

Multitime optimal control with area integral costs on boundary

Constantin Udriște, Andreea Bejenaru

Abstract. This paper joins some concepts that appear in Mechanics, Field Theory, Differential Geometry and Control Theory in order to solve multitime optimal control problems with area integral costs on boundary. Section 1 recalls the multitime maximum principle in the sense of the first author. The main results in Section 2 include the needle-shaped control variations, the adjoint PDEs, the behavior of infinitesimal deformations and other ingredients needed for the multitime maximum principle in case of no running cost and in case of running cost. Section 3 solves the previous multitime control problems based on techniques of variational calculus. Section 4 shows that concavity is a sufficient condition in multitime optimal control theory. Section 5 contains an example illustrating the utility of such a multitime optimal control theory.

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1 Multitime maximum principle

Let $N = \mathbb{R}^m$ with global coordinates (t^1, \dots, t^m) , $M = \mathbb{R}^n$ with global coordinates (x^1, \dots, x^n) and \mathbb{R}^k having global coordinates (u^1, \dots, u^k) . We consider the hyperparallelepiped $T = \Omega_{0t_0} \subset \mathbb{R}^m$ defined by the opposite diagonal points $0 = (0, \dots, 0)$ and $t_0 = (t_0^1, \dots, t_0^m)$ and a subset $U \subset \mathbb{R}^k$. For the multi-times $s = (s^1, \dots, s^m)$ and $t = (t^1, \dots, t^m)$ we denote $s \leq (<) t$ if and only if $s^\alpha \leq (<) t^\alpha$, $\alpha = 1, \dots, m$ (*product order*). We also consider the *L - type set* $[s] = \{t \in \mathbb{R}_+^m \mid t \leq s \text{ and } \exists \alpha = \overline{1, m} \text{ such that } t^\alpha = s^\alpha\}$. We shall use the following *L - type intervals*:

$$([s], [t]) = \Omega_{0,t} \setminus \Omega_{0,s}; \quad [[s], [t]] = ([s], [t]) \cup [s]; \quad ([s], [t]) = ([s], [t]) \setminus [t].$$

Let $X_\alpha = (X_\alpha^i) : T \times M \times U \rightarrow \mathbb{R}^n$ be C^1 vector fields. For a given control function $u : T \rightarrow \mathbb{R}^k$, suppose the evolution PDEs system (*controlled m-flow*)

$$(PDE) \quad \frac{\partial x^i}{\partial t^\alpha}(t) = X_\alpha^i(t, x(t), u(t)), \quad x(0) = x_0, \quad t \in \Omega_{0t_0} \subset \mathbb{R}_+^m.$$

has solution. As it is wellknown, this PDEs system has solutions if and only if the *complete integrability conditions*

$$(CIC) \quad \frac{\partial X_\alpha^i}{\partial t^\beta} + \frac{\partial X_\alpha^i}{\partial x^j} X_\beta^j + \frac{\partial X_\alpha^i}{\partial u^a} \frac{\partial u^a}{\partial t^\beta} = \frac{\partial X_\beta^i}{\partial t^\alpha} + \frac{\partial X_\beta^i}{\partial x^j} X_\alpha^j + \frac{\partial X_\beta^i}{\partial u^a} \frac{\partial u^a}{\partial t^\alpha}$$

are satisfied. The relations CIC define the set of *admissible controls*

$$\mathcal{U} = \{u(\cdot) : \mathbb{R}_+^m \rightarrow U \mid u(\cdot) \text{ is constrained by (CIC)}\}.$$

The multitime evolution system (PDE) is used as a constraint when we want to optimize a *multitime cost functional*

$$(J) \quad J[u(\cdot)] = \int_{\Omega_{0t_0}} X(t, x(t), u(t)) dt + \int_{\partial\Omega_{0t_0}} g(t, x(t)) d\sigma,$$

where the *running cost* $X : N \times M \times U \rightarrow \mathbb{R}$ is a C^2 function (*nonautonomous Lagrangian*), and $g : \partial N \times M \rightarrow \mathbb{R}$ is a C^1 *boundary cost*.

Multitime optimal control problem. Find

$$\begin{aligned} \max_{u(\cdot)} J[u(\cdot)] &= \int_{\Omega_{0t_0}} X(t, x(t), u(t)) dt + \int_{\partial\Omega_{0t_0}} g(t, x(t)) d\sigma \\ \text{subject to} \quad \frac{\partial x^i}{\partial t^\alpha}(t) &= X_\alpha^i(t, x(t), u(t)), \quad i = 1, \dots, n, \quad \alpha = 1, \dots, m, \\ u(t) &\in \mathcal{U}, \quad t \in \Omega_{0t_0}, \quad x(0) = x_0. \end{aligned}$$

The multitime maximum principle (necessary condition) asserts that the existence of an optimal control $u^*(\cdot)$ implies the existence of costate vector functions $(p_0^*, p^*)(\cdot) = (p_0^*(\cdot), p_i^{*\alpha}(\cdot))$, which together with the optimal m -sheet $x^*(\cdot)$ satisfy a suitable PDEs system. Similar to single-time theory, this multitime maximum principle involves an appropriate *control Hamiltonian*

$$H(t, x, p_0, p, u) = p_0 X(t, x, u) + p_i^\alpha X_\alpha^i(t, x, u).$$

Theorem 1.1. (multitime maximum principle) *Suppose $u^*(\cdot)$ is optimal for (PDE), (J) and that $x^*(\cdot)$ is the corresponding optimal m -sheet. Then there exists a map $(p_0^*, p^*) = (p_0^*, p_i^*) : \Omega_{0t_0} \rightarrow \mathbb{R}^{mn+1}$ such that*

$$(PDE) \quad \frac{\partial x^{*i}}{\partial t^\alpha}(t) = \frac{\partial H}{\partial p_i^\alpha}(t, x^*(t), p_0^*(t), p^*(t), u^*(t)),$$

$$(ADJ) \quad \frac{\partial p_i^{*\alpha}}{\partial t^\alpha}(t) = -\frac{\partial H}{\partial x^i}(t, x^*(t), p_0^*(t), p^*(t), u^*(t))$$

and

$$(M) \quad \frac{\partial H}{\partial u^a}(t, x^*(t), p_0^*(t), p^*(t), u^*(t)) = 0, \quad \forall t \in \Omega_{0t_0}.$$

