Structure of the indicatrix bundle of Finsler-Rizza manifolds

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Abstract. In this paper, we construct a framed f-structure on the slit tangent space of a Rizza manifold. This induces on the indicatrix bundle an almost contact metric. We find the conditions under which this structure reduces to a contact or to a Sasakian structure. Finally we study these structures on Kählerian Finsler manifolds.

M.S.C. 2010: 53B40, 53C60, 32Q60, 53C15.
Key words: Contact structures; framed f-manifolds; Kählerian structures; Rizza manifolds.

1 Introduction

In [15, 16], G. B. Rizza introduced on almost complex manifolds (M, J), the so-called *Rizza condition*. For Finsler structures (M, L) - where L is the Finsler metric, a triple (M, J, L) which satisfies the Rizza condition is called *a Finsler-Rizza manifold*. It was shown by Heil ([4]) that if the fundamental tensor g_{ij} of the Finsler metric L is compatible with the almost complex structure J, then the Finsler structure is Riemannian. This leads to considering a weaker assumption on the Finsler metric, like the Rizza condition. The notion of Rizza manifolds was developed further in Finslerian framework by Y. Ichijyō, who showed that every tangent space to a Rizza manifold is a complex Banach space and that Rizza condition does not necessarily reduce the Finsler metric to a Riemannian one ([7, 8]). He also defined a notion of Kähler Finsler metric and showed that for a Kähler Finsler manifold, the almost complex structure is integrable. Recently, several mathematicians studied Rizza and Kähler Finsler manifolds ([5, 9, 10, 13, 17]).

Let (M, L) be an n-dimensional Finsler manifold (n even), admitting an almost complex structure $J^i_j(x)$. Let $g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2(x, y)$ be the associated Finsler metric tensor field [11, 12, 14]. We say that the fundamental function $L(x, y)$ satisfies *the Rizza condition*, if

$$L(x, \phi_\theta y) = L(x, y), \quad \forall \theta \in \mathbb{R}, \quad \text{where} \quad \phi^i_{\theta j} = \delta^i_j \cos\theta + J^i_j \sin\theta.$$

Balkan Journal of Geometry and Its Applications, Vol.16, No.2, 2011, pp. 1-12.

In this case, M is called an *almost Hermitian Finsler manifold* or simply, a *Rizza manifold*. The Rizza condition practically requires that, for all $y \in T_xM$, the Finsler norm L should be constant on the orbit $y \to \phi_\theta(y),\ \theta \in \mathbb{R}$, which lives inside the fiber T_xM. We note that each such orbit is determined by the action on $y \in T_xM$ of a special element ϕ of the loop group $\Lambda Aut(T_xM)$ of $id-$automorphisms of the tangent bundle, defined by $\theta \in S^1 \xrightarrow{\phi} \phi_\theta = I \cdot \cos\theta + J \cdot \sin\theta \in Aut(T_xM)$.

In [6], it was shown that the Rizza condition holds if the following equivalent conditions hold true:[1]

- $g_{pq}(x, \phi_\theta y)\phi^p_{\theta i}\phi^q_{\theta j} = g_{ij}(x, y), \quad \forall\theta \in \mathbb{R}$;
- $g_{ij}(x, y)J^i_m(x)y^m y^j = 0; \quad (g_{im}(x, y) - g_{pq}(x, y)J^p_i(x)J^q_m(x))\, y^m = 0$;
- $g_{im}(x, y)J^m_j(x) + g_{jm}(x, y)J^m_i(x) + 2C_{ijm}(x, y)J^m_r(x)y^r = 0$, where C_{ijm} is the Cartan tensor of the Finsler metric.

As well, it was shown that if the Finsler metric L satisfies $g_{pq}(x, y)J^p_i(x)J^q_j(x) = g_{ij}(x, y)$, then it reduces to a Riemannian metric and accordingly, (J, g) reduces to an almost Hermitian structure.

Now let M be a Rizza manifold. Then

$$\widetilde{g}_{ij}(x, y) = \frac{1}{2}\left(\, g_{ij}(x, y) + g_{pq}(x, y)J^p_i(x)J^q_j(x)\,\right), \tag{1.1}$$

is a homogeneous symmetric generalized metric on TM, which is called a *generalized Finsler metric*; this satisfies the relation $\widetilde{g}_{pq}(x, y)J^p_i(x)J^q_j(x) = \widetilde{g}_{ij}(x, y)$. Also, in a Rizza manifold, the relation $g_{ij}(x, y) = \dot{\partial}_i\dot{\partial}_j\left(\frac{1}{2}\widetilde{g}_{pq}(x, y)y^p y^q\right)$ holds true.

Conversely, for M a manifold admitting an almost complex structure $J^i_j(x)$, Ichijyō showed ([6]) that M admits a Finsler metric which constructs a Rizza structure together with $J^i_j(x)$, if and only if M admits a generalized Finsler metric $\widetilde{g}_{ij}(x, y)$ satisfying the following conditions:

- $\widetilde{g}_{jk}(x, y) = \widetilde{g}_{pq}(x, y)J^p_j(x)J^q_k(x)$;
- $\dot{\partial}_k\widetilde{g}_{pq}(x, y)y^p y^q = 0$;
- $(\widetilde{g}_{jk}(x, y) + \dot{\partial}_k\widetilde{g}_{jm}(x, y)y^m)\zeta^j\zeta^k$ is positive definite.

