Fixed loci of the anticanonical complete linear systems of anticanonical rational surfaces

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Abstract. We determine the fixed locus of the anticanonical complete linear system of a given anticanonical rational surface. The case of a geometrically ruled rational surface is fully studied, e.g., the monoid of numerically effective divisor classes of such surface is explicitly determined and is minimally generated by two elements. On the other hand, as a consequence in the particular case where X is a smooth rational surface with $K_X^2 > 0$, the following expected result holds: every fixed prime divisor of the complete linear system $|-K_X|$ is a $(-n)$-curve, for some integer $n \geq 1$.

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1 Introduction

This note is mainly devoted to determine the integral curves of the fixed locus of the complete linear system $|-K_X|$ of an anticanonical rational surface X. Here X is anticanonical means that it is smooth and such that the complete linear system $|-K_X|$ is not empty, where K_X denotes a canonical divisor on X. Such linear system is worth studying, for example, Hironaka considers the unique fixed irreducible component of the anticanonical complete linear system of a very special anticanonical rational surface in order to give an example for which the contraction of an integral curve of strictly negative self-intersection on an algebraic surface is not necessarily an algebraic one (this contraction is always an analytic surface according to Grauert).

From Theorem 4.1 below, it appears that if the fixed locus is not the zero divisor - such situation is the general one - then its irreducible components are either smooth rational curves of strictly negative self-intersection or an integral curve of arithmetic genus equal to one which has in almost all cases a strictly negative self-intersection. The case where the fixed locus is zero implies that $-K_X$ is numerically effective. The nef-ness condition of $-K_X$ means that the intersection number of K_X and of any

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prime divisor on X is less than or equal to zero. Thus the inequality $K_X^2 \geq 0$ holds and consequently the Picard number $\rho(X)$ of X is less than or equal to ten.

On the other hand, from the Riemann-Roch Theorem (see Lemma 2.1 below), a smooth rational surface Y having a canonical divisor K_Y of self-intersection greater than or equal to zero is anticanonical. Such surfaces are studied intensively for different reasons in [17], [18], [16], [8], [11], [12], [13] and [14]. The case where the self-intersection of a canonical divisor is equal to zero is very special and leads to very interesting geometric phenomena, see for instance [17], [16], [12], [13] and [14]. Finally, when the self-intersetion of a canonical divisor is negative, one may determine the geometry of some specific projective rational surfaces, e.g. see [10], [12], [13], [3], [4], [5], [6] and [7].

In the case where $K_Y^2 > 0$ and if the fixed locus of the complete linear system $|-K_Y|$ is not equal to zero as a divisor, we will deduce mainly from Theorem 4.1 that its prime components are smooth rational curves of strictly negative self-intersection (see Corollary 4.2 below). Whereas in the case where $K_Y^2 = 0$, it may happen that $|-K_Y|$ is equal to a singleton, so in particular, the fixed locus is an integral curve of arithmetic genus equal to one and of self-intersection equal to zero.

This note is organized as follows. In section 2, we give some standard facts about smooth rational surfaces and fix our notations. Section 3 deals with the case when the Picard number of the smooth rational surface is equal to two, i.e., the case of geometrically ruled rational surfaces. We determine the fixed locus of the complete linear system associated to any effective divisor (see Proposition 3.2). Also, in this case, the monoid of numerically effective divisor classes of the geometrically ruled rational surface is explicitly determined, it is shown that it is minimally generated by two elements, see Lemma 3.1. Finally, section 4 contains our main result (see Theorem 4.1 below). It is shown that if the fixed locus of the anticanonical rational surface is not equal to zero, then every integral curve of the fixed locus is either a $(-n)$-curve for some integer $n \geq 1$, or an integral curve of arithmetic genus equals to one and of self-intersection less than or equal to zero. Whereas if the fixed locus is zero, then the self-intersection of the canonical divisor of the surface is larger than or equal to zero; thus gives an explicit description of the anticanonical rational surface.

2 Preliminaries

In this section, we mention the notions that we need. See [9] as a reference for these materials. Let X be a smooth algebraic surface defined over an algebraically closed field. A divisor on X is effective if it is a nonnegative linear combination of prime divisors. Similarly, a class of divisors modulo algebraic equivalence on X is effective if it contains an effective divisor. Moreover, if X is rational, then the class of divisors modulo algebraic equivalence containing the divisor D on X is effective if and only if the vector space of global sections of the invertible sheaf $\mathcal{O}_X(D)$ associated to D in the Picard group $Pic(X)$ of X is not trivial. Indeed, more generally the algebraic, the linear, the numerical and the rational equivalences of divisors on the smooth rational surface X are the same. On the other hand $Pic(X)$ is isomorphic to the group $Cl(X)$ of classes of divisors modulo linear equivalence on X.

