^j_\alpha(t)\frac{\partial X^i_\beta}{\partial x^j}(x(t), u(t)),\ y_\alpha(0) = y_{\alpha 0},\ t \in R^m_+.$$

Coming back to the dynamics (22) and simple control m-needle variation, we find

Lemma 3.2. (dynamics and simple control variations) *If $u_\epsilon(\cdot)$ is a simple variation of the control $u^*(\cdot)$, then*

$$x_\epsilon(t) = x(t) + \epsilon^\alpha y_\alpha(t) + o(\epsilon) \text{ as } \epsilon \to 0$$

uniformly for t in compact subsets of R^m_+, where $\quad y_\alpha(t) = 0,\ t \in \Omega_{0s} \quad$ *and*

$$\frac{\partial y^i_\alpha}{\partial t^\beta}(t) = y^j_\alpha(t)\frac{\partial X^i_\beta}{\partial x^j}(x(t), u^*(t)),\ y_\alpha(s) = y_{\alpha s},\ t \in R^m_+ \setminus \Omega_{0s}, \tag{23}$$

for

$$y_{\alpha s} = X_\alpha(x(s), v(s)) - X_\alpha(x(s), u^*(s)).$$

Proof. For simplicity, let us drop the superscript $*$. First we remark that $x_\epsilon(t) = x(t)$ for $t \in \Omega_{0s-\epsilon}$ and hence $y_\alpha(t) = 0$ for $t \in \Omega_{0s-\epsilon}$. For the multitime $t \in \Omega_{0s} \setminus \Omega_{0s-\epsilon}$, we can use the curvilinear integral

$$x_\epsilon(t) - x(t) = \int_{\Gamma_{s-\epsilon t}} (X_\alpha(x(s), v(s)) - X_\alpha(x(s), u(s)))ds^\alpha = o(\epsilon),$$

where $\Gamma_{s-\epsilon t}$ is a C^∞ curve joining the points $s-\epsilon$ and t. Hence we can put $y_\alpha(t) = 0$ for $t \in \Omega_{0s} \setminus \Omega_{0s-\epsilon}$.

We set $t = s$. Since the curvilinear integral is independent of the path, we select Γ as being the straight line joining the points $s - \epsilon$ and s, i.e.,

$$\Gamma : t^\alpha = s^\alpha - \epsilon^\alpha + \epsilon^\alpha\tau,\ \alpha = 1, ..., m,\ \tau \in [0, 1].$$

Consequently,

$$x_\epsilon(s) - x(s) = (X_\alpha(x(s), v(s)) - X_\alpha(x(s), u(s)))\epsilon^\alpha + o(\epsilon).$$

On the multitime box $R^m_+ \setminus \Omega_{0s}$, the functions $x(.)$ and $x_\epsilon(.)$ are solutions for the same PDEs, but with different initial conditions: $x(0) = x_0$ and $x_\epsilon(s) = x(s)+\epsilon^\alpha y_{\alpha s}+o(\epsilon)$, for $y_{\alpha s}$ defined by (23). The Lemma of changing initial conditions shows that

$$x_\epsilon(t) = x(t) + \epsilon^\alpha y_\alpha(t) + o(\epsilon)$$

for $y_\alpha(\cdot)$ solving (23) and for $t \in R^m_+ \setminus \Omega_{0s}$. □

3.2 Free endpoint problem, no running cost

Statement. Let us consider again the multitime dynamics

$$(PDE) \qquad \frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(x(t), u(t)),\ x(0) = x_0,\ t \in \Omega_{0t_0} \subset R^m_+$$

together the terminal cost functional

$$(P) \qquad P(u(\cdot)) = g(x(t_0)),$$

which must be maximized with respect to the control. We denote by $u^*(\cdot)$ respectively $x^*(\cdot)$ the optimal control and the optimal m-sheet of this problem. Since the running cost is zero, the control Hamiltonian 1-form is

$$H_\alpha(x, p, u) = p_i X^i_\alpha(x, u).$$

It remains to find a vector function $p^* = ({p^*}_i) : \Omega_{0t_0} \to R^n$ such that

$$(ADJ) \qquad \frac{\partial {p^*}_i}{\partial t^\alpha}(t) = -\frac{\partial H_\alpha}{\partial x^i}(x^*(t), p^*(t), u^*(t)),\ t \in \Omega_{0t_0},$$

and

$$(M) \qquad H_\alpha(x^*(t), p^*(t), u^*(t)) = \max_{u \in U} H_\alpha(x^*(t), p^*(t), u).$$

For simplicity, let us drop the superscript $*$. Also, we take into account the control variation $u_\epsilon(.)$.