In the following, for a given Rizza manifold (M, J, L) we shall denote

$$J_{ij}(x, y) = g_{im}(x, y)J^m_j(x), \qquad \widetilde{J}_{ij}(x, y) = \widetilde{g}_{im}(x, y)J^m_j(x). \tag{1.2}$$

Then we obtain the following relations ([6])

$$\widetilde{J}_{ij} = -\widetilde{J}_{ji}, \quad \widetilde{J}_{im}J^m_j = -\widetilde{g}_{ij}, \quad \widetilde{J}_{ij} = \frac{1}{2}(J_{ij} - J_{ji}). \tag{1.3}$$

By using [6, Theorem, eq. (3)] and denoting $y_i := g_{ij}y^j$, we infer

$$\widetilde{g}_{ij}y^j = \frac{1}{2}(g_{ij}y^j + g_{pq}J^p_i J^q_j y^j) = y_i, \tag{1.4}$$

[1] We shall assume that all the Latin indices run within the range $\overline{1, n}$, and that the Einstein summation rule for indices applies.

It follows that

$$\widetilde{g}_{ij}y^iy^j = L^2. \tag{1.5}$$

Also, the relations (1.2), (1.3) and (1.4) lead to

$$\widetilde{J}_{ij}y^j = -\widetilde{J}_{ji}y^j = -\widetilde{g}_{jm}J_i^m y^j = -J_i^m y_m = -J_i^j y_j. \tag{1.6}$$

By a *contact manifold* we mean a differentiable manifold M^{2n+1} endowed with with a 1-form η, such that $\eta \wedge (d\eta)^n \neq 0$. It is well known ([2]) that given the form η, there exist a unique vector field ξ, such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$; this field is called the *characteristic vector field* or the *Reeb vector field* of the contact form η.

We say that a Riemannian metric g is an *associated metric* for a contact form η if (i) $\eta(X) = g(X, \xi)$ and (ii) there exist a field of endomorphisms f, such that $f^2 = -I + \eta \otimes \xi$ and $d\eta(X, Y) = g(fX, Y)$. Then we refer to (f, ξ, η, g) as a *contact metric structure* and to M^{2n+1} with such a structure as a *contact metric manifold.*

An *almost contact structure*, (f, ξ, η), consists of a field of endomorphisms f, and a 1-form η such that $f^2 = -I+\eta\otimes\xi$ and $\eta(\xi) = 1$ and an *almost contact metric structure* owns a Riemannian metric which satisfies the compatibility condition $g(fX, fY) = g(X, Y) - \eta(X)\eta(Y)$.

The product $M^{2n+1} \times \mathbb{R}$ carries a natural almost complex structure defined by $J(X, g\frac{d}{dt}) = (fX - g\xi, \eta(X)\frac{d}{dt})$ and the underlying almost contact structure is said to be *normal* if J is integrable. The normality condition can be expressed as $N = 0$, where N is defined by $N = N_f + d\eta \otimes \xi$ and N_f is the Nijenhuis tensor of f. A *Sasakian* manifold is a normal contact metric manifold.

A framed f-structure is a natural generalization of an almost contact structure. It was introduced by S. I. Goldberg and K. Yano [3]. We recall its definition: let N be a $(2n + s)$-dimensional manifold endowed with an endomorphism f of rank $2n$ of the tangent bundle, satisfying $f^3 + f = 0$. If there exists on N the vector fields (ξ_b) and the 1-forms η^a $(a, b = 1, 2, \ldots, s)$ such that

$$\eta^a(\xi_b) = \delta^a_b, \quad f(\xi_a) = 0, \quad \eta^a \circ f = 0, \quad f^2 = -I + \sum_{a=1}^{s} \eta^a \otimes \xi_a,$$

where I is the identity automorphism of the tangent bundle, then N is said to be *a framed f-manifold* [1].

In this paper, by using an almost complex structure $J^i_j(x)$ and the generalized Finsler metric $\widetilde{g}_{ij}(x, y)$ defined by (1.1) we introduce an almost Hermitian structure (G, F) on TM. Then we obtain a framed f-structure on $\widetilde{TM} = TM\backslash\{0\}$ and by its restriction to the indicatrix bundle IM we introduce an almost contact metric structure on IM. We further find the conditions under which this structure is a contact metric structure and a Sasakian structure. Also, we study these structures on Kählerian Finsler manifolds.

2 A framed f- structure on $\widetilde{TM}$

Let M be a Rizza manifold, and let TM be the tangent bundle over M. We shall further use a local frame $(\delta_i, \dot{\partial}_i)$ of TM, where we put $\delta_i = \partial_i - G^m_i\dot{\partial}_m$, $\dot{\partial}_i = \frac{\partial}{\partial y^i}$,

$\partial_i = \frac{\partial}{\partial x^i}$ and G_i^m are the components of an Ehresmann connection on M. Then we can globally define on TM an (1,1)-tensor field F, such that

$$F(\delta_i) = J_i^k \dot{\partial}_k, \quad F(\dot{\partial}_i) = J_i^k \delta_k. \tag{2.1}$$

Since $J_k^i J_j^k = -\delta_j^i$, it is obvious that F defines an almost complex structure on TM. Moreover, we can introduce an inner product $\langle \cdot\,, \cdot \rangle$ such that

$$\langle \delta_i, \delta_j \rangle = \widetilde{g}_{ij}, \quad \langle \delta_i, \dot{\partial}_j \rangle = 0, \quad \langle \dot{\partial}_i, \dot{\partial}_j \rangle = \widetilde{g}_{ij}. \tag{2.2}$$

Then the inner product gives on TM a globally defined Riemann metric G, as follows

$$G = \widetilde{g}_{ij} dx^i dx^j + \widetilde{g}_{ij} \delta y^i \delta y^j, \tag{2.3}$$

where $(dx^i, \delta y^i)$ is the dual basis of $(\delta_i, \dot{\partial}_i)$ and $\delta y^i = dy^i + G_m^i dx^m$. Using (2.1) and (2.3), we obtain

$$G(F(\delta_i), F(\delta_j)) = J_i^k J_j^l G(\dot{\partial}_k, \dot{\partial}_l) = J_i^k J_j^l \widetilde{g}_{kl} = \widetilde{g}_{ij} = G(\delta_i, \delta_j).$$

