Multitime sine-Gordon solitons via geometric characteristics

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Abstract. Our paper introduces and studies the idea of multitime evolution in the context of solitons.

Section 1 presents some historical data about solitons. Section 2 defines the multitime sine-Gordon PDE, using a fundamental tensor and a linear connection. Section 3 describes the multitime sine-Gordon scalar solitons as special solutions of the multitime sine-Gordon PDE. Section 4 proves the existence of the multitime sine-Gordon biscalar solitons. Section 5 analyzes the geometric characteristics (fundamental tensor, linear connection) of the sine-Gordon PDE, showing the existence of an infinity of Riemannian or semi-Riemannian structures such that the new PDE is a prolongation of the sine-Gordon PDE. The "two-time" sine-Gordon geometric dynamics, which is presented here for the first time, shows that the sine-Gordon soliton is generated.

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1 Single-time solitons

The term *soliton* was introduced in the 1960's, but the idea to study the solitons had arisen in the 19th century, when Russell observed a large solitary wave in a canal near Edinburgh. Nowadays, many model PDEs of nonlinear phenomena are known to possess soliton solutions. *Physically, solitons are very stable independent solitary waves which have finite energy, and which behave like particles.* When they are located mutually far apart, each of them is approximately a traveling wave with constant shape and velocity. When two such solitary waves get closer, they interact, they gradually deform and finally merge into a single wave packet, a 2-soliton. However, this wave packet soon splits into two solitary waves with the same shape and velocity before collision. When a 2-soliton meet a 1-soliton, they merge into a 3-soliton and so on. This happens with the solutions of some nonlinear partial differential equations, which are named *soliton equations.* These PDEs include the sine-Gordon equations,

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with some versions and related equations, and the Korteweg-de Vries equations, with related equations too, etc. Reformulating, the *soliton* is *a solution of a non-linear evolution equation, which at every moment of time is localized in a bounded domain of space, such that the size of the domain remains bounded in time while the movement of the center of the domain can be interpreted as the movement of a particle.*

Issued from the geometry of surfaces of constant negative curvature and found later in the study of some physical phenomena, the *single-time sine-Gordon PDE* is

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \sin u. \tag{1.1}$$

2 Multitime sine-Gordon PDE

Partial differential equations (PDEs) arise in many areas of science and technology. Usually these equations have several variables. Some of them are called *spatial variables* and are denoted with $x^i, i = 1, ..., n$ or by a vector $x = (x^i)$ and others are called *time variables*, denoted with $t^\alpha, \alpha = 1, ..., m$ or by a vector $t = (t^\alpha)$. There are PDEs that contain only spatial variables and there are PDEs that contain both spatial variables and time variables. A PDE is called a *multitime partial differential equation* if the time (evolution) parameter t is multidimensional. In the last decade the study of multitime PDEs has received considerable attention (see [3]).

This paper introduces and studies a multitime version (for techniques, see also [12] - [10]) of the single-time sine-Gordon PDE (see [1], [5], [2]). We search for some particular solutions of the new equation, or some information about the form of these solutions. We study also some geometric characteristics of the single-time sine-Gordon PDE and of multitime sine-Gordon PDE.

A *multitime* $t = (t^1, ..., t^m)$ is a point in the manifold $\mathbb{R}^m$. We endow the manifold $\mathbb{R}^m$ with a *symmetric linear connection* $\Gamma^\gamma_{\alpha\beta}$ and with a *fundamental symmetric contravariant tensor field* $g = (g^{\alpha\beta})$ of constant signature (r, z, s), $r + z + s = m$. Using a C^2 function $u : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, we build the *Hessian operator*

$$(Hess_\Gamma u)_{\alpha\beta} = \frac{\partial^2 u}{\partial t^\alpha \partial t^\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial u}{\partial t^\gamma},$$

and its trace, called *ultra-parabolic-hyperbolic operator*,

$$\Box_{\Gamma,g} u = g^{\alpha\beta}\left(\frac{\partial^2 u}{\partial t^\alpha \partial t^\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial u}{\partial t^\gamma}\right), \quad \alpha, \beta, \gamma \in \{1, ..., m\}.$$