The costate. We define the *costate* $p : \Omega_{0t_0} \to R^n$, $p = (p_i)$, as the unique solution of the terminal value problem

$$(24) \qquad \frac{\partial p_i}{\partial t^\beta}(t) = -p_j(t)\frac{\partial X^j_\beta}{\partial x^i}(x(t), u(t)),\ t \in \Omega_{0t_0},\ p_i(t_0) = \frac{\partial g}{\partial x^i}(x(t_0)).$$

The solutions of the Cauchy problems (23)+(24) determine an 1-form of components $p_i y^i_\beta$. The PDEs (24) are called *adjoint equations* since we can verify by computation that the components (scalar products) $p_i y^i_\beta$ are first integrals of the PDEs system (23)+(24). The costate will be used to calculate the variation of the terminal cost.

Lemma 3.3. (variation of terminal cost) *The partial variations of the terminal cost are*

$$\frac{\partial}{\partial \epsilon^\beta} P(u_\epsilon(\cdot))|_{\epsilon=0} = p_i(s)\left(X^i_\beta(x(s), v(s)) - X^i_\beta(x(s), u(s))\right).$$

Proof. Since $P(u_\epsilon(\cdot)) = g(x(t_0) + \epsilon^\beta y_\beta(t_0) + o(\epsilon))$, where $y(\cdot)$ satisfies the previous Lemmas, we find

$$\frac{\partial}{\partial \epsilon^\beta} P(u_\epsilon(\cdot))|_{\epsilon=0} = \frac{\partial g}{\partial x^i}(x(t_0)) y^i_\beta(t_0).$$

Since the components $p_i y^i_\beta$ are first integrals of the PDEs system (23)+(24), we obtain

$$\frac{\partial g}{\partial x^i}(x(t_0)) y^i_\beta(t_0) = p_i(t_0) y^i_\beta(t_0) = p_i(s) y^i_\beta(s),\ \forall s \in \Omega_{0t_0}.$$

Finally, the functions $y_\beta(s) = X_\beta(x(s), v(s)) - X_\beta(x(s), u(s))$ give the desired formula. □

We restore the superscript $*$ and we formulate the next

Theorem 3.4. (multitime maximum principle) *There exists a function* $p^* : \Omega_{0t_0} \to R^n$ *satisfying the adjoint dynamics* (ADJ)*, the maximization principle* (M) *and the terminal (transversality) condition* (t_0)*.*

Proof. The adjoint dynamics and the terminal condition appear in the Cauchy problem (24). To prove (M), we fix $s \in \operatorname{int}\Omega_{0t_0}$ and $v(\cdot) \in \mathcal{U}$, as above. Since the function $\epsilon \to P(u_\epsilon(\cdot)),\ \epsilon \in \Omega_{0t_0}$, has a maximum at $\epsilon = 0$, we must have

$$0 \geq \frac{\partial P}{\partial \epsilon^\beta}(u_\epsilon(\cdot))|_{\epsilon=0} = {p^*}_i(s)\left(X^i_\beta(x^*(s), v(s)) - X^i_\beta(x^*(s), u^*(s))\right).$$

Consequently

$$H_\beta(x^*(s), p^*(s), v(s)) = {p^*}_i(s) X^i_\beta(x^*(s), v(s))$$
$$\leq {p^*}_i(s) X^i_\beta(x^*(s), u^*(s)) = H_\beta(x^*(s), p^*(s), u^*(s)),$$

for each $s \in \operatorname{int}\Omega_{0t_0}$ and $v(\cdot) \in \mathcal{U}$. Since the function $v(\cdot)$ is arbitrary, so it is the value $v(s)$ and therefore

$$H_\alpha(x^*(t), p^*(t), u^*(t)) = \max_{u \in U} H_\alpha(x^*(t), p^*(t), u).$$

□

3.3 Free endpoint problem with running costs

Let us consider that the cost functional include a running cost, i.e.,

$$(P) \qquad P(u(\cdot)) = \int_{\Gamma_{0t_0}} X^0_\beta(x(t), u(t))dt^\beta + g(x(t_0)),$$

where Γ_{0t_0} is an arbitrary C^1 curve joining the points 0 and t_0, the *running cost* $dx^0 = X^0_\beta(x(t), u(t))dt^\beta$ is a closed (completely integrable) 1-form, and g is the *terminal cost*. In this case the *control Hamiltonian* 1-*from* must have the form

$$H_\alpha(x, p_0, p, u) = p_0 X^0_\alpha(x, u) + p_i X^i_\alpha(x, u),$$

under the condition that we can built a costate function $\overline{p}^*(\cdot) = ({p^*}_0(\cdot), {p^*}_i(\cdot))$ satisfying (ADJ), (M) and (t_0).