Let Y be an anticanonical rational surface and let K_Y be a canonical divisor on

it. That Y is anticnonical means by definition that Y a smooth surface such that its anticanonical complete linear system $|-K_Y|$ is not empty. Following [9], we adopt in all this note the following notations:

- $Div(Y)$ is the group of divisors on Y.
- $D \sim D'$ means that D is linearly equivalent to D', where D and D' are elements of $Div(Y)$,
- $Cl(Y)$ is the quotient group $Div(Y)/\sim$ of $Div(Y)$ by $\sim$.
- $NS(Y)$ is the Néron-Severi group $NS(Y)$ of Y, i.e., the quotient group of $Div(Y)$ by the numerical equivalence classes of divisors on Y. Since Y is a rational surface, the linear and numerical equivalences are equivalents on $Div(Y)$. One has $NS(Y)$ is equal to $Cl(Y)$.
- $\rho(Y)$ is the rank of $NS(Y)$ and called the Picard number of Y.
- $\mathbb{F}_n$ is the Hirzebruch surface associated to the integer n, $n \geq 0$ (see [9, Section 2, p. 369]).
- $\mathcal{F}$ is the element of $NS(\mathbb{F}_n)$ associated to any fiber of the ruling of $\mathbb{F}_n$ if $n \neq 0$, and any fiber of any ruling of $\mathbb{F}_0$ if $n = 0$.
- $\mathcal{C}_n$ is the element of $NS(\mathbb{F}_n)$ determined by the unique integral curve of self-intersection equal to $-n$ if $n \neq 0$ or any fiber F' of the second ruling if $n = 0$.
- For a smooth rational surface Y, $\rho(Y) = 1$ if and only if Y is isomorphic to the projective plane $\mathbb{P}^2$. And $\rho(Y) = 2$ if and only if Y is isomorphic to $\mathbb{F}_n$ for some $n \geq 0$. This can be deduced from [9, Chapter 5]).

Now we state the Riemann-Roch Theorem for smooth algebraic surfaces, see [9, Theorem 1.6 (Riemann-Roch)., page 362].

Let X be a smooth algebraic surface. If D is a divisor on X and $\mathcal{O}_X(D)$ denotes the invertible sheaf associated to D in $Pic(X)$. Then the following equality holds.

$$h^0(X, \mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) + h^0(X, \mathcal{O}_X(K_X - D)) = \chi(\mathcal{O}_X) + \frac{1}{2}(D^2 - K_X.D),$$

where K_X and $\chi(X)$ denote a canonical divisor and the Euler characteristic of X respectively.
Notice that for smooth rational surfaces Z, one always has $\chi(\mathcal{O}_Z) = 1$.
The next lemma provides, in particular, an example of an anticanonical rational surface. It is a straightforward application of the Riemann-Roch Theorem to the invertible sheaf associated to an anticanonical divisor (see [9, Theorem1.6 (Riemann-Roch)., page 362]) and of the rationality criterion of Castelnuovo (see [9, Theorem 6.1., page 422] and [1]).

Lemma 2.1. *Let Y be a smooth rational surface such that $K_Y^2 \geq 0$. Then Y is anticanonical.*

Here we recall the notion of nefness of divisors on a smooth algebraic surface X.

Let D be a divisor on a smooth algebraic surface X. D is numerically effective (nef in short) if the intersection number of D with any prime divisor on X is larger than or equal to zero. Similarly, a class of divisors modulo algebraic equivalence on X is nef if this class contains a nef divisor.
To illustrate the last definition, the following examples are useful:

Example 2.1. Let $\pi : X \longrightarrow \mathbb{P}^2$ be the blow up the projective plane $\mathbb{P}^2$ at a finite set of points. Then, the class of a line pulled back to X via π is nef. However, the exceptional divisors are not nef.

Example 2.2. Let $\mathbb{F}_n$ be the Hirzebruch surface associated to the integer $n \geq 0$. Then $\mathcal{F}$ and $(\mathcal{C}_n + n\mathcal{F})$ are numerically effective.

The following example generalizes the useful remark stated in [2, Remarque utile III.5, p.35].