Similarly, we obtain $G(F(\dot{\partial}_i), F(\dot{\partial}_j)) = \widetilde{g}_{ij} = G(\dot{\partial}_i, \dot{\partial}_j)$ and $G(F(\delta_i), F(\dot{\partial}_j)) = 0 = G(\delta_i, \dot{\partial}_j)$, which ultimately lead to

Theorem 2.1. *On every Rizza manifold M, its tangent bundle TM admits an almost Hermitian structure (G, F), where F and G are defined by (2.1) and (2.3), respectively.*

Now we define the vector fields ξ_1, ξ_2 and 1-forms η^1, η^2 on $\widetilde{TM}$ respectively by

$$\xi_1 := y^m J_m^i \delta_i, \quad \xi_2 := y^i \dot{\partial}_i \tag{2.4}$$

and

$$\eta^1 := \frac{1}{L^2} y^m \widetilde{J}_{im} dx^i, \quad \eta^2 := \frac{1}{L^2} y_i \delta y^i. \tag{2.5}$$

Lemma 2.2. *Let F be defined by (2.1), ξ_1, ξ_2 be defined by (2.4) and η^1, η^2 be defined by (2.5). Then we have*

$$F(\xi_1) = -\xi_2, \quad F(\xi_2) = \xi_1, \tag{2.6}$$

and

$$\eta^1 \circ F = \eta^2, \quad \eta^2 \circ F = -\eta^1. \tag{2.7}$$

Proof. Relation (2.6) immediately follows from (2.1) and (2.4). To prove (2.7), we use the adapted basis $(\delta_i, \dot{\partial}_i)$ and (1.3), which infer

$$(\eta^1 \circ F)(\dot{\partial}_i) = J_i^k \eta^1(\delta_k) = \frac{1}{L^2} y^m J_i^k \widetilde{J}_{km} = \frac{1}{L^2} \widetilde{g}_{mi} y^m = \frac{1}{L^2} y_i = \eta^2(\dot{\partial}_i),$$
$$(\eta^2 \circ F)(\delta_i) = J_i^k \eta^2(\dot{\partial}_k) = \frac{1}{L^2} J_i^k y_k = \frac{1}{L^2} J_i^k \widetilde{g}_{kr} y^r = -\frac{1}{L^2} \widetilde{J}_{ir} y^r = -\eta^1(\delta_i),$$

and $(\eta^1 \circ F)(\delta_i) = 0 = \eta^2(\delta_i)$, $(\eta^2 \circ F)(\dot{\partial}_i) = 0 = \eta^1(\dot{\partial}_i)$, which lead to (2.7). $\square$

Lemma 2.3. *Let G be defined by (2.3), ξ_1, ξ_2 be defined by (2.4) and η_1, η_2 be defined by (2.5). Then we have*

$$\eta^1(X) = \frac{1}{L^2} G(X, \xi_1), \quad \eta^2(X) = \frac{1}{L^2} G(X, \xi_2), \tag{2.8}$$

and

$$\eta^a(\xi_b) = \delta^a_b, \ \ a, b = 1, 2, \tag{2.9}$$

where $X \in \mathcal{X}(\widetilde{TM})$.

Proof. In the adapted basis $(\delta_i, \dot{\partial}_i)$, we obtain

$$G(\delta_i, \xi_1) = G(\delta_i, y^k J^j_k \delta_j) = y^k J^j_k \widetilde{g}_{ij} = y^k \widetilde{J}_{ik} = L^2 \eta^1(\delta_i).$$

Similarly, we infer $G(\dot{\partial}_i, \xi_2) = L^2 \eta^2(\dot{\partial}_i)$, $G(\dot{\partial}_i, \xi_1) = 0 = L^2 \eta^1(\dot{\partial}_i)$ and $G(\delta_i, \xi_2) = 0 = L^2 \eta^2(\delta_i)$, which completes the proof of (2.8). As well, using (1.3) and (1.5) we get

$$\eta^1(\xi_1) = \eta^1(y^k J^i_k \delta_i) = \frac{1}{L^2} y^k J^i_k y^m \widetilde{J}_{im} = \frac{1}{L^2} y^k y^m \widetilde{g}_{mk} = 1,$$

and $\eta^1(\xi_2) = 0$, which prove the first relation of (2.9). In a similar way, one can prove the second relation of (2.9). $\square$

Using the almost complex structure F, we define a new tensor field f of type $(1,1)$ on $\widetilde{TM}$ by

$$f = F + \eta^1 \otimes \xi_2 - \eta^2 \otimes \xi_1. \tag{2.10}$$

By using the above relation, (2.6) and (2.9) we infer that $f(\xi_1) = f(\xi_2) = 0$. Similarly, from (2.7) and (2.9)we conclude that $(\eta^1 \circ f)(X) = (\eta^2 \circ f)(X) = 0$, where $X \in \mathcal{X}(\widetilde{TM})$. These relations and (2.6), (2.10) give us

$$f^2(X) = F(f(X)) = F^2(X) + \eta^1(X) F(\xi_2) - \eta^2(X) F(\xi_1) = -X + \eta^1(X)\xi_1 + \eta^2(X)\xi_1.$$

Therefore we have

Lemma 2.4. *The following properties hold true for tensor field f defined by (2.10):*

$$f(\xi_a) = 0, \quad \eta^a \circ f = 0, \quad a = 1, 2, \tag{2.11}$$

$$f^2(X) = -X + \eta^1(X)\xi_1 + \eta^2(X)\xi_2, \quad X \in \mathcal{X}(\widetilde{TM}). \tag{2.12}$$

Theorem 2.5. *Let $f, (\xi_a), (\eta^a), a = 1, 2$ be defined respectively by (2.10), (2.4) and (2.5). Then the triple $(f, \{\xi_a\}, \{\eta^a\})$ provides a framed f-structure on $\widetilde{TM}$.*