Adding a new variable. Introducing a new variable x^0, we convert the theory to the previous case.

Let $x^0 : \Omega_{0t_0} \to R$ be the solution of initial problem

$$(25) \qquad \frac{\partial x^0}{\partial t^\alpha}(t) = X^0_\alpha(x(t), u(t)),\ x^0(0) = 0,\ t \in \Omega_{0t_0},$$

where $x(\cdot) = (x^i(\cdot))$ is the solution of (PDE). Introduce

$$x = (x^1, ..., x^n),\ \overline{x} = (x^0, x),\ \overline{x}_0 = (0, x_0),\ \overline{x}(\cdot) = (x^0(\cdot), x(\cdot)),$$

and

$$\overline{X}_\alpha(\overline{x}, u) = (X^0_\alpha(x, u), X_\alpha(x, u)),\ \overline{g}(\overline{x}) = g(x) + a_0 x^0.$$

Then (PDE) and (25) give the dynamics

$$(\overline{PDE}) \qquad \frac{\partial \overline{x}}{\partial t^\alpha}(t) = \overline{X}_\alpha(x(t), u(t)),\ \overline{x}(0) = \overline{x}_0,\ t \in \Omega_{0t_0}.$$

Consequently, the actual control problem transforms into a new control problem with no running cost and the terminal cost

$$(\overline{P}) \qquad \overline{P}(u(\cdot)) = \overline{g}(\overline{x}(t_0)).$$

We apply the previous theorem of multitime maximum principle to obtain $\overline{p}^* : \Omega_{0t_0} \to R^{n+1}$, $\overline{p}^* = (\overline{p}^*{}_i)$ satisfying $(\overline{M}_\alpha)$ for the control Hamiltonian 1-form

$$\overline{H}_\alpha(\overline{x}, \overline{p}, u) = \overline{p}_i \overline{X}^i_\alpha(\overline{x}, u).$$

The adjoint equations $(\overline{ADJ})$ hold for the terminal transversality condition

$$(\overline{t_0}) \qquad \overline{p^*}_j(t_0) = \frac{\partial \overline{g}}{\partial \overline{x}^j}(\overline{x}(t_0)), j = 0, 1, 2, ..., n.$$

Since $\overline{X}_\alpha$ do not depend upon the variable x^0, the 0-th equation in the adjoint equations $(\overline{ADJ})$ is $\dfrac{\partial p_0}{\partial t^\beta} = 0$. On the other hand, the relation $\dfrac{\partial \overline{g}}{\partial x^0} = a_0$ implies $p_0 = a_0$. Consequently $H_\beta(x, p, u)$ and $p^*(\cdot) = (p^*{}_i(\cdot))$ satisfy (ADJ) and (M).

3.4 Multitime multiple control variations

To formulate and prove the multitime maximum principle for the next fixed endpoint problem, we need to introduce a multiple variation of the control.

Let us find how multiple changes in the control affect the response. We fix the multitimes $s_A = (s^1_A, ..., s^m_A)$, $A = 1, ..., N$, with $0 < s_1 < ... < s_N$, the control parameters $v_A(\cdot) \in \mathcal{U}$ and strictly positive numbers λ^A, $A = 1, 2, ..., N$. Select $\epsilon > 0$ so small that the domains $\Omega_{0s_A} \backslash \Omega_{0s_A - \lambda^A \epsilon}$ do not overlap. Define the modified control

$$u_\epsilon(t) = \begin{cases} v_A(t) & \text{if } t \in \Omega_{0s_A} \setminus \Omega_{0, s_A - \lambda^A \epsilon}, A = 1, ..., N \\ u^*(t) & \text{otherwise,} \end{cases}$$

which is called a *multiple m-needle variation* of the control $u^*(\cdot)$. We denote $x_\epsilon(\cdot)$ the corresponding response of the Cauchy problem

$$(26) \qquad \frac{\partial x^i_\epsilon}{\partial t^\alpha}(t) = X^i_\alpha(x_\epsilon(t), u_\epsilon(t)),\ x_\epsilon(0) = x_0,\ t \in R^m_+.$$