Define the *multitime sine-Gordon PDE* as

$$\frac{\partial^2 u}{\partial x^2}(x,t) - g^{\alpha\beta}(t)\left(\frac{\partial^2 u}{\partial t^\alpha \partial t^\beta}(x,t) - \Gamma^\gamma_{\alpha\beta}(t)\frac{\partial u}{\partial t^\gamma}(x,t)\right) = \sin u(x,t). \tag{2.1}$$

3 Multitime sine-Gordon scalar solitons

For the multitime sine-Gordon PDE (2.1), we seek for solutions in the *scalar form*

$$u(x,t) = \phi(x - v_\alpha t^\alpha) = \phi(\xi), \tag{3.1}$$

where (v_α), $\alpha \in \{1, ..., m\}$, is a constant vector and $\xi = x - v_\alpha t^\alpha$. These solutions do not depend on (x, t) coordinates in explicit form. The partial derivatives of the unknown function $u(x, t)$ are

$$\frac{\partial^2 u}{\partial x^2} = \phi''(\xi), \ \frac{\partial u}{\partial t^\alpha} = \phi'(\xi)(-v_\alpha), \ \frac{\partial^2 u}{\partial t^\alpha \partial t^\beta} = \phi''(\xi) v_\alpha v_\beta.$$

We substitute these derivatives in the PDE (2.1) and we obtain

$$\phi''(\xi)[1 - g^{\alpha\beta}(t) v_\alpha v_\beta] - \phi'(\xi) g^{\alpha\beta}(t) \Gamma^\gamma_{\alpha\beta}(t) v_\gamma = \sin\phi(\xi).$$

Let (v_α) be a constant vector. We choose a constant fundamental tensor $g^{\alpha\beta}$ such that $g^{\alpha\beta} v_\alpha v_\beta = V \neq 1$, and the constant connection $\Gamma^\gamma_{\alpha\beta} = 0$. With this choice, the PDE (2.1) becomes

$$\phi''(\xi)(1 - V) = \sin\phi(\xi).$$

We multiply both sides by $\phi'(\xi)$ and write the equation in the form

$$\phi''\phi' - \frac{1}{1 - V}\phi' \sin\phi = 0.$$

Taking the primitive, we find $\frac{\phi'^2}{2} + \frac{\cos\phi}{1-V} = B, \quad B \in \mathbb{R}$, that is $\phi' = \pm\sqrt{\frac{2B(1-V) - 2\cos\phi}{1-V}}$, which is equivalent to $\frac{d\phi}{d\xi} = \pm\sqrt{\frac{2}{1-V}}\,\sqrt{A - \cos\phi}$, where $A = B(1 - V)$. We integrate again, selecting $A = 1$. We get the next sequence of equivalences:

$$\pm \int \frac{d\phi}{\sqrt{1 - \cos\phi}} = \sqrt{\frac{2}{1 - V}} \int d\xi \Leftrightarrow \ \pm \int \frac{d\phi}{\sin\dfrac{\phi}{2}} = \frac{2}{\sqrt{1 - V}} \int d\xi$$

$$\Leftrightarrow \ \pm \ln\left|\mathrm{tg}\frac{\phi}{4}\right| = \frac{\xi}{\sqrt{1 - V}} + a, \quad a \in \mathbb{R}.$$

Finally, we find two families of solutions, as follows:

a) $\ln\left|\mathrm{tg}\frac{\phi}{4}\right| = \frac{\xi}{\sqrt{1-V}} + a \Rightarrow \mathrm{tg}\frac{\phi}{4} = c\, e^{\frac{\xi}{\sqrt{1-V}}}, \quad c \in \mathbb{R}$, which leads to

$$\phi(\xi) = 4\arctan\left(c\, e^{\frac{\xi}{\sqrt{1-V}}}\right), \Rightarrow u(x, t) = 4\arctan\left(c\, e^{\frac{x - v_\alpha t^\alpha}{\sqrt{1 - g^{\alpha\beta} v_\alpha v_\beta}}}\right), \quad c \in \mathbb{R}.$$