Proof. Considering Lemma 2.4 and relation (2.9), in order to complete the proof, we need to prove $f^3 + f = 0$ and to show that f is of rank $2n-2$. Since $f(\xi_1) = f(\xi_2) = 0$, by using (2.11) we derive that $f^3(X) = -f(X)$, for all $X \in \mathcal{X}(\widetilde{TM})$. Now we need to show that $\ker f = Span\{\xi_1, \xi_2\}$. It is clear that $Span\{\xi_1, \xi_2\} \subseteq \ker f$, because $f(\xi_1) = f(\xi_2) = 0$. Now let $X \in \ker f$; then $f(X) = 0$ implies that $F(X) + \eta^1(X)\xi_2 - \eta^2(X)\xi_1 = 0$. Thus we infer that $F^2(X) = \eta^2(X)F(\xi_1) - \eta^1(X)F(\xi_2)$. Since $F^2 = -I$, it follows from Lemma 2.2 that $X = \eta^1(X)\xi_1 + \eta^2(X)\xi_2$, i.e., $X \in Span\{\xi_1, \xi_2\}$. $\square$

Theorem 2.6. *The Riemannian metric G defined by (2.3) satisfies*

$$G(f(X), f(Y)) = G(X,Y) - L^2\eta^1(X)\eta^1(Y) - L^2\eta^2(X)\eta^2(Y),$$

for $X, Y \in \mathcal{X}(\widetilde{TM})$.

Proof. By using (2.10), we get the local expression of f as follows:

$$f(\delta_i) = \Big(J_i^k + \frac{1}{L^2}\widetilde{J}_{ir}y^r y^k\Big)\dot{\partial}_k, \tag{2.13}$$

$$f(\dot{\partial}_i) = \Big(J_i^k - \frac{1}{L^2}J_r^k y^r y_i\Big)\delta_k. \tag{2.14}$$

By using (2.3), (2.13) we obtain

$$G(f(\delta_i), f(\delta_j)) = J_i^k J_j^h \widetilde{g}_{kh} + \frac{y^r}{L^2}\Big[\widetilde{J}_{jr}J_i^k y^h \widetilde{g}_{kh} + \widetilde{J}_{ir}J_j^h y^k \widetilde{g}_{kh} + \frac{1}{L^2}\widetilde{J}_{ir}\widetilde{J}_{jl}y^l y^k y^h \widetilde{g}_{kh}\Big].$$

By using (1.5) and (1.6), the above equation leads to

$$G(f(\delta_i), f(\delta_j)) = \widetilde{g}_{ij} - \frac{1}{L^2}\widetilde{J}_{ir}\widetilde{J}_{jk}y^r y^k. \tag{2.15}$$

But we have the relations

$$G(\delta_i, \delta_j) = \widetilde{g}_{ij}, \quad \eta^1(\delta_i)\eta^2(\delta_j) = \frac{1}{L^4}y^r\widetilde{J}_{ir}y^k\widetilde{J}_{jk} \quad \eta^2(\delta_i)\eta^2(\delta_j) = 0,$$

which allow to rewrite (2.15) as follows

$$G(f(\delta_i), f(\delta_j)) = G(\delta_i, \delta_j) - L^2\eta^1(\delta_i)\eta^1(\delta_j) - L^2\eta^2(\delta_i)\eta^2(\delta_j).$$

We similarly obtain

$$G(f(\dot{\partial}_i), f(\dot{\partial}_j)) = \widetilde{g}_{ij} - \frac{1}{L^2}y_i y_j = G(\dot{\partial}_i, \dot{\partial}_j) - L^2\eta^1(\dot{\partial}_i)\eta^1(\dot{\partial}_j) - L^2\eta^2(\dot{\partial}_i)\eta^2(\dot{\partial}_j),$$

and

$$G(f(\delta_i), f(\dot{\partial}_j)) = 0 = G(\delta_i, \dot{\partial}_j) - L^2\eta^1(\delta_i)\eta^1(\dot{\partial}_j) - L^2\eta^2(\delta_i)\eta^2(\dot{\partial}_j),$$

which completes the proof. □

Let us set $\Omega(X,Y) = G(fX, Y)$ for $X, Y \in \mathcal{X}(\widetilde{TM})$. Then we have

Theorem 2.7. *The map Ω is a 2-form on $\widetilde{TM}$. Further, the annihilator of Ω is $Span\{\xi_1, \xi_2\}$, which is an integrable distribution.*

Proof. By using (1.4), (2.13) and (2.14), we obtain

$$\Omega(\delta_i, \dot{\partial}_j) = G(f(\delta_i), \dot{\partial}_j) = J_i^k\widetilde{g}_{kj} + \frac{1}{L^2}y^r\widetilde{J}_{ir}y^k\widetilde{g}_{kj} = \widetilde{J}_{ji} + \frac{1}{L^2}\widetilde{J}_{ir}y^r y_j, \tag{2.16}$$

$$\Omega(\dot{\partial}_i, \delta_j) = G(f(\dot{\partial}_i), \delta_j) = (J_i^k - \frac{1}{L^2}y_i y^r J_r^k)\widetilde{g}_{kj} = \widetilde{J}_{ji} - \frac{1}{L^2}y_i y^r\widetilde{J}_{jr}, \tag{2.17}$$

$$\Omega(\delta_i, \delta_j) = \Omega(\dot{\partial}_i, \dot{\partial}_j) = 0. \tag{2.18}$$

Since $\widetilde{J}_{ij} = -\widetilde{J}_{ji}$, then from (2.16) and (2.17), we derive that $\Omega(\delta_i, \dot{\partial}_j) = -\Omega(\dot{\partial}_j, \delta_i)$. Then, using (2.16) we obtain $\Omega(X, Y) = -\Omega(Y, X)$, $\forall X, Y \in \mathcal{X}(\widetilde{TM})$. Thus Ω is a 2-form on $\widetilde{TM}$. We further show that the annihilator of Ω is $Span\{\xi_1, \xi_2\}$. Let $X = X^i\delta_i + X^{\bar{i}}\dot{\partial}_i \in \mathcal{X}(\widetilde{TM})$. By using (1.4) and (2.16)-(2.16) we get

$$\Omega(X, \xi_1) = X^{\bar{i}} y^m J^j_m (\widetilde{J}_{ji} - \frac{1}{L^2} y_i y^r \widetilde{J}_{jr}) = X^{\bar{i}} y^m (\widetilde{g}_{im} - \frac{1}{L^2} y_i y^r \widetilde{g}_{rm}) = 0.$$