Let us try to understand how the choices of s_A and $v_A(\cdot)$ cause $x_\epsilon(\cdot)$ to differ from $x(\cdot)$, for small $\epsilon > 0$. Firstly, we set $y_\alpha(t) = Y_\alpha(t, s) y_{\alpha s},\ t \in \Omega_{s\infty}$, for the solution of the Cauchy variational problem (linear PDE system)

$$\frac{\partial y^i_\alpha}{\partial t^\beta}(t) = y^j_\alpha(t) \frac{\partial X^i_\beta}{\partial x^j}(x(t), u(t)),\ y_\alpha(s) = y_{\alpha s},\ t \in \Omega_{s\infty},$$

where the points $y_{\alpha s} \in R^n$ are given and $Y_\alpha(t, s)$ is the *transition matrix.*

We define

$$y_{\alpha s_A} = X_\alpha(x(s_A), v_A(s)) - X_\alpha(x(s_A), u(s_A)),\ A = 1, 2, ..., N.$$

and we replace the Lemma of dynamics and simple control variations with

Lemma 3.5. (dynamics and multiple control variations) *If $u_\epsilon(.)$ is a multiple variation of the control $u(\cdot)$, then*

$$x_\epsilon(t) = x(t) + \epsilon^\alpha y_\alpha(t) + o(\epsilon) \text{ as } \epsilon \to 0$$

uniformly for t in compact subsets of R^m_+, where

$$\begin{cases} y_\alpha(t) = 0 & t \in \Omega_{0s_1} \\ y_\alpha(t) = \sum_{A=1}^P \lambda^A Y_\alpha(t, s_A) y_{\alpha s_A} & t \in \Omega_{0s_{A+1}} \setminus \Omega_{0s_A},\ P = 1, 2, ..., N-1 \\ y_\alpha(t) = \sum_{A=1}^N \lambda^A Y_\alpha(t, s_A) y_{\alpha s_A} & t \in \Omega_{s_N \infty}. \end{cases}$$

Definition 3.1. (cones of variations) Let $0 < s_1 \leq s_2 \leq ... \leq s_N < t$ and $y_{\alpha s_A} \in R^n,\ A = 1, ..., N$. For each α, the set

$$K_\alpha(t) = \{\sum_{A=1}^N \lambda^A Y_\alpha(t, s_A) y_{\alpha s_A} | N = 1, 2, ...; \lambda^A > 0\}.$$

is called the *cone of variations at multitime t*.

We remark that each $K_\alpha(t)$ is a convex cone in R^n, consisting in all changes in the state $x(t)$ (up to order ϵ) we can make by multiple variations of the control $u(\cdot)$ (see the previous Lemma). To study the geometry of $K_\alpha(t)$, we need the following topological Lemma:

Lemma 3.6. (zeroes of a vector field) *Let S be a closed, bounded, convex subset of R^n and $p \in Int S$. If $Y : S \to R^n$ is a continuous vector field satisfying $||Y(x) - x|| < ||x - p||,\ \forall x \in \partial S$, then there exists a point $x \in S$ such that $Y(x) = p$.*

Proof. For the general case, we assume after a translation that $p = 0$, and $0 \in \operatorname{Int} S$. We map S onto $B(0, 1)$ by a radial dilation, and map Y by rigid motion. This process convert the general case to the next case.

Suppose that S is the unit ball $B(0, 1)$ and $p = 0$. The inequality in hypothesis is equivalent to $(Y(x), x) > 0,\ \forall x \in \partial B(0, 1)$. Consequently, for small t, the continuous mapping $Z(x) = x - tY(x)$ maps $B(0, 1)$ into itself. According Brouwer Fixed Point Theorem, the mapping Z has a fixed point, let say $Z(x^*) = x^*$, and hence $Y(x^*) = 0$. □

3.5 Fixed endpoint problem

The fixed endpoint problem is characterized by the constraint $x(\tau) = x_1$, where $\tau = \tau(u(\cdot))$ is the first multitime that $x(\cdot)$ hits the target point x_1. In this context, the cost functional is

$$P(u(\cdot)) = \int_{\gamma_{0\tau}} X^0_\beta(x(t), u(t)) dt^\beta.$$

Adding a new variable. We define again $x^0 : \Omega_{0t_0} \to R$ as the solution of initial problem

$$\frac{\partial x^0}{\partial t^\alpha}(t) = X^0_\alpha(x(t), u(t)),\ x^0(0) = 0,\ t \in \Omega_{0\tau},$$

and reintroduce

$$x = (x^1, ..., x^n),\ \overline{x} = (x^0, x),\ \overline{x}_0 = (0, x_0),\ \overline{x}(\cdot) = (x^0(\cdot), x(\cdot)),$$

$$\overline{X}_\alpha(\overline{x}, u) = (X^0_\alpha(x, u), X_\alpha(x, u)),\ \overline{g}(\overline{x}) = a_0 x^0.$$