Finally, the boundary conditions

$$(t_0) \quad n_\alpha \mathcal{P}_i^{*\alpha} |_{\partial\Omega_{0t_0}} = \frac{\partial g}{\partial x^i} |_{\partial\Omega_{0t_0}}$$

are satisfied, where, for each multi-time s , n denotes the covector corresponding to the unit normal vector on $\partial\Omega_{0s}$, that is $n = (n_\alpha)$, where

$$n_\alpha(t) = \begin{cases} 1, & \text{if } t^\alpha = s^\alpha; \\ -1, & \text{if } t^\alpha = 0; \\ 0, & \text{otherwise.} \end{cases}$$

We call $x^*(\cdot)$ the state of the optimally controlled system and $(p_0^*, p^*(\cdot))$ the costate map. Even more, we can consider $p_0^* = 1$.

Remark 1.2. 1) Explicitly, (PDE) means the identities

$$\frac{\partial x^{*i}}{\partial t^\beta}(t) = X_\beta^i(t, x^*(t), u^*(t)), \quad \beta = 1, \dots, m; \quad i = 1, \dots, n,$$

and (ADJ) means the identities

$$\frac{\partial p_i^{*\alpha}}{\partial t^\alpha}(t) = - \left(p_0^*(t) \frac{\partial X}{\partial x^i} + p_j^{*\alpha}(t) \frac{\partial X_\alpha^j}{\partial x^i} \right) (t, x^*(t), u^*(t)).$$

2) The identities (PDE) reveal the controlled evolution PDEs, the identities (ADJ) suggest the adjoint PDEs, the relation (M) represents the multitime maximum principle and the relation (t_0) means the transversability (boundary) condition.

3) The multitime maximum principle states necessary conditions that must hold on an optimal m -sheet of evolution.

2 Needle-shaped control variations and adjoint PDEs

The general proofs of multitime maximum principle rely on a special type of variations, called *needle-shaped control variations*.

Suppose $u^*(\cdot)$ is a candidate optimal control and that $x^*(\cdot)$ is the corresponding m -sheet. Fixing a multitime $s \in ([0], [t_0])$ and $u(\cdot) \in \mathcal{U}$, an m -needle variation is a family of controls u_ϵ obtained replacing u^* with u on $([s - \epsilon], [s])$. In other words, given the multitime $s \in ([0], [t_0])$ and an admissible control $u(t)$, we set $\epsilon \in [[0], [s]]$ and define the modified control

$$u_\epsilon(t) = \begin{cases} u(t) & \text{if } t \in ([s - \epsilon], [s]) \\ u^*(t) & \text{otherwise.} \end{cases}$$

We also denote $x_\epsilon(\cdot)$ the corresponding response of our system, i.e.

$$\frac{\partial x_\epsilon^i}{\partial t^\alpha}(t) = X_\alpha^i(t, x_\epsilon(t), u_\epsilon(t)), \quad x_\epsilon(0) = x_0, \quad t \in \Omega_{0t_0} \subset \mathbb{R}_+^m.$$

Let then $y_\alpha^i(t) = \frac{\partial x_\epsilon^i(t)}{\partial \epsilon^\alpha} |_{\epsilon=0}$ be the infinitesimal gradient deformation of the m -sheet $x^*(t)$ induced by the previous control variation.

Lemma 2.1. Let $\varphi : \Omega_{0,s} \times (-\delta, \delta)^m \rightarrow \mathbb{R}$, $\varphi = \varphi(t, \epsilon)$ be a differentiable parametrized function. Then

$$\frac{\partial}{\partial \epsilon^\alpha} \int_{[[s-\epsilon], [s]]} \varphi(t, \epsilon) dt \Big|_{\epsilon=0} = \int_{[s]} \varphi(t, 0) n_\alpha(t) d\sigma.$$

Proof. Successively, we can write

$$\begin{aligned} & \frac{\partial}{\partial \epsilon^\alpha} \int_{[[s-\epsilon], [s]]} \varphi(t, \epsilon) dt \Big|_{\epsilon=0} = \frac{\partial}{\partial \epsilon^\alpha} \left[\int_{\Omega_{0s}} \varphi(t, \epsilon) dt - \int_{\Omega_{0s-\epsilon}} \varphi(t, \epsilon) dt \right] \Big|_{\epsilon=0} \\ &= \frac{\partial}{\partial \epsilon^\alpha} \left[\int_{\Omega_{0s}} \varphi(t, \epsilon) dt - \int_0^{s^m - \epsilon^m} \dots \int_0^{s^1 - \epsilon^1} \varphi(t, \epsilon) dt^1 \dots dt^m \right] \Big|_{\epsilon=0} \\ &= \int_{\Omega_{0s}} \frac{\partial \varphi}{\partial \epsilon^\alpha}(t, 0) dt - \int_{\Omega_{0s-\epsilon}} \frac{\partial \varphi}{\partial \epsilon^\alpha}(t, \epsilon) dt \Big|_{\epsilon=0} \\ &+ \int_0^{s^m - \epsilon^m} \dots \int_0^{s^{\alpha+1} - \epsilon^{\alpha+1}} \int_0^{s^{\alpha-1} - \epsilon^{\alpha-1}} \dots \int_0^{s^1 - \epsilon^1} \varphi(t^1, \dots, s^\alpha - \epsilon^\alpha, \dots, t^m, \epsilon) dt_\alpha \Big|_{\epsilon=0} \\ &= \int_{[s]} \varphi(t, 0) n_\alpha(t) d\sigma. \end{aligned}$$