Example 2.3. Let Z be a smooth algebraic surface. Let $\Gamma_1, \ldots, \Gamma_p$ be the irreducible components of the effective divisor D on Z. Then the followings are equivalents:

1. The intersection number of D and Γ_i is larger than or equal to zero for every $i = 1, \ldots, p$.
2. D is nef.

We are interested to answer the following question: let Y be an anticanonical rational surface and let K_Y be a canonical divisor on Y. What kind of fixed integral curves may have the anticanonical complete linear system $|-K_Y|$ if it has some? More specially, we are interested in the curves which are fixed components in $|-K_Y|$.

Since the anticanonical complete linear system $|\mathcal{O}_{\mathbb{P}^2}(3)|$ of the projective plane $\mathbb{P}^2$ does not have a fixed component, we will focus in the case $\rho(Y) \geq 2$. Firstly in the next section, we will review the case of geometrically ruled rational surfaces, i.e., those smooth rational surfaces with Picard number equal to two.

Here we give a useful result.

Lemma 2.2. *Let Γ be a prime divisor on an anticanonical rational surface Z. If $\Gamma^2 > 0$, then $h^0(Z, O_Z(\Gamma)) \geq 2$.*

Proof. The Riemann-Roch Theorem applied to the invertible sheaf $\mathcal{O}_Z(\Gamma)$ gives the following inequality:

$$h^0(Z, \mathcal{O}_Z(\Gamma)) \geq 1 + \frac{1}{2}(\Gamma^2 - K_Z.\Gamma).$$

An application of Example 2.3 to Γ shows that Γ is nef. Taking into account that Z is anticanonical leads to the inequality: $\Gamma.K_Z \leq 0$. Then the result follows obviously if $\Gamma.K_Z \leq -1$. Whereas if $\Gamma.K_Z = 0$, then the adjunction formula implies that $\Gamma^2 \geq 2$. And we are done. □

Remark 2.4. If one allows that $\Gamma^2 = 0$ in the above Lemma 2.2, then the inequality $h^0(Z, O_Z(\Gamma)) \geq 2$ may fail to hold.

3 The case of a geometrically ruled rational surface

Let $\mathbb{F}_n$ be the Hirzebruch surface associated to the integer $n \in \mathbb{N}$. The Néron-Severi group $NS(\mathbb{F}_n)$ of $\mathbb{F}_n$ is a free abelian group generated by $\mathcal{C}_n$ and $\mathcal{F}$ and it is endowed with the intersection form denoted by . which is given on the generators by (see [9, proposition 3.2., p. 386]):

- ${\mathcal{C}_n}^2 = -n$;
- $\mathcal{F}^2 = 0$;
- $\mathcal{C}_n.\mathcal{F} = 1$.

The following lemma shows that $\mathcal{C}_n$ and $\mathcal{F}$ generate also the monoid $M(\mathbb{F}_n)$ of effective divisor classes of $\mathbb{F}_n$ and that the monoid $NEF(\mathbb{F}_n)$ of numerically effective divisors classes of $\mathbb{F}_n$ is generated by two elements, namely $(\mathcal{C}_n + n\mathcal{F})$ and $\mathcal{F}$. Note that both $\mathcal{C}_n$, $(\mathcal{C}_n + n\mathcal{F})$ and $\mathcal{F}$ are all of them prime classes, i.e., each of them is the class in $NS(\mathbb{F}_n)$ of a prime divisor on $\mathbb{F}_n$. For completeness, we give a proof of it.

Lemma 3.1. *Let $NS(\mathbb{F}_n)$ be as above. Then*

1. *$M(\mathbb{F}_n) = \mathbb{N}\mathcal{C}_n + \mathbb{N}\mathcal{F}$. Moreover, $M(\mathbb{F}_n)$ can not be generated by one element.*
2. *$NEF(\mathbb{F}_n) = \mathbb{N}(\mathcal{C}_n + n\mathcal{F}) + \mathbb{N}\mathcal{F}$. Moreover, $NEF(\mathbb{F}_n)$ can not be generated by one element.*

Proof. 1. The inclusion $\mathbb{N}\mathcal{C}_n + \mathbb{N}\mathcal{F} \subset M(\mathbb{F}_n)$ is clear. Let us see why the other inclusion is true. Take an element z in $M(\mathbb{F}_n) \subset NS(\mathbb{F}_n)$, it follows that $z = u\mathcal{C}_n + v\mathcal{F}$ for some integers u and v. The fact that $\mathcal{F}$ and $(\mathcal{C}_n + n\mathcal{F})$ (see Example 2.2) are numerically effective gives the required inequalities $u = z.\mathcal{F} \geq 0$ and $v = z.(\mathcal{C}_n + n\mathcal{F}) \geq 0$. This proves the first statement. Since $\mathcal{C}_n$ and $\mathcal{F}$ are linearly independents, the submonoid $M(\mathbb{F}_n)$ of $NS(\mathbb{F}_n)$ can not be generated by one element.