The problem is replaced to the controlled dynamics

$$(\overline{PDE}) \qquad \frac{\partial \overline{x}}{\partial t^\alpha}(t) = \overline{X}_\alpha(x(t), u(t)),\ \overline{x}(0) = \overline{x}_0,\ t \in \Omega_{0\tau}.$$

and maximizing

$$(\overline{P}), \qquad \overline{P}(u(\cdot)) = \overline{g}(\overline{x}(\tau)) = a_0 x^0(\tau).$$

τ being the first multitime that $x(\tau) = x_1$. More precisely, the last n components of $\overline{x}(\tau)$ are prescribed, and we want to maximize the first component x^0.

Now we suppose that $u^*(\cdot)$ is an optimal control for this problem, corresponding to the optimal m-sheet $x^*(\cdot)$. Let us construct the associate costate $p^*(\cdot)$, satisfying the maximization principle (M). To simplify the notations, we drop the superscript $*$.

Cones of variations. We use the previous theory, replacing the n variables with $n+1$ variables, and the nm variables with $(n+1)m$ variables, i.e., overlining the mathematical objects. Also, we denote $\overline{y}_\alpha(t) = \overline{Y}^\beta_\alpha(t, s)\overline{y}_{\beta s}$ for the solution of the Cauchy problem

$$\frac{\partial \overline{y}^i_\alpha}{\partial t^\beta}(t) = \overline{y}^j_\alpha(t)\frac{\partial \overline{X}^i_\beta}{\partial \overline{x}^j}(\overline{x}(t), u(t)),\ \overline{y}_\alpha(s) = \overline{y}_{\alpha s},\ t \in \Omega_{\tau\infty} \setminus \Omega_{0s},$$

where the points $\overline{y}_{\alpha s} \in R^{n+1}$ are given and $\overline{Y}^\beta_\alpha(t, s) = \left(\overline{Y}^\beta_{\alpha i}(t, s)\right)$ is the *fundamental (transition) operator*. In this way, for $0 < s_1 \leq s_2 \leq ... \leq s_N < \tau$, the cones of variations are

$$\overline{K}_\alpha(\tau) = \{\sum_{A=1}^{N} \lambda^A \overline{Y}^\beta_\alpha(\tau, s_A)\overline{y}_{\beta s_A} | N = 1, 2, ...; \lambda^A > 0\},$$

where

$$\overline{y}_{\beta s_A} = \overline{X}_\beta(\overline{x}(s_A), u_A) - \overline{X}_\beta(\overline{x}(s_A), u(s_A)),\ u_A \in U_x.$$

Let us show that each cone $\overline{K}_\alpha$ does not occupy all the space R^{n+1}, and consequently it stays aside of a hyperplane.

Lemma 3.7. (geometry of one cone of variations) *The $(n+1)$-dimensional versor $e_{\alpha 0} = (1, 0, ..., 0)$ is outside Int$\overline{K}_\alpha$.*

Proof. **Step 1**. Suppose $e_{\alpha 0} \in \text{Int}\,\overline{K}_\alpha$. Then there exist $n+1$ linearly independent vectors $z_{\alpha 0}, z_{\alpha 1}, ..., z_{\alpha n} \in \overline{K}_\alpha$ such that $e_{\alpha 0} = \lambda^A z_{\alpha A}$ with positive constants λ^A and $z_{\alpha A} = \overline{Y}^\beta_\alpha(\tau, s_A)\overline{y}_{\beta s_A}$ for suitable multitimes $0 < s_0 < s_1 < ... < s_n < \tau$ and vectors

$$\overline{y}_{\beta s_A} = \overline{X}_\beta(\overline{x}(s_A), u_A) - \overline{X}_\beta(\overline{x}(s_A), u(s_A)),\ A = 0, 1, 2, ..., n.$$

Step 2. Since for each $\overline{K}_\alpha$ we must follow the same rules, we drop the index α and we write $x = \lambda^A z_A$. For small $\eta > 0$, we introduce the closed and convex set $S = \{x = \lambda^A z_A | 0 \leq \lambda^A \leq \eta\}$. Since the vectors $z_0, z_1, ..., z_n$ are linearly independent, the interior of S is nonvoid.