□

Lemma 2.2. The infinitesimal deformation y induced by the needle-shaped control variation satisfies the following relations:

$$y_\alpha^i(t) = 0, \text{ if } t \in [[0], [s]],$$

$$\int_{[s]} y_\beta^i(t) n_\alpha(t) d\sigma = \int_{[s]} [X_\alpha^i(t, x_*(t), u(t)) - X_\alpha^i(t, x^*(t), u^*(t))] n_\beta(t) d\sigma,$$

$$\frac{\partial y_\beta^i}{\partial t^\alpha}(t) = \frac{\partial X_\alpha^i}{\partial x^j}(t, x^*(t), u^*(t)) y_\beta^j(t), \text{ if } t \in ([s], [t_0]),$$

$$\forall \alpha, \beta = 1, \dots, m, \forall i = 1, \dots, n.$$

Proof. We recall $y_\alpha^i(\epsilon, t) = \frac{\partial x_\epsilon^i}{\partial \epsilon^\alpha}(t)$. Since $x_\epsilon(t) = x^*(t)$, $\forall t \in [[0], [s-\epsilon]]$, we have $y_\alpha(\epsilon, t) = 0$, $\forall t \in [[0], [s-\epsilon]]$. Let us consider $t \in ([s-\epsilon], [s])$ a fixed multi-time. The PDE $\frac{\partial x_\epsilon^i}{\partial t^\alpha}(t) = X_\alpha^i(t, x_\epsilon(t), u(t))$ generates the variational PDE

$$\frac{\partial y_\beta^i}{\partial t^\alpha}(\epsilon, t) = \frac{\partial X_\alpha^i}{\partial x^j}(t, x_\epsilon(t), u(t)) y_\beta^j(\epsilon, t).$$

Since we are interested on what it happens starting with the multi-time s , we can chose $y_\alpha(\epsilon, t) = 0$, $\forall t \in ([s-\epsilon], [s])$. Therefore $y_\alpha(\epsilon, t) = 0$, $\forall t \in [[0], [s])$ and, when making $\epsilon = 0$, we obtain $y_\alpha(t) = 0$, $\forall t \in [[0], [s])$.

In the following, we take $t = s$. Successively, we have

$$\begin{aligned} & \int_{[[s-\epsilon],[s]]} [X_\alpha^i(t, x_\epsilon(t), u(t)) - X_\alpha^i(t, x^*(t), u^*(t))] dt \\ &= \int_{[[s-\epsilon],[s]]} \left(\frac{\partial x_\epsilon^i}{\partial t^\alpha} - \frac{\partial x^{*i}}{\partial t^\alpha} \right) dt = \int_{\partial[[s-\epsilon],[s]]} (x_\epsilon^i - x^{*i}) n_\alpha(t) d\sigma \\ &= \int_{[s]} (x_\epsilon^i - x^{*i}) n_\alpha(t) d\sigma \end{aligned}$$

and, by applying Lemma 2.1, it follows

$$\int_{[s]} y_\beta^i(t) n_\alpha(t) d\sigma = \int_{[s]} [X_\alpha^i(t, x^*(t), u(t)) - X_\alpha^i(t, x^*(t), u^*(t))] n_\beta(t) d\sigma.$$

On $([s], [t_0])$, we have again

$$\frac{\partial y_\beta^i}{\partial t^\alpha}(t) = \frac{\partial X_\alpha^i}{\partial x^j}(t, x^*(t), u^*(t)) y_\beta^j(t).$$

On the other hand, the m -dimensional flow of the infinitesimal deformation,

$$\frac{\partial y_\alpha^i}{\partial t^\beta}(t) = y_\beta^j(t) \frac{\partial X_\alpha^i}{\partial x^j}(x(t)) \quad \text{or} \quad \frac{\partial y_\alpha^i}{\partial t^\beta}(t) = y_\alpha^j(t) \frac{\partial X_\beta^i}{\partial x^j}(x(t)),$$

on the jet bundle of order one $J^1(T, M)$, determines a *dual m -flow*

$$(ADJ) \quad \frac{\partial p_i^\alpha}{\partial t^\alpha}(t) = -p_j^\alpha(t) \frac{\partial X_\alpha^j}{\partial x^i}(x(t))$$

on the dual space $J^{1*}(T, M)$. These PDEs systems are *adjoint* in the sense of *zero total divergence* of the tensor field $Q_\beta^\alpha = p_i^\alpha y_\beta^i$ produced by their solutions. The adjoint system (ADJ) has solutions since it contains n PDEs with nm unknown functions p_i^α . \square

2.1 Free boundary problem, no running cost

The basic problem here is to find

$$\begin{aligned} & \max_{u(\cdot)} J[u(\cdot)] = \int_{\partial\Omega_{0t_0}} g(t, x(t)) d\sigma \\ & \text{subject to} \quad \frac{\partial x^i}{\partial t^\alpha}(t) = X_\alpha^i(t, x(t), u(t)), \quad i = 1, \dots, n, \quad \alpha = 1, \dots, m, \\ & \quad \quad \quad u(t) \in \mathcal{U}, \quad t \in \Omega_{0t_0}, \quad x(0) = x_0. \end{aligned}$$

We denote by $u^*(\cdot)$ respectively $x^*(\cdot)$ the optimal control and the optimal m -sheet of this problem and we consider the control Hamiltonian

$$(H) \quad H(t, x, p, u) = p_i^\alpha X_\alpha^i(t, x, u), \quad p = (p_i^\alpha).$$

Lemma 2.3. (fundamental inequality) Let $\varphi : \Omega_{0t_0} \rightarrow \mathbb{R}$ be a continuous (measurable) function. If

$$\int_{[s]} \varphi(t)n_\alpha(t)d\sigma \geq 0, \quad \forall s \in \Omega_{0t_0},$$

then

$$\int_{\Omega_{0s}} \varphi(t)dt \geq 0, \quad \forall s \in \Omega_{0t_0}.$$

Proof. Since we can write

$$\int_{[s]} \varphi(t)n_\alpha(t)d\sigma = \frac{\partial}{\partial s^\alpha} \left(\int_{[[0],[s]]} \varphi(t)dt \right) = \frac{\partial}{\partial s^\alpha} \left(\int_{\Omega_{0s}} \varphi(t)dt \right),$$

the hypotheses ensure us that $s \rightarrow \int_{\Omega_{0s}} \varphi(t)dt$ is a partial increasing function. Therefore