2. It is obvious that $\mathbb{N}(\mathcal{C}_n + n\mathcal{F}) + \mathbb{N}\mathcal{F} \subset NEF(\mathbb{F}_n)$. Now, let x be an element of $NEF(\mathbb{F}_n) \subset NS(\mathbb{F}_n)$, there exist then two integers a and b such that $x = a\mathcal{C}_n + b\mathcal{F}$. Since $\mathcal{F}$ and $\mathcal{C}_n$ are effective and x is numerically effective, we get $0 \leq \mathcal{F}.x = a$ and $0 \leq x.\mathcal{C}_n = b - na$. So, $x = a\mathcal{C}_n + b\mathcal{F} = a(\mathcal{C}_n + n\mathcal{F}) + (b - na)\mathcal{F}$ and we are done. Again as above,$NEF(\mathbb{F}_n)$ can not be generated by one element. □

Next, we determine the fixed locus of any complete linear system $|aC_n + bF|$ associated to an effective divisor $D_{(a,b)}$ whose class in the Néron-Severi group $NS(\mathbb{F}_n)$ is $a\mathcal{C}_n + b\mathcal{F}$. Our result is:

Proposition 3.2. *Let $a\mathcal{C}_n + b\mathcal{F}$ be an effective element of $NS(\mathbb{F}_n)$, where n is an integer greater than or equal to zero. Then, the complete linear system $|aC_n + bF|$ does not have a fixed component if both inequalities $b \geq an$ and $n \geq 1$ hold. Moreover if $b < an$, then there is only one fixed component. In this case the fixed component and the mobile component of $|aC_n + bF|$ are jC_n and $(a - j)C_n + bF$ respectively, where j is the unique integer j such that $1 \leq j \leq a$ and $(a-j)n \leq b \leq (a-j+1)n-1$. For $n = 0$, $a\mathcal{C}_0 + b\mathcal{F}$ does not have a fixed component.*

Proof. Assuming that $b \geq an$ and $n \geq 1$, it follows from [9, Corollary 2.18., page 380] that the complete linear system $|aC_n + bF|$ has no fixed component. Now if $b < an$, then from

$jC_n.(aC_n + bF) = j(b - an) < 0$ we deduce that jC_n is a fixed component of $|aC_n + bF|$, even it is the fixed component since $|(a - j)C_n + bF|$ contains an integral curve. To end the proof, it is straightforward from [9, Corollary 2.18., page 380] that if $n = 0$, then the effective class $a\mathcal{C}_0 + b\mathcal{F}$ has a zero fixed locus. □

A direct application of the last proposition to an anticanonical divisor $2C_n + (2+n)F$ on $\mathbb{F}_n$ gives the following.

Corollary 3.3. *The complete anticanonical linear system of $\mathbb{F}_n$ does not have a fixed component if n takes the values zero, one or two. And, it has C_n as the fixed component for $n \geq 3$.*

Proof. Taking into account that the complete linear system of the anticanonical class of $\mathbb{F}_0$, $\mathbb{F}_1$ and $\mathbb{F}_2$ respectively are $|2C_0 + 2F|$, $|2C_n + 3F|$ and $|2C_n + 4F|$ respectively; and these complete linear systems contains integral curves, the result holds in the case of $\mathbb{F}_n$ with $0 \leq n \leq 2$. Now, assume that $n \geq 3$. From $C_n.(2C_n + (2+n)F) = 2 - n < 0$, we deduce that C_n is a fixed component of the complete linear system $|2C_n + (2+n)F|$ of the anticanonical class of $\mathbb{F}_n$. On the other hand, since the complete linear system $|C_n + (2+n)F|$ contains a smooth curve, we deduce that C_n is the fixed component of $|2C_n + (2+n)F|$. □

4 The case of a blow up a geometrically ruled surface

Here, we consider the case when the Picard number $\rho(Y)$ of the anticanonical rational surface Y is greater than or equal to three. In the following theorem, we determine in particular the fixed components of the anticanonical complete linear system of Y if it has some. If this system does not have any, then the nature of Y can be also determined.

Theorem 4.1. *Let Y be an anticanonical rational surface with Picard number $\rho(Y) \geq 3$. Two cases may occur:*

1. If the anticanonical complete linear system $|-K_Y|$ has a fixed component, then it is either a $(-n)$-curve or an integral curve of arithmetic genus equal to one and of self-intersection less than or equal to zero. Moreover, the second case occurs with an integral curve of self-intersection equal to zero only if $K_Y^2 = 0$.