b) $-\ln\left|\mathrm{tg}\frac{\phi}{4}\right| = \frac{\xi}{\sqrt{1-V}} + a \Rightarrow \mathrm{ctan}\frac{\phi}{4} = c\, e^{\frac{\xi}{\sqrt{1-V}}}, \quad c \in \mathbb{R}$, which leads to

$$\phi(\xi) = 4\mathrm{arcctan}\left(c\, e^{\frac{\xi}{\sqrt{1-V}}}\right), \Rightarrow u(x, t) = 4\mathrm{arcctan}\left(c\, e^{\frac{x - v_\alpha t^\alpha}{\sqrt{1 - g^{\alpha\beta} v_\alpha v_\beta}}}\right), \quad c \in \mathbb{R}.$$

Thus, in the particular case $g^{\alpha\beta}(t) = \text{constant}$, $\Gamma^\gamma_{\alpha\beta}(t) = 0$, we found two families of solutions of the equation (2.1), which corresponds to the *multitime solitons* and respectively to the *multitime anti-solitons.*

Theorem 3.1. *The scalar solutions of the PDE (2.1) split as family of multitime solitons and family of multitime anti-solitons.*

4 Multitime sine-Gordon biscalar solitons

Let us consider again the multitime sine-Gordon PDE (2.1). We search for solutions in the *biscalar form*

$$u(x,t) = 4 \text{ arctan} \frac{\phi(x)}{\psi(t)}. \tag{4.1}$$

The partial derivatives are

$$u_x = \frac{4\psi(t)\phi'(x)}{\psi^2(t)+\phi^2(x)},$$

$$u_{t^\alpha} = \frac{-4\phi(x)}{\psi^2(t)+\phi^2(x)} \frac{\partial \psi}{\partial t^\alpha}(t),$$

$$u_{xx} = 4\psi(t) \frac{\phi''(x)(\psi^2(t)+\phi^2(x)) - 2(\phi'(x))^2\phi(x)}{(\psi^2(t)+\phi^2(x))^2},$$

$$u_{t^\alpha t^\beta} = -4\phi(x) \frac{(\psi^2(t)+\phi^2(x)) \dfrac{\partial^2\psi}{\partial t^\alpha \partial t^\beta}(t) - 2\psi(t) \dfrac{\partial \psi}{\partial t^\alpha}(t) \dfrac{\partial \psi}{\partial t^\beta}(t)}{(\psi^2(t)+\phi^2(x))^2}. \tag{4.2}$$

After replacing the derivatives (4.2) and ranging the terms, the equation (2.1) becomes

$$\frac{4}{(\psi^2(t)+\phi^2(x))^2}\left[\psi(t)\phi''(x) + g^{\alpha\beta}(t)\phi(x)\left(\frac{\partial^2\psi}{\partial t^\alpha \partial t^\beta}(t) - \Gamma^\gamma_{\alpha\beta}(t)\frac{\partial\psi}{\partial t^\gamma}(t)\right)\right]\cdot$$

$$\cdot\left(\psi^2(t)+\phi^2(x)\right) - \frac{8\phi(x)\psi(t)}{(\psi^2(t)+\phi^2(x))^2}\left((\phi'(x))^2 + g^{\alpha\beta}(t)\frac{\partial\psi}{\partial t^\alpha}(t)\frac{\partial\psi}{\partial t^\beta}(t)\right) =$$

$$= \sin\left(4 \arctan \frac{\phi(x)}{\psi(t)}\right). \tag{4.3}$$

On the other hand,

$$\sin(4\arctan\alpha) = \frac{2\text{tg}(2\arctan\alpha)}{1+\text{tg}^2(2\arctan\alpha)} = \frac{2\,\dfrac{2\alpha}{1-\alpha^2}}{1+\left(\dfrac{2\alpha}{1-\alpha^2}\right)^2} = \frac{4\alpha(1-\alpha^2)}{(1+\alpha^2)^2}.$$