$$\int_{\Omega_{0s}} \varphi(t)dt \geq 0, \quad \forall s \in \Omega_{0t_0}.$$

□

Theorem 2.4. (multitime maximum principle, no running cost) Suppose $u^*(\cdot)$ is optimal for (PDE), (J) and that $x^*(\cdot)$ is the corresponding optimal m -sheet. Then there exists the dual functions $p_i^{*\alpha} : \Omega_{0t_0} \rightarrow \mathbb{R}$ such that

$$(PDE) \quad \frac{\partial x^{*i}}{\partial t^\alpha}(t) = \frac{\partial H}{\partial p_i^\alpha}(t, x^*(t), p^*(t), u^*(t)),$$

$$(ADJ) \quad \frac{\partial p_i^{*\alpha}}{\partial t^\alpha}(t) = -\frac{\partial H}{\partial x^i}(t, x^*(t), p^*(t), u^*(t)),$$

$$(M) \quad \frac{\partial H}{\partial u^a}(t, x^*(t), p^*(t), u^*(t)) = 0, \quad \forall t \in \Omega_{0t_0}$$

and satisfy the boundary conditions

$$(t_0) \quad n_\alpha p_i^{*\alpha}|_{\partial\Omega_{0t_0}} = \frac{\partial g}{\partial x^i}|_{\partial\Omega_{0t_0}}.$$

Proof. For each map p , the control Hamiltonian $H(t, x, p, u) = p_i^\alpha X_\alpha^i$ satisfies

$$(1) \quad H(t, x^*(t), p(t), u^*(t)) + \frac{\partial p_i^\alpha}{\partial t^\alpha}(t)x^{*i}(t) = \frac{\partial(p_i^\alpha x^{*i})}{\partial t^\alpha}, \quad \forall t \in [[0], [t_0]];$$

$$(2) \quad H(t, x_\epsilon(t), p(t), u(t)) + \frac{\partial p_i^\alpha}{\partial t^\alpha}(t)x_\epsilon^i(t) = \frac{\partial(p_i^\alpha x_\epsilon^i)}{\partial t^\alpha}, \quad \forall t \in ([s - \epsilon], [s]);$$

$$(3) \quad H(t, x_\epsilon(t), p(t), u^*(t)) + \frac{\partial p_i^\alpha}{\partial t^\alpha}(t)x_\epsilon^i(t) = \frac{\partial(p_i^\alpha x_\epsilon^i)}{\partial t^\alpha}, \quad \forall t \in ([s], [t_0]).$$

Therefore, by taking the difference (2) – (1) on $([s - \epsilon], [s])$ and integrating afterwards, we obtain

$$\begin{aligned} \int_{[s]} (x_\epsilon^i - x^{*i}) p_i^\alpha n_\alpha d\sigma &= \int_{[[s-\epsilon],[s]]} [H(t, x_\epsilon(t), p(t), u(t)) \\ &\quad - H(t, x^*(t), p(t), u^*(t)) + \frac{\partial p_i^\alpha}{\partial t^\alpha}(t)(x_\epsilon^i - x^{*i})] d\sigma. \end{aligned}$$

Computing the partial derivative with respect to ϵ^β (see Lemma 2.1), we obtain

$$\int_{[s]} y_\beta^i p_i^\alpha n_\alpha d\sigma = \int_{[s]} [H(t, x^*(t), p(t), u(t)) - H(t, x^*(t), p(t), u^*(t))] n_\beta d\sigma.$$

If the costate vector p^* is the solution for the adjoint system (*ADJ*) with boundary conditions (t_0) , then, on $([s], [t_0])$, we have $\frac{\partial(p_i^{*\alpha} y_\beta^i)}{\partial t^\alpha} = 0$. Denoting by n the normal vector field on $\partial\Omega_{0t_0}$, respectively on $\partial\Omega_{0s}$, we obtain

$$\begin{aligned} 0 &= \int_{([s],[t_0])} \frac{\partial(p_i^{*\alpha} y_\beta^i)}{\partial t^\alpha} dt = \int_{\partial\Omega_{0t_0}} y_\beta^i p_i^{*\alpha} n_\alpha d\sigma - \int_{[s]} y_\beta^i p_i^{*\alpha} n_\alpha d\sigma \\ &= \int_{\partial\Omega_{0t_0}} \frac{\partial g}{\partial x^i} y_\beta^i d\sigma - \int_{[s]} y_\beta^i p_i^{*\alpha} n_\alpha d\sigma. \end{aligned}$$

Since u^* is an optimal control, it follows that $\epsilon = 0$ is a maximum point for the function $\epsilon \rightarrow \int_{\partial\Omega_{0t_0}} (g \circ x_\epsilon^i) d\sigma$ and, therefore, $\int_{\partial\Omega_{0t_0}} \frac{\partial g}{\partial x^i} y_\beta^i d\sigma \leq 0$. We find

$$\int_{[s]} [H(t, x^*(t), p^*(t), u(t)) - H(t, x^*(t), p^*(t), u^*(t))] n_\beta d\sigma \leq 0.$$

Applying Lemma 2.3, we obtain the maximum principle inequality in functional integral form. Consequently, using the Euler-Lagrange relation, it appears the critical point condition. \square

2.2 Free boundary problem, with running cost

We suppose that the functional includes a running cost, i.e.,

$$(J) \quad J[u(\cdot)] = \int_{\Omega_{0t_0}} X(t, x(t), u(t)) dt + \int_{\partial\Omega_{0t_0}} (g(t, x(t))) d\sigma.$$

In this case, the control Hamiltonian has the following expression:

$$(H) \quad H(t, x, p_0, p, u) = p_0 X(t, x, u) + p_i^\alpha X_\alpha^i(t, x, u).$$

Theorem 2.5. (Multitime maximum principle with running cost) *Suppose $u^*(\cdot)$ is optimal for (PDE), (J) and that $x^*(\cdot)$ is the corresponding optimal m -sheet. Then there exist some functions $p_0^*, p_i^{*\alpha} : \Omega_{0t_0} \rightarrow \mathbb{R}$ such that*

$$(PDE) \quad \frac{\partial x^{*i}}{\partial t^\alpha}(t) = \frac{\partial H}{\partial p_i^\alpha}(t, x^*(t), p_0^*(t), p^*(t), u^*(t)),$$

$$(ADJ) \quad \frac{\partial p_i^{*\alpha}}{\partial t^\alpha}(t) = -\frac{\partial H}{\partial x^i}(t, x^*(t), p_0^*(t), p^*(t), u^*(t))$$

and

$$(M) \quad \frac{\partial H}{\partial u^a}(t, x^*(t), p^*(t), u^*(t)) = 0, \quad \forall t \in \Omega_{0t_0}$$

Finally, the boundary conditions

$$(t_0) \quad n_\alpha p_i^{*\alpha}|_{\partial\Omega_{0t_0}} = \frac{\partial g}{\partial x^i}|_{\partial\Omega_{0t_0}}$$

are satisfied.