Now, for small $||\epsilon|| > 0$, we define

$$Z^\epsilon : S \to R^{n+1},\ Z^\epsilon(x) = \overline{x}_\epsilon(\tau) - \overline{x}(\tau),\ x = \lambda^A z_A,$$

where $\overline{x}_\epsilon(\cdot)$ solves (27) for the multiple control variation $u_\epsilon(\cdot)$. If $\mu, \eta, \epsilon > 0$ are small enough, then we conjecture $Z^\epsilon(x) = p = \mu e_0 = (\mu, 0, ..., 0)$. This is true since

$$||Z^\epsilon(x) - x|| = ||\overline{x}_\epsilon(\tau) - \overline{x}(\tau) - x|| = o(||x||),\ \text{as } x \to 0, x \in S$$

$$||Z^\epsilon(x) - x|| = ||\overline{x}_\epsilon(\tau) - \overline{x}(\tau) - x|| < ||x - p||,\ \forall x \in \partial S.$$

Step 3. Consequently we can build a control $u_\epsilon(\cdot)$, having a multiple variation with the associated response $\overline{x}_\epsilon(\cdot) = (x^0_\epsilon(\cdot), x_\epsilon(\cdot))$ satisfying $x_\epsilon(\tau) = x_1$ and $x^0_\epsilon(\tau) > x^0(\tau)$. This is in contradiction to the optimality of $u(\cdot)$ since the last inequality says that we can increase the cost. □

Existence of the costate. We restore the superscript $*$ and we formulate

Theorem 3.8. (multitime maximum principle) *Suppose the problem is not abnormal. Then there exists a function $p^* : \Omega_{0\tau^*} \to R^n$ satisfying the adjoint dynamics (ADJ) and the maximization principle (M).*

Proof. **Step 1**. The geometry of each cone of variations shows the existence of a nonzero vector w of components $(w_j) \in R^{n+1}$, $j = 0, 1, ..., n$ such that $w_j z^j_\alpha \leq 0,\ \forall z = (z^j_\alpha) \in \overline{K}_\alpha(t)$, and $w_j e^j_{\alpha 0} = w_0 \geq 0$.

Let $\overline{p}^*(\cdot)$ be the solution of $(\overline{ADJ})$, with the terminal condition $\overline{p}^*(\tau) = w$. Then $p^*{}_0 = w_0 \geq 0$.

We fix the multitime $0 \leq s < \tau$, the control $u(\cdot) \in \mathcal{U}$, and we set

$$\overline{y}_{\alpha s} = \overline{X}_\alpha(\overline{x}^*(s), u(s)) - \overline{X}_\alpha(\overline{x}^*(s), u^*(s)).$$

Solving

$$\frac{\partial \overline{y}^i_\alpha}{\partial t^\beta}(t) = \overline{y}^j_\alpha(t) \frac{\partial \overline{X}^i_\beta}{\partial \overline{x}^j}(\overline{x}(t), u(t)),\ \overline{y}_\alpha(s) = \overline{y}_{\alpha s},\ t \in \Omega_{\tau\infty} \setminus \Omega_{0s},$$

we can write (see Lemma of variation of cost)

$$0 \geq w_j \overline{y}^j_\beta(\tau) = \overline{p}^*{}_j(\tau) \overline{y}^j_\beta(\tau) = \overline{p}^*{}_j(s) \overline{y}^j_\beta(s).$$

Consequently

$$\overline{p}^*{}_i(s)(\overline{X}^i_\beta(\overline{x}^*(s), u(s)) - \overline{X}^i_\beta(\overline{x}^*(s), u^*(s))) \leq 0$$

or

$$\overline{H}_\beta(\overline{x}^*(s), \overline{p}^*(s), u(s)) \leq \overline{H}_\beta(\overline{x}^*(s), \overline{p}^*(s), u^*(s)).$$

Step 2. If the number w_0 is zero, then we have an abnormal problem and the multitime maximum principle must be reformulated.