In view of this, the right side of (4.3) takes the form

$$\sin\left(4\arctan\frac{\phi(x)}{\psi(t)}\right) = \frac{4\,\dfrac{\phi(x)}{\psi(t)}\left(1-\dfrac{\phi^2(x)}{\psi^2(t)}\right)}{\left(1+\dfrac{\phi^2(x)}{\psi^2(t)}\right)^2} = \frac{4\phi(x)\psi(t)\left(\psi^2(t)-\phi^2(x)\right)}{(\psi^2(t)+\phi^2(x))^2}.$$

The equation (4.3) becomes, after some simplifications

$$\left[\psi(t)\phi''(x) + g^{\alpha\beta}(t)\phi(x)\left(\frac{\partial^2\psi}{\partial t^\alpha \partial t^\beta}(t) - \Gamma^\gamma_{\alpha\beta}(t)\frac{\partial\psi}{\partial t^\gamma}(t)\right)\right]\left(\psi^2(t)+\phi^2(x)\right) -$$

$$-2\phi(x)\psi(t)\left((\phi'(x))^2+g^{\alpha\beta}(t)\frac{\partial\psi}{\partial t^\alpha}(t)\frac{\partial\psi}{\partial t^\beta}(t)\right)=\phi(x)\psi(t)\left(\psi^2(t)-\phi^2(x)\right).$$

Dividing by $\phi(x)\psi(t)$, we obtain

$$\left[\frac{\phi''(x)}{\phi(x)}+g^{\alpha\beta}(t)\ \frac{1}{\psi(t)}\ \left(\frac{\partial^2\psi}{\partial t^\alpha\partial t^\beta}(t)-\Gamma^\gamma_{\alpha\beta}(t)\frac{\partial\psi}{\partial t^\gamma}(t)\right)\right]\left(\psi^2(t)+\phi^2(x)\right)-$$

$$-2(\phi'(x))^2-2g^{\alpha\beta}(t)\frac{\partial\psi}{\partial t^\alpha}(t)\frac{\partial\psi}{\partial t^\beta}(t)=\psi^2(t)-\phi^2(x).$$

It follows that

$$\phi''(x)\frac{\psi^2(t)}{\phi(x)}+g^{\alpha\beta}(t)\frac{\phi^2(x)}{\psi(t)}\left(\frac{\partial^2\psi}{\partial t^\alpha\partial t^\beta}(t)-\Gamma^\gamma_{\alpha\beta}(t)\frac{\partial\psi}{\partial t^\gamma}(t)\right)=$$

$$=\left[\psi^2(t)+2g^{\alpha\beta}(t)\frac{\partial\psi}{\partial t^\alpha}(t)\frac{\partial\psi}{\partial t^\beta}(t)-g^{\alpha\beta}(t)\psi(t)\left(\frac{\partial^2\psi}{\partial t^\alpha\partial t^\beta}(t)-\Gamma^\gamma_{\alpha\beta}(t)\frac{\partial\psi}{\partial t^\gamma}(t)\right)\right]+$$

$$+\left(-\phi^2(x)+2(\phi'(x))^2-\phi(x)\phi''(x)\right). \tag{4.4}$$

Differentiating (4.4) with respect to x, the equation becomes

$$\phi'''(x)\frac{\psi^2(t)}{\phi(x)}-\phi''(x)\phi'(x)\frac{\psi^2(t)}{\phi^2(x)}+g^{\alpha\beta}(t)\frac{2\phi(x)\phi'(x)}{\psi(t)}\left(\frac{\partial^2\psi}{\partial t^\alpha\partial t^\beta}(t)-\Gamma^\gamma_{\alpha\beta}(t)\frac{\partial\psi}{\partial t^\gamma}(t)\right)=$$

$$=0+(A(x))'.$$

To differentiate with respect to t^δ, $\delta=1,...,m$, we consider the particular case when $g^{\alpha\beta}(t)=constant$ and $\Gamma^\gamma_{\alpha\beta}(t)=constant$. We obtain the equation