Proof. We begin by adding new variables in order to transform the running cost into a terminal cost. We introduce some supplementary state-variables $x^{(\alpha)} : \Omega_{0t_0} \rightarrow \mathbb{R}$, solutions for the PDE system

$$(4) \quad \frac{\partial x^{(\alpha)}}{\partial t^\beta}(t) = \frac{1}{m} \delta_\beta^\alpha X(t, x(t), u(t)), \quad x^{(\alpha)}(0) = 0, \quad t \in \Omega_{0t_0}.$$

We also consider

$$\bar{x} = (x^{(\alpha)}, x^i); \quad \bar{x}_0 = (0, x_0); \quad \bar{x}(\cdot) = (x^{(\alpha)}(\cdot), x^i(\cdot))$$

and

$$\bar{X}_\alpha(t, \bar{x}, u) = \frac{1}{m} \delta_\alpha^\beta X(t, x, u) \frac{\partial}{\partial x^{(\beta)}} + X_\alpha^i(t, x, u) \frac{\partial}{\partial x^i}; \quad \bar{g}(t, \bar{x}) = n_\beta(t) x^{(\beta)} + g(t, x).$$

Then (PDE) and relation (4) give the dynamics

$$\overline{PDE} \quad \frac{\partial \bar{x}}{\partial t^\alpha} = \bar{X}_\alpha(t, x(t), u(t)), \quad \bar{x}(0) = \bar{x}_0, \quad t \in \Omega_{0t_0}.$$

Consequently, the initial control problem transforms into a new control problem with boundary cost functional

$$(\bar{J}) \quad \bar{J}[u(\cdot)] = \int_{\partial\Omega_{0t_0}} \bar{g}(t, \bar{x}(t)) d\sigma.$$

The control Hamiltonian associated to this new problem is

$$\bar{H}(t, \bar{x}, \bar{p}, u) = \frac{1}{m} p_\beta^\alpha \delta_\alpha^\beta X(t, x, u) + p_i^\alpha X_\alpha^i(t, x, u) = H(t, x, p_0, p, u),$$

where $p_0 = \frac{1}{m} Tr(p_\beta^\alpha)$.

We apply the multitime maximum principle with no running cost and we obtain a costate vector $\bar{p}^* = (p_\beta^{*\alpha}, p_i^{*\alpha})$ such that \bar{H} satisfies (PDE) , (ADJ) , (M) and (t_0) . Let $p_0^* = \frac{1}{m}Tr(p_\beta^{*\alpha})$. Relation (PDE) can be rewritten as

$$\frac{\partial x^{*i}}{\partial t^\alpha}(t) = \frac{\partial H}{\partial p_i^\alpha}(t, x^*(t), p_0^*(t), p^*(t), u^*(t))$$

and

$$\frac{\partial x^{*(\beta)}}{\partial t^\alpha}(t) = \frac{\partial \bar{H}}{\partial p_\beta^\alpha}(t, \bar{x}^*(t), \bar{p}^*(t), u^*(t)) = \frac{1}{m} \delta_\beta^\alpha \frac{\partial H}{\partial p_0}(t, x^*(t), p_0^*(t), p^*(t), u^*(t)).$$

The immediate consequence of the previous relation is

$$\frac{\partial x^{*(\alpha)}}{\partial t^\alpha}(t) = \frac{\partial H}{\partial p_0}(t, x^*(t), p_0^*(t), p^*(t), u^*(t)).$$

We analyze next the adjoint equation (ADJ) . We obtain

$$(ADJ) \quad \frac{\partial p_i^{*\alpha}}{\partial t^\alpha}(t) = -\frac{\partial H}{\partial x^i}(t, x^*(t), p_0^*(t), p^*(t), u^*(t))$$

and

$$\frac{\partial p_\beta^{*\alpha}}{\partial t^\alpha}(t) = 0.$$

The maximization principle can be rewritten

$$(M) \quad \frac{\partial H}{\partial u^a}(t, x^*(t), p_0^*(t), p^*(t), u^*(t)) = 0, \quad \forall t \in \Omega_{0t_0}$$

and the boundary conditions are

$$(t_0) \quad n_\alpha p_i^{*\alpha}|_{\partial\Omega_{t_0}} = \frac{\partial g}{\partial x^i}|_{\partial\Omega_{t_0}}; \quad n_\alpha p_\beta^{*\alpha}|_{\partial\Omega_{t_0}} = n_\beta$$

Gathering together the restrictions related to $p_\beta^{*\alpha}$,

$$p_0^* = \frac{1}{m}tr(p_\beta^{*\alpha}); \quad \frac{\partial p_\beta^{*\alpha}}{\partial t^\alpha}(t) = 0; \quad n_\alpha p_\beta^{*\alpha}|_{\partial\Omega_{t_0}} = n_\beta,$$

the immediate consequence is that we can choose $p_\beta^{*\alpha} = \delta_\beta^\alpha$ and $p_0^* = 1$. \square

3 Optimal control theory based on techniques of variational calculus

We consider a smooth vector field $v = (v^a) : \Omega_{0t_0} \rightarrow \mathbb{R}^k$ satisfying $v^a(0) = 0, \forall a = 1, \dots, k$. Let $u^*(\cdot) \in \mathcal{U}$ denote the optimal control. Moreover, we suppose that $u^*(t) \in \text{Int}U$. Then, we consider the control variation

$$u_\delta(t) = u^*(t) + \delta v(t).$$

Since $u^*(t) \in \text{Int}U$, there is $\delta_0 > 0$ such that $u_\delta(t) \in \text{Int}U$, $\forall |\delta| < \delta_0$. Let $x_\delta(t)$ be the state variable corresponding to the control variable $u_\delta(t)$, that is $x_\delta(t)$ is solution for the following PDEs system

$$(PDE) \quad \frac{\partial x_\delta^i}{\partial t^\alpha}(t) = X_\alpha^i(t, x_\delta(t), u_\delta(t)), \quad x_\delta(0) = x_0, \quad t \in \Omega_{0t_0} \subset \mathbb{R}_+^m.$$