We similarly obtain $\Omega(X, \xi_2) = 0$. Therefore the annihilator of Ω contains $Span\{\xi_1, \xi_2\}$. Now let X belong to the annihilator of Ω. Then we have $\Omega(X, \delta_j) = 0$ and $\Omega(X, \dot{\partial}_j) = 0$. If we assume $X = X^i\delta_i + X^{\bar{i}}\dot{\partial}_i$, then these equations give us the relations

$$X^i(\widetilde{J}_{ji} + \frac{1}{L^2}\widetilde{J}_{im} y^m y_j) = 0, \tag{2.19}$$

$$X^{\bar{i}}(\widetilde{J}_{ji} - \frac{1}{L^2}\widetilde{J}_{jm} y^m y_i) = 0. \tag{2.20}$$

Since $\widetilde{J}_{ij} = \widetilde{g}_{im} J^m_j$, then by direct calculation we obtain $\widetilde{J}_{ji} J^j_r \widetilde{g}^{rh} = \delta^h_i$ and $\widetilde{J}_{sr}\widetilde{g}^{rh} = -J^h_s$. Transvecting (2.19) and (2.20) with $J^j_r \widetilde{g}^{rh}$ and using these equations we derive that $X^i = \frac{1}{L^2} X^h \widetilde{J}_{hm} y^m y^s J^i_s$ and $X^{\bar{i}} = \frac{1}{L^2} X^{\bar{h}} y_h y^i$. Thus we deduce that

$$X = \frac{1}{L^2} X^h \widetilde{J}_{hm} y^m y^s J^i_s \delta_i + \frac{1}{L^2} X^{\bar{h}} y_h y^i \dot{\partial}_i = \frac{1}{L^2} X^h \widetilde{J}_{hm} y^m \xi_1 + \frac{1}{L^2} X^{\bar{h}} y_h \xi_2.$$

Therefore $Span\{\xi_1, \xi_2\}$ contains the annihilator of Ω, and consequently the annihilator of Ω is $Span\{\xi_1, \xi_2\}$. Also we obtain $[\xi_1, \xi_2] = y^j J^i_j \delta_i = \xi_1$. Hence the distribution $Span\{\xi_1, \xi_2\}$ is integrable. $\square$

3 Almost contact structure on the indicatrix bundle

Let (M, J, L) be a Finsler-Rizza manifold, and let IM be its indicatrix bundle of (M, L), i.e.,

$$IM = \{(x, y) \in \widetilde{TM} | L(x, y) = 1\},$$

which is a submanifold of dimension $2n - 1$ of $\widetilde{TM}$. Note that ξ_2 defined in (2.4) is a unit vector field on IM, since $G(\xi_2, \xi_2) = 1$. It is easy to show that ξ_2 is a normal vector field on IM with respect to the metric G. Indeed, if the local equations of IM in $\widetilde{TM}$ are given by

$$x^i = x^i(u^\gamma), \quad y^i = y^i(u^\gamma), \quad \gamma \in \{1, \ldots, 2n-1\},$$

then we have

$$\frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial u^\gamma} + \frac{\partial L}{\partial y^i}\frac{\partial y^i}{\partial u^\gamma} = 0.$$

Since F is a horizontally covariant constant, i.e., $\frac{\partial L}{\partial x^i} = N^k_i \frac{\partial L}{\partial y^k}$, it follows that

$$\Big(N^k_i \frac{\partial x^i}{\partial u^\gamma} + \frac{\partial y^k}{\partial u^\gamma}\Big)\frac{\partial L}{\partial y^k} = 0. \tag{3.1}$$

The natural frame field $\{\frac{\partial}{\partial u^\gamma}\}$ on IM is given by

$$\frac{\partial}{\partial u^\gamma}=\frac{\partial x^i}{\partial u^\gamma}\frac{\partial}{\partial x^i}+\frac{\partial y^i}{\partial u^\gamma}\frac{\partial}{\partial y^i}=\frac{\partial x^i}{\partial u^\gamma}\delta_i+\Big(N_i^k\frac{\partial x^i}{\partial u^\gamma}+\frac{\partial y^k}{\partial u^\gamma}\Big)\frac{\partial}{\partial y^k}. \tag{3.2}$$

Thus by using (3.1) and the equality $\frac{y_k}{L}=\frac{\partial L}{\partial y^k}$, we obtain

$$G\Big(\frac{\partial}{\partial u^\gamma},\xi_2\Big)=L\Big(N_i^k\frac{\partial x^i}{\partial u^\gamma}+\frac{\partial y^k}{\partial u^\gamma}\Big)\frac{\partial L}{\partial y^k}=0. \tag{3.3}$$

Thus ξ_2 is orthogonal to any tangent to IM vector. Also, the vector field ξ_1 is tangent to IM, since $G(\xi_1,\xi_2)=0$.