2. If the anticanonical complete linear system $|-K_Y|$ does not have a fixed component, then $K_Y^2 \geq 0$ and Y is isomorphic to a blow up the projective plane at r points, may be infinitely near, r is an integer less than or equal to nine.

Proof. Since a blow up of $\mathbb{F}_0$ or of $\mathbb{F}_1$ at a nonempty set of points (may be infinitely near) has the projective plane $\mathbb{P}^2$ as a minimal model, and since a blow up of $\mathbb{F}_2$ at a nonempty set of points (may be infinitely near) has $\mathbb{P}^2$ or $\mathbb{F}_3$ as a minimal model, we may assume that the surface Y has either $\mathbb{P}^2$ or $\mathbb{F}_n$, with $n \geq 3$, as a minimal model.
Let us prove the item $(1-)$. Assume first that $\mathbb{P}^2$ is a minimal model of Y and let ϕ be a projective birational morphism from Y to $\mathbb{P}^2$. Let Γ be a fixed irreducible component of the complete linear system $|-K_Y|$. Two possibilities may occur: $\phi(\Gamma)$ is either a point of $\mathbb{P}^2$ or an integral curve on $\mathbb{P}^2$.
Assume that $\phi(\Gamma)$ is a point, then by [9, Exercise 5.4. (a), page 419], we deduce that Γ

is a smooth rational curve of self-intersection strictly negative, i.e. a $(-n)$-curve on Y for some integer $n \geq 1$. Now assume that $\phi(\Gamma)$ is an irreducible curve on $\mathbb{P}^2$, let denote by d its degree. Since $\phi(-K_Y)$ has degree equal to three. It follows that $1 \leq d \leq 3$. If $d = 3$, then we have $\Gamma + \sum_{i=1}^{i=u} n_i E_i = -K_Y$ for some integers $n_i \geq 0$ and some smooth rational curves E_i of self-intersection strictly negative, where $u \geq 1$ is an integer. On the other hand, it follows from the fact that Γ is a fixed irreducible component of $|-K_Y|$ that $\Gamma^2 \leq 0$. Otherwise, we would get that $\Gamma^2 > 0$, in particular Γ (see Lemma 2.2) moves which is a contradiction with the fact that Γ does not move.
If $d = 2$, then $\phi(\Gamma)$ is an irreducible conic on $\mathbb{P}^2$. Hence, it is a smooth rational curve. It follows from [9, Corollary 5.4., page 411] that Γ is also a smooth rational curve on Y. And Γ should be of self-intersection strictly negative. The same argument prove that if $d = 1$, then Γ is a smooth rational curve of self-intersection strictly negative.
Now let $n \geq 3$ be a fixed integer, assume that $\mathbb{F}_n$ is a minimal model of Y. Then consider ψ be a projective birational morphism from Y to $\mathbb{F}_n$. Let Γ be a fixed irreducible component of $|-K_Y|$, then we can assume that $\psi(\Gamma)$ is an irreducible curve on $\mathbb{F}_n$. Otherwise, it should be a point of $\mathbb{F}_n$; so by proceeding as in the above case for $\mathbb{P}^2$, we get the result.
Thus assuming that $\psi(\Gamma)$ is an irreducible curve, in particular, it is an irreducible component of $-K_{\mathbb{F}_n} = 2C_n + (2+n)F$. Thus taking into account of the results of Proposition 3.2 , Γ may be one of the following irreducible curves: C_n, F, $C_n + nF$, $C_n + (1+n)F$, and $C_n + (2+n)F$. So the result follows.

The item $(2-)$ follows at once by remarking that an anticanonical divisor $-K_Y$ of Y is numerically effective. □

In particular, the following result holds:

Corollary 4.2. *Let X be a smooth rational surface such that $K_X^2 \geq 0$, where K_X denotes a canonical divisor on X. Assume that the anticanonical complete linear system has a fixed component Γ. Two cases may occur:*

- *If $K_X^2 > 0$, then Γ is a $(-n)$-curve, where $n \geq 1$ is an integer;*
- *If $K_X^2 = 0$, then Γ is either an integral curve of arithmetic genus equal to one and of self-intersection equal to zero, or a smooth rational curve of strictly negative self-intersection.*

Another useful result, see for instance [10] and [13], is:

Corollary 4.3. *Let X be a smooth rational surface such that $K_X^2 \geq 0$, where K_X denotes a canonical divisor on X. If $-K_X$ is not numerically effective, then the anticanonical complete linear system has a $(-n)$-curve, n being an integer greater than or equal to three, as a fixed component.*

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