$$\phi'''(x)\frac{2\psi(t)\ \dfrac{\partial\psi}{\partial t^\delta}(t)}{\phi(x)}-\phi''(x)\phi'(x)\frac{2\psi(t)\ \dfrac{\partial\psi}{\partial t^\delta}(t)}{\phi^2(x)}-$$

$$-g^{\alpha\beta}\frac{2\phi(x)\phi'(x)}{\psi^2(t)}\ \frac{\partial\psi}{\partial t^\delta}(t)\ \left(\frac{\partial^2\psi}{\partial t^\alpha\partial t^\beta}(t)-\Gamma^\gamma_{\alpha\beta}\frac{\partial\psi}{\partial t^\gamma}(t)\right)+$$

$$+g^{\alpha\beta}\frac{2\phi(x)\phi'(x)}{\psi(t)}\ \left(\frac{\partial^3\psi}{\partial t^\alpha\partial t^\beta\partial t^\delta}(t)-\Gamma^\gamma_{\alpha\beta}\frac{\partial^2\psi}{\partial t^\gamma\partial t^\delta}(t)\right)=0,$$

or, equivalently,

$$\frac{\left[\psi(t)\left(\dfrac{\partial^3\psi}{\partial t^\alpha\partial t^\beta\partial t^\delta}(t)-\Gamma^\gamma_{\alpha\beta}\dfrac{\partial^2\psi}{\partial t^\gamma\partial t^\delta}(t)\right)-\dfrac{\partial\psi}{\partial t^\delta}(t)\left(\dfrac{\partial^2\psi}{\partial t^\alpha\partial t^\beta}(t)-\Gamma^\gamma_{\alpha\beta}\dfrac{\partial\psi}{\partial t^\gamma}(t)\right)\right]}{\psi^2(t)}\cdot$$

$$\cdot 2g^{\alpha\beta}\phi(x)\phi'(x)+\frac{\phi'''(x)\phi(x)-\phi''(x)\phi'(x)}{\phi^2(x)}\ 2\psi(t)\frac{\partial\psi}{\partial t^\delta}(t)=0,\quad \delta\in\{1,...,m\}.$$

We divide the both sides by $2\phi(x)\psi(t)\phi'(x)\dfrac{\partial\psi}{\partial t^\delta}(t)$:

$$\left(\frac{\phi''(x)}{\phi(x)}\right)' \cdot \frac{1}{\phi(x)\phi'(x)} + \frac{\partial}{\partial t^\delta}\left(\frac{\dfrac{\partial^2\psi}{\partial t^\alpha\partial t^\beta}(t) - \Gamma^\gamma_{\alpha\beta}\dfrac{\partial\psi}{\partial t^\gamma}(t)}{\psi(t)}\right) g^{\alpha\beta}\frac{1}{\psi(t)\dfrac{\partial\psi}{\partial t^\delta}(t)} = 0.$$

It follows that

$$\frac{\left(\dfrac{\phi''(x)}{\phi(x)}\right)'}{\phi(x)\phi'(x)} = -g^{\alpha\beta}\,\frac{\dfrac{\partial}{\partial t^\delta}\left(\dfrac{\dfrac{\partial^2\psi}{\partial t^\alpha\partial t^\beta}(t) - \Gamma^\gamma_{\alpha\beta}\dfrac{\partial\psi}{\partial t^\gamma}(t)}{\psi(t)}\right)}{\psi(t)\dfrac{\partial\psi}{\partial t^\delta}(t)} = -4k,\ k\in\mathbb{R}, \tag{4.5}$$

$$\delta\in\{1,...,m\}.$$

A function in x and a function in t can be equal if they are equal in fact with the same constant. Let this constant be $-4k$, for a suitable form of the relations. We also note that in (4.5) the function in x is the same for every $\delta\in\{1,...,m\}$ and from here it follows that the value of k is the same for every choice of δ.