Lemma 3.1. *The hypersurface IM is invariant with respect to f, i.e., $f(T_u(IM))\subseteq T_u(IM)$, $\forall u\in IM$.*

Proof. By using (2.8) and the second equation of (2.11), we get

$$G\left(f\left(\frac{\partial}{\partial u^\gamma}\right),\xi_2\right)=(\eta^2\circ f)\left(\frac{\partial}{\partial u^\gamma}\right)=0.,\quad \forall\gamma=1,2,\ldots,2n-1.$$

Thus the hypersurface IM is invariant with respect to f. □

Lemma 3.2. *Let the framed f-structure be given by Theorem 2.5. Then restricting this to IM, we have*

$$\eta^1=y^m\widetilde{J}_{im}dx^i,\quad \eta^2=0,\quad f(X)=F(X)+\eta^1(X)\xi_2,\quad \forall X\in\mathcal{X}(IM).$$

Proof. Since $L^2=1$ on IM and $\eta^2(X)=G(X,\xi_2)=0$, the claim follows. □

Denoting $\bar\eta=\eta^1|_{IM}$, $\bar\xi=\xi|_{IM}$, $\bar f=f|_{IM}$ and $\bar G=G|_{IM}$, then from Theorem 2.6 we get $\bar G(\bar f(X),\bar f(Y))=\bar G(X,Y)-\bar\eta(X)\bar\eta(Y)$. Therefore Theorem 2.5 and Lemma 3.2 imply that

Theorem 3.3. *Let the framed f-structure be given by Theorem 2.5. Then $(\bar f,\bar\xi,\bar\eta,\bar G)$ defines an almost contact metric structure on IM.*

If we put $\dot\delta_j=\bar f(\delta_j)$, then we get n local vector fields which are tangent to IM, since IM is an invariant hypersurface. These, together with $\delta_i, i=1,\ldots,n$, are all tangent to IM and they are not linearly independent. But, considering $\delta_i, i=1,\ldots,n$ and $\dot\delta_j$ with $j=1,\ldots,n-1$, we obtain a set $(\delta_i,\dot\delta_j)$ of local vector fields which form a local bases in the fibers of the tangent bundle to IM. By using (2.13) and the definition of $\bar\eta$, we obtain

$$d\bar\eta(\delta_i,\dot\delta_j)=-\Big(J_j^k\widetilde{J}_{ik}+y^m(\dot\partial_k\widetilde{J}_{im})J_j^k+y^ry^k\widetilde{J}_{jr}\widetilde{J}_{ik}+y^r\widetilde{J}_{jr}y^my^k\dot\partial_k\widetilde{J}_{im}\Big). \tag{3.4}$$

It is easy to see that $\widetilde{J}_{im}$ is positive homogenous of degree 0. Thus we have $y^k\dot\partial_k\widetilde{J}_{im}=0$. Also, we have $J_j^k\widetilde{J}_{ik}=-\widetilde{g}_{ij}$. By using these relations and (3.4), we obtain

$$d\bar\eta(\delta_i,\dot\delta_j)=\widetilde{g}_{ij}-\widetilde{J}_{ik}\widetilde{J}_{jr}y^ry^k-y^m(\dot\partial_k\widetilde{J}_{im})J_j^k.$$

It is known that $\nabla_i y^m = 0$, where ∇ means h-covariant derivative with respect to the Cartan Finsler connection (Γ^i_{jk}, G^i_j). From this equation we derive that $\delta_i y^m = -y^r \Gamma^m_{ri}$. Thus we get

$$d\bar{\eta}(\delta_i, \delta_j) = y^m(\nabla_i \widetilde{J}_{jm} - \nabla_j \widetilde{J}_{im}).$$

Also, we infer $d\bar{\eta}(\dot{\delta}_i, \dot{\delta}_j) = 0$. On the other hand, the relation

$$\bar{G}(\bar{f}(X), \bar{f}(Y)) = \bar{G}(X, Y) - \bar{\eta}(X)\bar{\eta}(Y)$$

gives us

$$\Omega(\delta_i, \dot{\delta}_j) = \widetilde{g}_{ij} - \widetilde{J}_{ir}\widetilde{J}_{js} y^r y^s, \quad \text{and} \quad \Omega(\delta_i, \delta_j) = \Omega(\dot{\delta}_i, \dot{\delta}_j) = 0.$$

These relations imply that $\Omega = d\bar{\eta}$ if and only if

$$\nabla_i \widetilde{J}_{jm} - \nabla_j \widetilde{J}_{im} = 0, \quad \text{and} \quad y^m(\dot{\partial}_k \widetilde{J}_{im}) J^k_j = 0.$$

Thus we have the following.

Theorem 3.4. *Let M be a Rizza manifold endowed with the Cartan Finsler connection. Then $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{G})$ is a contact metric structure on IM if and only if $y^m \dot{\partial}_j \widetilde{J}_{im} = 0$ and $\nabla_i \widetilde{J}_{jm} = \nabla_j \widetilde{J}_{im}$.*

It is known that a Rizza manifold satisfying $\nabla_k J^i_j = 0$ is said to be a *Kählerian Finsler manifold*. Further, if g_{ij} is a Riemannian metric, then we call it *Kählerian Riemann manifold*. It was proved (Ichijyō, [6]) that if M be a Kählerian Finsler manifold, then M is Landsberg space, and that the following relations hold true

$$\nabla_k J^i_j = 0, \quad \nabla_k g_{ij} = \nabla_k \widetilde{g}_{ij} = 0, \quad \nabla_k \widetilde{J}_{ij} = 0$$

on Kählerian Finsler manifolds. These relations and Theorem 3.4 lead to

Corollary 3.5. *If M is a Kählerian Finslerian manifold, then $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{G})$ is a contact Riemannian structure on IM if and only if $y^m \dot{\partial}_j \widetilde{J}_{im} = 0$.*

Now, let g_{ij} be a Riemannian metric. Then we have $\dot{\partial}_k g_{ij} = 0$, and consequently, $\dot{\partial}_k \widetilde{g}_{ij} = 0$. Therefore we obtain $\dot{\partial}_k \widetilde{J}_{im} = 0$, since $\widetilde{J}_{im} = \widetilde{g}_{ij} J^j_m$ and J^j_m is a function depending on (x^h) only. This leads to the following

Corollary 3.6. *If M is a Kählerian Riemann manifold, then $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{G})$ is a contact Riemannian structure on IM.*

We further obtain sufficiency conditions for the contact metric structure $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{G})$ given by Theorem 3.4, to be Sasakian.