If $w_0 > 0$, then the maximization formulas reduces to

$$H_\beta(x^*(s), p^*(s), u) \leq H_\beta(x^*(s), p^*(s), u^*(s)).$$

But this is the maximization principle (M). $\square$

Missing discussions: measurability concerns, how we proceed when some multitimes s_A are equal, how we prove that the functions

$$t \to H_\beta(x^*(t), p^*(t), u(t))$$

are constants in free endpoint problems, and respectively

$$H_\beta(x^*(t), p^*(t), u(t)) = 0$$

in fixed endpoint problems?

4 Sufficiency of the multitime maximum principle

Let us consider the controlled functional

$$F(x(\cdot), u(\cdot)) = \int_{\Gamma_{0t_0}} X^0_\beta(t, x(t), x_\alpha(t), u(t)) dt^\beta.$$

Definition 4.1. A point $(x^*(\cdot), u^*(\cdot))$ is called critical point of the functional $F(x(\cdot), u(\cdot))$ if

$$\frac{\partial X^0_\beta}{\partial x^i}(t, x^*(t), x^*_\alpha(t), u^*(t)) - D_\gamma \left(\frac{\partial X^0_\beta}{\partial x^i_\gamma} \right) (t, x^*(t), x^*_\alpha(t), u^*(t)) = 0$$

$$\frac{\partial X^0_\beta}{\partial u^a}(t, x^*(t), x^*_\alpha(t), u^*(t)) = 0.$$

Definition 4.2. Let $(x^*(\cdot), u^*(\cdot))$ be a critical point of the functional $F(x(\cdot), u(\cdot))$. If there exists a vector function $\eta(t, x(t), x^*(t), x_\alpha(t), x^*_\alpha(t), u(t), u^*(t))$ such that

$$\eta(t, x(t), x^*(t), x_\alpha(t), x^*_\alpha(t), u(t), u^*(t))|_{x(t)=x^*(t)} = 0$$

and a vector function $\xi(t, x(t), x^*(t), x_\alpha(t), x^*_\alpha(t), u(t), u^*(t))$ with the property

$$F(x(\cdot), u(\cdot)) - F(x^*(\cdot), u^*(\cdot)) \leq$$

$$\int_{\Gamma_{0t_0}} \Big(\eta^i(t, x(t), x^*(t), x_\alpha(t), x^*_\alpha(t), u(t), u^*(t)) \frac{\partial X^0_\beta}{\partial x^i}(t, x^*(t), x^*_\alpha(t), u^*(t))$$

$$+ D_\gamma \eta^i(t, x(t), x^*(t), x_\alpha(t), x^*_\alpha(t), u(t), u^*(t)) \frac{\partial X^0_\beta}{\partial x^i_\gamma}(t, x^*(t), x^*_\alpha(t), u^*(t))$$

$$+ \xi^a(t, x(t), x^*(t), x_\alpha(t), x^*_\alpha(t), u(t), u^*(t)) \frac{\partial X^0_\beta}{\partial u^a}(t, x^*(t), x^*_\alpha(t), u^*(t)) \Bigg) dt^\beta,$$

then the functional is called incave at x, u on Ω_{0t_0} with respect to η and ξ.

Theorem 4.1. *The functional $F(x(\cdot), u(\cdot))$ is incave if and only if each critical point $(x^*(\cdot), u^*(\cdot))$ is a global maximum point.*

Now we come back to our original control problem P, PDE with the control Hamiltonian 1-form

$$H_\beta(t, x(t), u(t), p(t)) = X^0_\beta(t, x(t), u(t)) + p_i(t)x^i_\beta(t).$$

The functional can be written

$$P(x(\cdot), u(\cdot)) = \int_{\Gamma_{0t_0}} \left(H_\beta(t, x(t), x_\alpha(t), u(t)) - p_i(t)x^i_\beta(t)\right) dt^\beta.$$

If $u^*(\cdot)$ is a C^1 optimal control, and $x^*(\cdot)$ is the optimal evolution, then

$$\frac{\partial H_\beta}{\partial x^i}(t, x^*(t), u^*(t)) + \frac{\partial p_i}{\partial t^\beta}(t) = 0, \frac{\partial H}{\partial u^a}(t, x^*(t), u^*(t)) = 0,$$

i.e., $(x^*(\cdot), u^*(\cdot))$ is a critical point of the functional $p(x(\cdot), u(\cdot))$. The point $(x^*(\cdot), u^*(\cdot))$ is a global maximum point if and only if the functional $P(x(\cdot), u(\cdot))$ is incave.