If $y = \frac{\partial x}{\partial \delta}|_{\delta=0}$ is the infinitesimal deformation induced by the previous control variation, then

$$\frac{\partial y^i}{\partial t^\beta}(t) = \frac{\partial X_\beta^i}{\partial x^j}(t)y^j(t) + \frac{\partial X_\beta^i}{\partial u^a}(t)v^a(t), \quad y^i(0) = 0, \quad t \in \Omega_{0t_0}.$$

3.1 Free boundary problem, no running cost

In this subsection, we consider again the boundary cost functional

$$(J) \quad J[u(\cdot)] = \int_{\partial\Omega_{0t_0}} g(t, x(t))d\sigma,$$

restricted by (PDE). We use again the control Hamiltonian

$$(H) \quad H(t, x, p, u) = p_i^\alpha X_\alpha^i(t, x, u).$$

We also introduce the *control tensor field*

$$T_\beta^\alpha(t, x, p, u) = p_i^\alpha X_\beta^i(t, x, u)$$

and we prove next a simplified maximum principle.

Theorem 3.1. (simplified multitime maximum principle, no running cost)

Suppose $u^*(\cdot)$ is an interior optimal control for (PDE), (J) and that $x^*(\cdot)$ is the corresponding optimal m -sheet. Then there exist the dual functions $p_i^{*\alpha} : \Omega_{0t_0} \rightarrow \mathbb{R}$ such that

$$(PDE) \quad \frac{\partial x^{*i}}{\partial t^\alpha}(t) = \frac{\partial H}{\partial p_i^\alpha}(t, x^*(t), p^*(t), u^*(t)),$$

$$(ADJ) \quad \frac{\partial p_i^{*\alpha}}{\partial t^\alpha}(t) = -\frac{\partial H}{\partial x^i}(t, x^*(t), p^*(t), u^*(t)),$$

$$(M) \quad \frac{\partial H}{\partial u^a}(t, x^*(t), p^*(t), u^*(t)) = 0$$

and satisfy the boundary conditions

$$(t_0) \quad n_\alpha p_i^{*\alpha}|_{\partial\Omega_{0t_0}} = \frac{\partial g}{\partial x^i}|_{\partial\Omega_{0t_0}}.$$

Moreover

$$D_\alpha[T_\beta^\alpha(t, x^*(t), p^*(t), u^*(t))] = \frac{\partial H}{\partial t^\beta}(t, x^*(t), p^*(t), u^*(t)),$$

where D_α denotes the total derivative with respect to t^α .

Proof. From (H) and (PDE) we have

$$H(t, x_\delta(t), p(t), u_\delta(t)) + \frac{\partial p_i^\alpha}{\partial t^\alpha}(t) x_\delta^i(t) = \frac{\partial(p_i^\alpha x_\delta^i)}{\partial t^\alpha}(t),$$

therefore

$$\begin{aligned} \frac{\partial(p_i^\alpha y^i)}{\partial t^\alpha}(t) &= \left[\frac{\partial H}{\partial x^i}(t, x^*(t), p(t), u^*(t)) + \frac{\partial p_i^\alpha}{\partial t^\alpha}(t) \right] y^i(t) \\ &+ \frac{\partial H}{\partial u^a}(t, x^*(t), p(t), u^*(t)) v^a(t). \end{aligned}$$

We choose p^* solution for (ADJ), (t_0) . When integrating on Ω_{0t_0} , we obtain

$$\begin{aligned} \int_{\Omega_{0t_0}} \frac{\partial H}{\partial u^a}(t, x^*(t), p^*(t), u^*(t)) v^a(t) dt &= \int_{\Omega_{0t_0}} \frac{\partial(p_i^\alpha y^i)}{\partial t^\alpha}(t) \\ &= \int_{\partial\Omega_{0t_0}} y^i(t) p_i^\alpha(t) n_\alpha(t) d\sigma = \int_{\partial\Omega_{0t_0}} \frac{\partial g}{\partial x^i}(t, x^*(t)) y^i(t) d\sigma. \end{aligned}$$

Since $u^*(\cdot)$ is an optimal control, it follows that $\delta = 0$ is a critical point for the function $\int_{\Omega_{0t_0}} g(x_\delta(t)) d\sigma$ and, therefore

$$\int_{\partial\Omega_{0t_0}} \frac{\partial g}{\partial x^i}(t, x^*(t)) y^i(t) d\sigma = 0.$$

It follows that

$$\int_{\Omega_{0t_0}} \frac{\partial H}{\partial u^a}(t, x^*(t), p^*(t), u^*(t)) v^a(t) dt = 0$$

and, since v is an arbitrary vector field, we conclude that

$$\frac{\partial H}{\partial u^a}(t, x^*(t), p^*(t), u^*(t)) = 0, \quad \forall t \in \Omega_{0t_0}.$$

Let us compute next the divergence of the control tensor field,

$$\begin{aligned} D_\alpha T_\beta^\alpha &= \frac{\partial p_i^\alpha}{\partial t^\alpha} X_\beta^i + p_i^\alpha D_\alpha X_\beta^i = \frac{\partial p_i^\alpha}{\partial t^\alpha} X_\beta^i + p_i^\alpha D_\beta X_\alpha^i \\ &= -\frac{\partial H}{\partial x^i} X_\beta^i + \frac{\partial H}{\partial x^i} X_\beta^i + \frac{\partial H}{\partial u^a} \frac{\partial u^a}{\partial t^\beta} + \frac{\partial H}{\partial t^\beta} = \frac{\partial H}{\partial t^\beta}. \end{aligned}$$