Theorem 3.7. *Let M be a Rizza manifold endowed with the Cartan Finsler connection. Then $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{G})$ on IM is Sasakian if and only if*

$$\nabla_i J^h_j + J^r_j L^h_{ir} = \nabla_j J^h_i + J^r_i L^h_{jr}, \tag{3.5}$$

$$R^k{}_{ji} = (J^k_j \widetilde{J}_{il} - J^k_i \widetilde{J}_{jl}) y^l, \tag{3.6}$$

$$y^m \widetilde{J}_{im} \Big[y^h J^r_h \nabla_r J^k_j + y^h (J^k_l + \widetilde{J}_{ls} y^s y^k) \nabla_j J^l_h + y^h y^l y^k J^r_h \nabla_r \widetilde{J}_{jl} \Big] = 0, \tag{3.7}$$

$$
\begin{aligned}
&y^r y^m\Big[(\widetilde{J}_{ir}\nabla_j J^k_m - \widetilde{J}_{jr}\nabla_i J^k_m) + J^h_m y^n J^k_n y^s(\widetilde{J}_{ir}\nabla_h \widetilde{J}_{js} - \widetilde{J}_{jr}\nabla_h \widetilde{J}_{is}) \\
&+J^h_m(\widetilde{J}_{jr}\nabla_h J^l_i - \widetilde{J}_{ir}\nabla_h J^l_j) + J^h_m L^s_{hl} J^k_s(\widetilde{J}_{jr}J^l_i - \widetilde{J}_{ir}J^l_j)\Big] = 0,
\end{aligned} \tag{3.8}
$$

where $L^s_{hl} = y^m\nabla_m C^s_{hl}$ *is the Landsberg tensor.*

Proof. The Nijenhuis tensor field of $\bar{f}$ is defined by

$$
N_{\bar{f}}(\delta_i,\delta_j) = [\dot{\delta}_i,\dot{\delta}_j] - \bar{f}[\dot{\delta}_i,\delta_j] - \bar{f}[\delta_i,\dot{\delta}_j] + \bar{f}^2[\delta_i,\delta_j]. \tag{3.9}
$$

By using (2.13), (2.14) and the relation $y^h(\dot{\partial}_h\widetilde{J}_{ir}) = 0$, we obtain

$$
[\dot{\delta}_i,\dot{\delta}_j] = \Big[J^h_i(\dot{\partial}_h\widetilde{J}_{jl})y^l y^k + J^k_i\widetilde{J}_{jl}y^l - J^h_j(\dot{\partial}_h\widetilde{J}_{il})y^l y^k - J^k_j\widetilde{J}_{il}y^l\Big]\dot{\partial}_k. \tag{3.10}
$$

Also, (2.13), (2.14) and the relation $\delta_j y^l = -y^r\Gamma^l_{rj}$ give us

$$
\begin{aligned}
\bar{f}[\dot{\delta}_i,\delta_j] + \bar{f}[\delta_i,\dot{\delta}_j] = &\Big[\nabla_i J^h_j - \nabla_j J^h_i + y^l y^h(\nabla_i\widetilde{J}_{jl} - \nabla_j\widetilde{J}_{il}) \\
&+J^r_j L^h_{ir} - J^r_i L^h_{jr}\Big]J^k_h\delta_k.
\end{aligned} \tag{3.11}
$$

Similarly, we infer

$$
\bar{f}^2[\delta_i,\delta_j] = -R^k{}_{ij}\dot{\partial}_k = R^k{}_{ji}\dot{\partial}_k, \tag{3.12}
$$

where $R^k{}_{ij} = \delta_j G^k_i - \delta_i G^k_j$. Setting (3.10), (3.11) and (3.12) into (3.9), we get

$$
\begin{aligned}
N_{\bar{f}}(\delta_i,\delta_j) = &\Big[\nabla_j J^h_i - \nabla_i J^h_j + y^l y^h(\nabla_j\widetilde{J}_{il} - \nabla_i\widetilde{J}_{jl}) + J^r_i L^h_{jr} - J^r_j L^h_{ir}\Big]J^k_h\delta_k \\
&+\Big[R^k{}_{ji} + J^h_i(\dot{\partial}_h\widetilde{J}_{jl})y^l y^k + J^k_i\widetilde{J}_{jl}y^l - J^h_j(\dot{\partial}_h\widetilde{J}_{il})y^l y^k - J^k_j\widetilde{J}_{il}y^l\Big]\dot{\partial}_k.
\end{aligned} \tag{3.13}
$$

By consideration of $d\bar{\eta}(\delta_i,\delta_j) = y^m(\nabla_i\widetilde{J}_{jm} - \nabla_j\widetilde{J}_{im})$, we obtain

$$
(d\bar{\eta}\otimes\bar{\xi})(\delta_i,\delta_j) = y^l(\nabla_i\widetilde{J}_{jl} - \nabla_j\widetilde{J}_{il})y^h J^k_h\delta_k.
$$

Therefore we get

$$
\begin{aligned}
N(\delta_i,\delta_j) = &\Big[(\nabla_j J^h_i - \nabla_i J^h_j + J^r_i L^h_{jr} - J^r_j L^h_{ir})J^k_h\Big]\delta_k + \Big[R^k{}_{ji} + J^h_i(\dot{\partial}_h\widetilde{J}_{jl})y^l y^k \\
&+J^k_i\widetilde{J}_{jl}y^l - J^h_j(\dot{\partial}_h\widetilde{J}_{il})y^l y^k - J^k_j\widetilde{J}_{il}y^l\Big]\dot{\partial}_k.
\end{aligned} \tag{3.14}
$$