Theorem 4.2. *The problem P, PDE has a solution $(x^*(\cdot), u^*(\cdot))$ if and only if the functional $P(x(\cdot), u(\cdot))$ is incave.*

If we add some concavity restrictions to the components of the control tensor and the constrained set, then we can prove the sufficiency of the conditions of multitime maximum principle.

Definition 4.3. A function $f : R^n \to R$ is called *concave* if its Hessian matrix $(f_{x^i x^j})$ is negative definite at each point x^*, i.e., the associated quadratic form $f_{x^i x^j}(x^*)(x^{*i} - x^i)(x^{*j} - x^j)$ is negative, for an arbitrary point x^*.

A concave function satisfies the inequality $f(x*) - f(x) \geq f_{x^i}(x^*)(x^{*i} - x^i)$.

Theorem 4.3. *If the triplet (x^*, p^*, u^*) satisfies the conditions of multitime maximum principle and each component of the control tensor evaluated at $p = p^*$ is (strictly) concave in the pair (x^*, u^*), then (x^*, p^*, u^*) is the (unique) solution of the control problem.*

Proof. Let us have in mind that we must maximize the functional

$$P(u(\cdot)) = \int_{\Gamma_{0t_0}} X^0_\beta(t, x(t), u(t))dt^\beta$$

subject to the evolution system. We fix a pair (x^*, u^*), where u^* is a candidate optimal m-sheet of the controls and x^* is a candidate optimal m-sheet of the states. Calling P^* the values of the functionals for (x^*, u^*), let us prove that

$$P^* - P = \int_{\Gamma_{0t_0}} (X^{*0}_\beta - X^0_\beta)dt^\beta \geq 0,$$

where the strict inequality holds under strict concavity. Denoting $H^*_\alpha = H_\alpha(t, x^*, p^*, u^*)$ and $H_\alpha = H_\alpha(t, x, p^*, u)$, we find

$$P^* - P = \int_{\Gamma_{0t_0}} (X^{*0}_\beta - X^0_\beta) dt^\beta = \int_{\Gamma_{0t_0}} \left((H^*_\beta - p^*_i \frac{\partial x^{*i}}{\partial t^\beta}) - (H_\beta - p^*_i \frac{\partial x^i}{\partial t^\beta}) \right) dt^\beta.$$

Integrating by parts, we obtain

$$P^* - P = \int_{\Gamma_{0t_0}} \left((H^*_\beta + x^{*i} \frac{\partial p^*_i}{\partial t^\beta}) - (H_\beta + x^i \frac{\partial p^*_i}{\partial t^\beta}) \right) dt^\beta$$

$$- \left(p^*_i(t_0) x^{*i}(t_0) - p^*_i(0) x^{*i}(0) \right) + \left(p^*_i(t_0) x^i(t_0) - p^*_i(0) x^i(0) \right).$$

Taking into account that any admissible m-sheet has the same initial and terminal conditions as the optimal m-sheet, we derive

$$P^* - P = \int_{\Gamma_{0t_0}} \left((H^*_\beta - H_\beta) + \frac{\partial p^*_i}{\partial t^\beta} (x^{*i} - x^i) \right) dt^\beta.$$

The definition of concavity implies

$$\int_{\Gamma_{0t_0}} \left((H^*_\beta - H_\beta) + \frac{\partial p^*_i}{\partial t^\beta} (x^{*i} - x^i) \right) dt^\beta$$

$$\geq \int_{\Gamma_{0t_0}} \left((x^{*i} - x^i) \frac{\partial H^*_\beta}{\partial x^i} + (u^{*a} - u^a) \frac{\partial H^*_\beta}{\partial u^a} + \frac{\partial p^*_i}{\partial t^\beta} (x^{*i} - x^i) \right) dt^\beta$$

$$= \int_{\Gamma_{0t_0}} \left((x^{*i} - x^i)(\frac{\partial H^*_\beta}{\partial x^i} + \frac{\partial p^*_i}{\partial t^\beta}) + (u^{*a} - u^a) \frac{\partial H^*_\beta}{\partial u^a} \right) dt^\beta.$$

This last equality follows from that all " $*$ " variables satisfy the conditions of the multitime maximum principle. In this way, $P^* - P \geq 0$. □

Remark 4.4. *One can ensure the sufficiency of the multitime maximum principle from more primitive assumptions on the functions X^0_α and X^i_α. For example, if X^0_α are concave in (x, u) and X^i_α are concave (convex) in (x, u), and $p^*_i \geq 0 (\leq 0)$, then H_β evaluated at p^*_i are concave in the pair (x, u).*

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