Thus, we obtain a system of equations

$$\frac{d}{dx}\left(\frac{\phi''(x)}{\phi(x)}\right) = -4k\phi(x)\phi'(x) \tag{4.6}$$

$$g^{\alpha\beta}\,\frac{\partial}{\partial t^\delta}\left(\frac{\dfrac{\partial^2\psi}{\partial t^\alpha\partial t^\beta}(t) - \Gamma^\gamma_{\alpha\beta}\dfrac{\partial\psi}{\partial t^\gamma}(t)}{\psi(t)}\right) = 4k\psi(t)\frac{\partial\psi}{\partial t^\delta}(t),\quad \delta\in\{1,...,m\}, \tag{4.7}$$

an ODE in the unknown $\phi=\phi(x)$ and m PDEs in the unknown $\psi=\psi(t)$.

Let us transform the ODE (4.6). After integrating the two sides, the equation becomes

$$\frac{\phi''(x)}{\phi(x)} = -4k\,\frac{\phi^2(x)}{2} + a,\quad a\in\mathbb{R},$$

that is $\phi''(x) = -2k\phi^3(x)+a\phi(x)$. We multiply by $\phi'(x)$, and apply a new integration of the two sides of the equation. We find

$$\phi''(x)\phi'(x) = -2k\phi^3(x)\phi'(x) + a\phi(x)\phi'(x) \Leftrightarrow$$

$$\left(\phi'(x)\right)^2 = -k\phi^4(x) + a\phi^2(x) + b,\quad k,a,b\in\mathbb{R}. \tag{4.8}$$

Since always we can relate the constants $k,a,b\in\mathbb{R}$ via the condition $a^2+4kb=0$, the equation (4.8) is a Riccati equation. Taking the curvilinear primitive, the PDE system (4.7) is transformed into the PDE

$$g^{\alpha\beta}\left(\frac{\dfrac{\partial^2\psi}{\partial t^\alpha\partial t^\beta}(t) - \Gamma^\gamma_{\alpha\beta}\dfrac{\partial\psi}{\partial t^\gamma}(t)}{\psi(t)}\right) = 4k\frac{\psi^2(t)}{2} + c,\quad c\in\mathbb{R},$$

that is

$$g^{\alpha\beta}\left(\frac{\partial^2\psi}{\partial t^\alpha \partial t^\beta}(t) - \Gamma^\gamma_{\alpha\beta}\frac{\partial\psi}{\partial t^\gamma}(t)\right) = 2k\psi^3(t) + c\ \psi(t), \quad c \in \mathbb{R}. \tag{4.9}$$

Theorem 4.1. *The biscalar solutions of the multitime sine-Gordon PDE are determined by the solutions of ODE (4.8) and PDEs (4.9).*

5 Geometric characteristics of the sine-Gordon PDE

The geometric characteristics of some PDEs give relevant link between Differential Geometry and Applied Sciences, for geometrical methods in Statistics, for mathematical modeling in Ecology, for optimization methods on Riemannian or semi-Riemannian manifolds etc (see also [12] - [10]).

5.1 Case of single-time sine-Gordon PDE

Let g^{ij} be a fundamental symmetric contravariant tensor field of constant signature $(1,1)$ and Γ^i_{jk} be a symmetric linear connection on the manifold $\mathbb{R}\times\mathbb{R}$. Denoting $x^1 = x$, $x^2 = t$; $i,j,k = 1,2$, and using a C^2 function $u : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, we introduce the hyperbolic PDE

$$g^{ij}\left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial u}{\partial x^k}\right) = \sin\ u.$$

Identifying this PDE to the single-time sine-Gordon PDE (1.1), we obtain

$$g^{11} = 1,\ g^{12} = g^{21} = 0,\ g^{22} = -\frac{1}{c^2},\ \Gamma^k_{11} - \frac{1}{c^2}\Gamma^k_{22} = 0.$$

The Lorentzian connection $\Gamma^i_{jk} = 0$, associated to the previous Lorentzian metric g_{ij}, is a particular case.