Similarly, we obtain

$$
\begin{aligned}
N(\dot{\delta}_i,\delta_j) = &-\bar{f}N_{\bar{f}}(\delta_i,\delta_j) + \Big[y^m y^r J^k_r(J^h_i\dot{\partial}_h\widetilde{J}_{jm} - J^h_j\dot{\partial}_h\widetilde{J}_{im}) + y^m\widetilde{J}_{im}(\delta^k_j \\
&-\widetilde{J}_{js}y^s J^k_h y^h - y^r J^s_r R^h{}_{sj}J^k_h)\Big]\delta_k + \Big[y^m\widetilde{J}_{im}y^h(J^r_h\nabla_r J^k_j \\
&+J^k_l\nabla_j J^l_h + \widetilde{J}_{ls}y^s y^k\nabla_j J^l_h + y^l y^k J^r_h\nabla_r\widetilde{J}_{jl})\Big]\dot{\partial}_k,
\end{aligned} \tag{3.15}
$$

and

$$
\begin{aligned}
N(\dot{\delta}_i, \dot{\delta}_j) = & -N_{\bar{f}}(\delta_i, \delta_j) + \Big[y^r y^m J^k m(\nabla_j \widetilde{J}_{ir} - \nabla_i \widetilde{J}_{jr}) + y^r y^m (\widetilde{J}_{ir} \nabla_j J^k_m - \widetilde{J}_{jr} \nabla_i J^k_m) \\
& + y^r y^m J^h_m y^n J^k_n y^s (\widetilde{J}_{ir} \nabla_h \widetilde{J}_{js} - \widetilde{J}_{jr} \nabla_h \widetilde{J}_{is}) + y^r y^m J^h_m (\widetilde{J}_{jr} \nabla_h J^l_i - \widetilde{J}_{ir} \nabla_h J^l_j) \\
& + y^r y^m J^h_m L^s_{hl} J^k_s (\widetilde{J}_{jr} J^l_i - \widetilde{J}_{ir} J^l_j) \Big] \delta_k + \Big[y^r y^m J^s_m (\widetilde{J}_{ir} R^k{}_{js} - \widetilde{J}_{jr} R^k{}_{is}) \\
& + y^r \widetilde{J}_{ir} y^s \widetilde{J}_{js} y^m J^l_m y^n J^h_n R^k{}_{lh} - y^r \widetilde{J}_{ir} J^k_j + y^r \widetilde{J}_{jr} J^k_i \Big] \dot{\partial}_k. \qquad (3.16)
\end{aligned}
$$

Since $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{G})$ is a contact structure, then from Theorem 3.4 we have $y^m \dot{\partial}_j \widetilde{J}_{im} = 0$ and $\nabla_i \widetilde{J}_{jm} = \nabla_j \widetilde{J}_{im}$. Setting these equations into (3.14), (3.15) and (3.16), we imply that $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{G})$ is Sasakian if and only if

$$
J^k_h (\nabla_i J^h_j + J^r_j L^h_{ir}) = J^k_h (\nabla_j J^h_i + J^r_i L^h_{jr}), \qquad (3.17)
$$

$$
R^k{}_{ji} = (J^k_j \widetilde{J}_{il} - J^k_i \widetilde{J}_{jl}) y^l, \qquad (3.18)
$$

$$
y^m \widetilde{J}_{im} \delta^k_j - \widetilde{J}_{im} y^m \widetilde{J}_{js} y^s J^k_h y^h - y^m \widetilde{J}_{im} y^r J^s_r R^h{}_{sj} J^k_h = 0, \qquad (3.19)
$$

$$
y^m [\widetilde{J}_{im} y^h J^r_h \nabla_r J^k_j + \widetilde{J}_{im} y^h (J^k_l + \widetilde{J}_{ls} y^s y^k) \nabla_j J^l_h + \widetilde{J}_{im} y^h y^l y^k J^r_h \nabla_r \widetilde{J}_{jl}] = 0, \qquad (3.20)
$$

$$
\begin{aligned}
& y^r y^m \Big[(\widetilde{J}_{ir} \nabla_j J^k_m - \widetilde{J}_{jr} \nabla_i J^k_m) + J^h_m y^n J^k_n y^s (\widetilde{J}_{ir} \nabla_h \widetilde{J}_{js} - \widetilde{J}_{jr} \nabla_h \widetilde{J}_{is}) \\
& \quad + J^h_m (\widetilde{J}_{jr} \nabla_h J^l_i - \widetilde{J}_{ir} \nabla_h J^l_j) + J^h_m L^s_{hl} J^k_s (\widetilde{J}_{jr} J^l_i - \widetilde{J}_{ir} J^l_j) \Big] = 0, \qquad (3.21)
\end{aligned}
$$

$$
\begin{aligned}
& y^r y^m J^s_m (\widetilde{J}_{ir} R^k{}_{js} - \widetilde{J}_{jr} R^k{}_{is}) + y^r \widetilde{J}_{ir} y^s \widetilde{J}_{js} y^m J^l_m y^n J^h_n R^k{}_{lh} \\
& \quad - y^r \widetilde{J}_{ir} J^k_j + y^r \widetilde{J}_{jr} J^k_i = 0. \qquad (3.22)
\end{aligned}
$$

It is easy to check that if (3.18) holds, then (3.19) and (3.22) hold true as well. This completes the proof. □

Now, if M is a Kählerian Finsler manifold, then we have $\nabla_i J^k_j = \nabla_i \widetilde{J}_{jk} = 0$ and $L^k_{ij} = 0$. Thus, the relations (3.5), (3.7) and (3.8) hold true, and we have the following

Theorem 3.8. *Let (M, F) be a Kählerian Finsler manifold with the Cartan Finsler connection. Then $(\bar{f}, \bar{\xi}, \bar{\eta}, \bar{G})$ on IM is Sasakian if and only if the following relation holds:*

$$
R^k{}_{ji} = (J^k_j \widetilde{J}_{il} - J^k_i \widetilde{J}_{jl}) y^l.
$$

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