□

Remark 3.2. *If the control Hamiltonian is autonomous, that is H doesn't depend explicitly on t , then we obtain a conservation law asserting that the control tensor has null divergence.*

3.2 Free boundary problem, with running cost

We suppose the functional includes a running cost:

$$(J) \quad J[u(\cdot)] = \int_{\Omega_{0t_0}} X(t, x(t), u(t)) dt + \int_{\partial\Omega_{0t_0}} g(t, x(t)) d\sigma.$$

In this case, the control Hamiltonian has the following expression:

$$(H) \quad H(t, x, p_0, p, u) = p_0 X(t, x, u) + p_i^\alpha X_\alpha^i(t, x, u).$$

Theorem 3.3. (simplified multitime maximum principle with running cost)

Suppose $u^*(\cdot)$ is an interior optimal control for (PDE), (J) and that $x^*(\cdot)$ is the corresponding optimal m -sheet. Then there exist some costate functions $p_0^*, p_i^{*\alpha} : \Omega_{0t_0} \rightarrow \mathbb{R}$ such that

$$(PDE) \quad \frac{\partial x^{*i}}{\partial t^\alpha}(t) = \frac{\partial H}{\partial p_i^\alpha}(t, x^*(t), p_0^*(t), p^*(t), u^*(t)),$$

$$(ADJ) \quad \frac{\partial p_i^{*\alpha}}{\partial t^\alpha}(t) = -\frac{\partial H}{\partial x^i}(t, x^*(t), p_0^*(t), p^*(t), u^*(t))$$

and

$$(M) \quad \frac{\partial H}{\partial u^a}(t, x^*(t), p_0^*(t), p^*(t), u^*(t)) = 0.$$

Finally, the boundary conditions

$$(t_0) \quad n_\alpha p_i^{*\alpha}|_{\partial\Omega_{0t_0}} = \frac{\partial g}{\partial x^i}|_{\partial\Omega_{0t_0}}$$

are satisfied. Moreover, the control tensor field

$$T_\beta^\alpha(t, x, p_0, p, u) = p_0 \delta_\beta^\alpha X(t, x, u) + p_i^\alpha X_\beta^i(t, x, u)$$

satisfies the relation

$$D_\alpha [T_\beta^\alpha(t, x^*(t), p_0^*(t), p^*(t), u^*(t))] = \frac{\partial H}{\partial t^\beta}(t, x^*(t), p_0^*(t), p^*(t), u^*(t)).$$

Proof. Same arguments as in the proof of Theorem 2.5. □

4 Sufficient conditions in multitime optimal control theory

If we add some concavity restrictions to the components of the control tensor, to the boundary cost and the constrained set, then we can prove the sufficiency of the conditions of multitime maximum principle.

Definition 4.1. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called *concave* if its Hessian matrix is negative definite at each point.

A concave function satisfies the inequality

$$f(y) - f(x) \leq df_x(y - x).$$

We consider the more general optimal control problem with running cost and we shall use the control Hamiltonian

$$H(t, x, p_0, p, u) = p_0 X(t, x, u) + p_i^\alpha X_\alpha^i(t, x, u).$$

Moreover, we can suppose that $p_0 = 1$.

Theorem 4.1. (sufficient condition in multitime optimal control) *If (x^*, p^*, u^*) satisfies the conditions of simplified multitime maximum principle and the control Hamiltonian evaluated at $p = p^*$ is (strictly) concave in the pair (x^*, u^*) and the boundary cost g is (strictly) concave at x^* , then (x^*, p^*, u^*) is the (unique) solution of the control problem.*

Proof. We must maximize the functional

$$J(u(\cdot)) = \int_{\Omega_{0t_0}} X(t, x(t), u(t)) dt + \int_{\partial\Omega_{0t_0}} g(t, x(t)) d\sigma.$$

subject to the evolution PDEs system. We fix a pair (x^*, u^*) , where u^* is a candidate optimal m -sheet of the controls and x^* is a candidate optimal m -sheet of the states. Calling J^* the value of the functional for (x^*, u^*) , let us prove that

$$J^* - J = \int_{\Omega_{0t_0}} (X^* - X) dt + \int_{\partial\Omega_{0t_0}} (g^* - g) d\sigma \geq 0,$$

where the strict inequality holds under strict concavity. Denoting $H^* = H(t, x^*, p^*, u^*)$ and $H = H(t, x, p^*, u)$, we find

$$\begin{aligned} J^* - J &= \int_{\Omega_{0t_0}} (X^* - X) dt + \int_{\partial\Omega_{0t_0}} (g^* - g) d\sigma \\ &= \int_{\Omega_{0t_0}} \left((H^* - p_i^{*\alpha} \frac{\partial x^{*i}}{\partial t^\alpha}) - (H - p_i^{*\alpha} \frac{\partial x^i}{\partial t^\alpha}) \right) dt + \int_{\partial\Omega_{0t_0}} (g^* - g) d\sigma. \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} J^* - J &= \int_{\Omega_{0t_0}} \left((H^* + x^{*i} \frac{\partial p_i^{*\alpha}}{\partial t^\alpha}) - (H + x^i \frac{\partial p_i^{*\alpha}}{\partial t^\alpha}) \right) dt \\ &\quad + \int_{\partial\Omega_{0t_0}} ((g^* - g) - (x^{*i} - x^i) p_i^{*\alpha} n_\alpha) d\sigma. \end{aligned}$$

Taking into account that p^* satisfies the boundary condition (t_0) , we infer

$$J^* - J = \int_{\Omega_{0t_0}} \left((H^* - H) + \frac{\partial p_i^{*\alpha}}{\partial t^\alpha} (x^{*i} - x^i) \right) dt + \int_{\partial\Omega_{0t_0}} \left((g^* - g) - \frac{\partial g^*}{\partial x^i} (x^{*i} - x^i) \right) d\sigma.$$