5.2 Two-time sine-Gordon geometric dynamics

Now let us show that the Lorentzian metric $\{g^{11} = 1, g^{12} = g^{21} = 0, g^{22} = -\frac{1}{c^2}\}$ connects the theory of geometric dynamics to the theory of solitons (see [12]). For that we introduce the source as a Lorentzian manifold

$$(\mathbb{R}\times\mathbb{R},\ g^{11} = 1,\ g^{12} = \ = g^{21} = 0,\ g^{22} = -\frac{1}{c^2})$$

and the target as the Riemannian manifold $(\mathbb{R}, g = 1)$. Let $u : \mathbb{R}\times\mathbb{R}\to R$, $(x,t)\to u(x,t)$ be a C^2 function. On the Riemannian manifold $(\mathbb{R}, g = 1)$, we introduce two vector fields

$$X_1(u) = 2\ \sin\ \frac{u}{2},\ \ X_2(u) = 2\ \sin\ \frac{u}{2},$$

which determine the 2-flow

$$u_x = 2\ \sin\ \frac{u}{2},\ \ u_t = 2\ \sin\ \frac{u}{2}. \tag{5.1}$$

The complete integrability condition is the sine-Gordon PDE $u_{xt} = \sin u$. The flow is characterized by $\ln|\mathrm{tg}\,\frac{u}{4}| = x + t + \ln k$, which yields the general solution

$$u(x,t) = 4\arctan(k\,e^{x+t}). \tag{5.2}$$

We build the Lagrangian $2L = (u_x - 2\sin\frac{u}{2})^2 - \frac{1}{c^2}(u_t - 2\sin\frac{u}{2})^2$. The Euler-Lagrange PDE generated by L is the sine-Gordon PDE

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = (1 - \frac{1}{c^2})\sin u.$$

This sine-Gordon PDE is a prolongation of the 2-flow (5.1) since the function (5.2) is a solution. The theory in this subsection shows that the sine-Gordon soliton PDE is *generated by a 2-flow and an appropriate geometric structure.*

5.3 Case of multitime sine-Gordon PDE

Let $\Gamma^\gamma_{\alpha\beta}$ be a symmetric linear connection and $g = (g^{\alpha\beta})$ be a fundamental symmetric contravariant tensor field of constant signature (r, z, s), $r+z+s = m$ on the manifold $\mathbb{R}^m$. If the tensor field $g = (g_{\alpha\beta})$ is nondegenerate (a Riemannian or semi-Riemannian metric; z = 0) on R^m and $(g^{\alpha\beta})$ is it inverse, then the induced linear connection is

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\lambda\gamma}\left(\frac{\partial g_{\alpha\lambda}}{\partial t^\beta} + \frac{\partial g_{\beta\lambda}}{\partial t^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial t^\lambda}\right).$$

Now we look for pairs (g, Γ), that is (fundamental tensor, connection), such that the solutions of multitime sine-Gordon PDE (2.1) are just the solutions of the single-time PDE (1.1).

Theorem 5.1. *There exists an infinity of Riemannian or semi-Riemannian structures $g_{\alpha\beta}$ on R^m such that a solution of the PDE (1.1) is also a solution of the PDE (2.1).*

Proof. Suppose $t^1 = t$ and $u = u(x, t^1)$ a solution of PDE (1.1). The function $u = u(x, t^1)$ is a solution of the PDE (2.1) if the family of Riemannian structures $(g^{\alpha\beta})$ is fixed by $g^{11} = \frac{1}{c^2}$; $\Gamma^\gamma_{11} = 0$, $\gamma = 1, ..., m$. A family of solutions is $g_{11} = c^2$; $g_{1\alpha} = 0$, for $\alpha = 1, ..., m$; $g_{\alpha\beta} =$ arbitrary for $\alpha, \beta \geq 2$. □

In the sense of the previous Theorem, we can say that the PDE (2.1) is a prolongation of the PDE (1.1).

6 Conclusion

This paper gives solutions to a generalized sine-Gordon PDE, different from the *supersymmetric extension of the sine-Gordon model.* The general expression of this equation is determined by a geometric structure given by a fundamental tensor field $g^{\alpha\beta}$ and a linear connection $\Gamma^\gamma_{\alpha\beta}$. The geometry induced by some of these geometrical objects as well as *vector solitons* will be studied in a further coming paper. In fact, the integrable models and their multitime formulation find a natural and universal setting in terms of Differential Geometry.

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