The definition of concavity implies

$$\int_{\partial\Omega_{0t_0}} \left((g^* - g) - \frac{\partial g^*}{\partial x^i} (x^{*i} - x^i) \right) d\sigma \geq 0$$

and

$$\begin{aligned} & \int_{\Omega_{0t_0}} \left((H^* - H) + \frac{\partial p_i^{*\alpha}}{\partial t^\alpha} (x^{*i} - x^i) \right) dt \\ & \geq \int_{\Omega_{0t_0}} \left((x^{*i} - x^i) \left(\frac{\partial H^*}{\partial x^i} + \frac{\partial p_i^{*\alpha}}{\partial t^\alpha} \right) + (u^{*a} - u^a) \frac{\partial H^*}{\partial u^a} \right) dt = 0. \end{aligned}$$

This last equality follows from the fact that all " * " variables satisfy the conditions of the multi-time maximum principle. In this way, $J^* - J \geq 0$. \square

5 Example of optimal control problem with area integral cost on boundary

In the previous sections, we considered the parallelepiped Ω_{0t_0} to be the domain of multitime. Next, we give an idea of how to extend our theory for arbitrary compact domains from \mathbb{R}^m . Let $\Omega \subset \mathbb{R}^m$ be a connected and compact subset, with a piecewise smooth $(m - 1)$ -dimensional boundary $U = \partial\Omega$. The new optimal control problem with area integral boundary costs asks for finding

$$\begin{aligned} \max_{u(\cdot)} \quad & J[u(\cdot)] = \int_{\Omega} X(t, x(t), u(t)) dt + \int_U g(t, x(t)) d\sigma \\ \text{subject to} \quad & \frac{\partial x^i}{\partial t^\alpha}(t) = X_\alpha^i(t, x(t), u(t)), \quad i = 1, \dots, n, \quad \alpha = 1, \dots, m, \\ & u(t) \in \mathcal{U}, \quad t \in \Omega, \quad x(0) = x_0. \end{aligned}$$

Solving this problem using needle-shaped control variations requires the introduction of a temporal orientation on Ω . In order to do so, we consider a fixed point $t_0 \in \Omega$. Moreover, for simplicity, we assume $t_0 = 0$. For each point $t \in \Omega$, we denote by $\alpha_t : [0, 1] \rightarrow \Omega$ the line segment starting from 0, passing through t , such that $\alpha_t(1) \in U$. We also consider the function $\tau : \Omega \rightarrow [0, 1]$ satisfying the relation $\alpha_t(\tau(t)) = t$. For two multitime s and t in Ω , we denote $s < (\leq) t$ if $\tau(s) < (\leq) \tau(t)$. Using the function τ , we can also define the ∂ -type set

$$[s] = \{t \in \Omega \mid \tau(t) = \tau(s)\}$$

and the ∂ -type intervals

$$[[0], [s]] = \{t \in \Omega \mid 0 \leq \tau(t) \leq \tau(s)\}; \quad ([s], [t]) = [[0], [t]] - [[0], [s]].$$

By considering needle-shaped control variations relative to the above intervals, we regain the multitime maximum principle.

There is some time now since we look for a variational proof of the fact that, *amongst all the bodies of constant surface, the sphere maximizes the volume*. Recently,

we have solved this problem, using multitime calculus of variations and taking *the normal vector field as state variable*. In our opinion, this is an important example since it emphasizes the utility of considering and studying a multitime variational theory. We reconsider this problem now, using multitime optimal control theory.

If D is a compact set of $\mathbb{R}^m = \{(t^1, \dots, t^m)\}$ with a piecewise smooth $(m-1)$ -dimensional boundary ∂D , we can write the volume $\int_D dt^1 \dots dt^m$ of the domain D using the position vector $t = (t^\alpha)$ and the exterior unit normal vector field $N = (N^\beta)$ on ∂D , via Gauss-Ostragradski formula, as

$$m \int_D dt^1 \dots dt^m = \int_{\partial D} \delta_{\alpha\beta} t^\alpha N^\beta d\sigma.$$

Moreover, the area of ∂D is $\int_{\partial D} d\sigma$. Introducing a parametrization on D , whose domain is $\Omega \subset \mathbb{R}^m$ and denoting $U = \partial\Omega$, we have $d\sigma = \|\mathcal{N}\|d\eta$, where $\mathcal{N} = \|\mathcal{N}\|N$ and η is a differential $(m-1)$ -form.

Let us show next, that of all solids having a given surface area, the sphere is the one having the greatest volume. To prove this statement, we formulate the multitime optimal control problem with isoperimetric constraint

$$\max_{\mathcal{N}} \int_U \delta_{\alpha\beta} t^\alpha \mathcal{N}^\beta(t) d\eta \quad \text{subject to} \quad \int_U \sqrt{\delta_{\alpha\beta} \mathcal{N}^\alpha(t) \mathcal{N}^\beta(t)} d\eta = \text{const.}$$

In order to solve this problem, we also add the evolution system

$$\frac{\partial \mathcal{N}^\alpha}{\partial t^\beta}(t) = u_\beta^\alpha(t), \quad \forall t \in \Omega,$$

which does not interfere with the quality of the solutions on the boundary. Using the Hamiltonian

$$H = p_\alpha^\beta u_\beta^\alpha$$

and the boundary cost $g(t, \mathcal{N}) = \delta_{\alpha\beta} t^\alpha \mathcal{N}^\beta - p \sqrt{\delta_{\alpha\beta} \mathcal{N}^\alpha \mathcal{N}^\beta}$, $p = \text{const.}$, the critical point condition, in the multitime maximum principle, gives

$$\frac{\partial H}{\partial u_\beta^\alpha} = p_\alpha^\beta(t) = 0, \quad \forall t \in \Omega$$

and the boundary condition writes

$$0 = p_\beta^\alpha N^\beta = \frac{\partial g}{\partial N^\alpha} = t^\alpha - p N^\alpha, \quad \forall t \in U.$$

Since the boundary cost g is a concave function of \mathcal{N} , the critical point is a maximum point. This confirms that D is the sphere $\|t\|^2 \leq p^2$ in \mathbb{R}^m .

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Authors' address:

Constantin Udriște, Andreea Bejenaru
University Politehnica of Bucharest, Faculty of Applied Sciences,
Department of Mathematics-Informatics I, Splaiul Independentei 313,
RO-060042 Bucharest, Romania
E-mail: anet.udri@yahoo.com ; e-mail: bejenaru.andreea@yahoo.com