Conformal vector fields and conformal transformations on a Riemannian manifold

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Abstract. In this paper first it is proved that if ξ is a nontrivial closed conformal vector field on an n-dimensional compact Riemannian manifold (M, g) with constant scalar curvature S satisfying $S \leq \lambda_1(n-1)$, λ_1 being first nonzero eigenvalue of the Laplacian operator Δ on M and Ricci curvature in direction of a certain vector field is non-negative, then M is isometric to the n-sphere $S^n(c)$, where $S = n(n-1)c$. Finally we show that a conformal transformation $F : M \to M$ of a Riemannian manifold (M, g) that preserves the eigenfunctions that is $\Delta' h = -\lambda h$ whenever $\Delta h = -\mu h$, for constants λ, μ, $(g' = F^* g$ and Δ' and Δ are Laplacian operators on (M, g') and (M, g) respectively), then F is a homothety.

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1 Introduction

Lichnerowicz's result states that if the Ricci curvature of a compact Riemannian manifold (M, g) satisfies $Ric \geq (n-1)c$ for a constant c, then the first nonzero eigenvalue λ_1 satisfies $\lambda_1 \geq nc$. Then Obata [9] has proved that the equality $\lambda_1 = nc$ holds if and only if M is isometric to $S^n(c)$. There are other results estimating eigenvalues of the Laplacian operator on different compact Riemannian manifolds (cf. [5, 10]). A smooth vector field ξ on a Riemannian manifold (M, g) is said to a conformal vector field if there exists a smooth function f on M called potential function that satisfies

$$\pounds_\xi g = 2fg,$$

where $\pounds_\xi g$ is the Lie derivative of g with respect ξ. We say that ξ nontrivial conformal vector field if the potential function f is a nonconstant function. If in addition ξ is a closed vector field, ξ is said to be a closed conformal vector field. If the conformal vector field ξ is gradient of a smooth function, then ξ is said to be a conformal

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gradient vector field. Riemannian manifolds admitting closed conformal vector fields or conformal gradient vector fields have been investigated in (cf. [1, 2, 3, 6, 7, 11]).

Recall that the sphere $S^n(c)$ of constant curvature c has positive Ricci curvature and admit many conformal gradient vector fields, with its first nonzero eigenvalue satisfying $S = \lambda_1(n-1)$. In [11], it is proved that on a compact Riemannian manifold of positive Ricci curvature and constant scalar curvature if there exists a nontrivial conformal gradient vector field, then it is isometric to a sphere. A natural question arises can we relax the condition that Ricci curvature being positive in this result. In this paper, we consider closed conformal vector fields (slightly general than conformal gradient vector fields) and answer this question, indeed we prove the following :

Theorem 1.1. *Let ξ be a nontrivial closed conformal vector field on an n-dimensional compact Riemannian manifold (M, g) of constant scalar curvature S. If the Ricci curvature of M in the direction of the vector field $\nabla f + c\xi$ is non-negative, where $S = n(n-1)c$ and the inequality $(n-1)\lambda_1 \geq S$ holds (λ_1 is the first nonzero eigenvalue of the Laplacian operator Δ), then M is isometric to the sphere $S^n(c)$.*

A conformal transformation F of a Riemannian manifold (M, g) is a diffeomorphism $F : M \to M$ that satisfies $F^*(g) = e^{-2f}g$ for a smooth function f on M. If f is a constant, the conformal transformation F is said to be a homothety. One of the interesting questions is to obtain conditions under which a conformal transformation is a homothety. For instance in [13], Xu has shown that if the Ricci tensors $\overline{Ric}$, Ric of the compact Riemannian manifolds $(M, F^*(g))$, (M, g) satisfy $\overline{Ric} = Ric$, then F is a homothety. We are interested in a conformal transformation $F : M \to M$ of a Riemannian manifold (M, g) that preserves the eigenfunctions that is $\Delta' h = -\lambda h$ whenever $\Delta h = -\mu h$ for a constants λ, μ and a smooth function h, where Δ', Δ are the Laplacian operators on the Riemannian manifolds $(M, F^*(g))$, (M, g) respectively. Our next result is the following :

Theorem 1.2. *Let (M, g) be an n-dimensional compact Riemannian manifold and $F : M \to M$ be a conformal transformation with $F^*(g) = g' = e^{-2f}g$. If for each eigenfunction h of Δ is also an eigenfunction of Δ', where Δ and Δ' are Laplacian operators on the Riemannian manifolds (M, g) and (M, g') respectively, then F is a homothety.*

2 Preliminaries

Let (M, g) be a Riemannian manifold with Lie algebra $\mathfrak{X}(M)$ of smooth vector fields on M. A vector field $\xi \in \mathfrak{X}(M)$ is said to be a conformal vector field if

$$\pounds_\xi g = 2fg, \tag{2.1}$$

for a smooth function $f : M \to R$ called the potential function, where $\pounds_\xi$ is the Lie derivative with respect to ξ. If ∇ is the Riemannian connection on the Riemannian manifold (M, g), then using Koszul's formula (cf. [3]), we obtain, for a vector field ξ on M,

$$2g(\nabla_X \xi, Y) = (\pounds_\xi g)(X, Y) + d\eta(X, Y), \quad X, Y \in \mathfrak{X}(M), \tag{2.2}$$

where η is the 1-form dual to ξ that is $\eta(X) = g(X, \xi)$, $X \in \mathfrak{X}(M)$. Define a skew symmetric tensor field φ of type $(1,1)$ on M by

$$d\eta(X, Y) = 2g(\varphi X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.3}$$

Then using equations (2.1), (2.2) and (2.3) we immediately get the following :

Lemma 2.1. *Let ξ be a conformal vector field on a Riemannian manifold (M, g) with potential function f. Then,*

$$\nabla_X \xi = fX + \varphi X, \quad X \in \mathfrak{X}(M).$$

Using Lemma 2.1, we immediately arrive at the following expression of the curvature tensor R of the Riemannian manifold (M, g)

$$R(X, Y)\xi = g(\nabla f, X)Y - g(\nabla f, Y)X + (\nabla\varphi)(X, Y) - (\nabla\varphi)(Y, X), \tag{2.4}$$

where $(\nabla\varphi)(X, Y) = \nabla_X \varphi Y - \varphi(\nabla_X Y)$, $X, Y \in \mathfrak{X}(M)$.

For a smooth function h on M, we define an operator $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $A(X) = \nabla_X \nabla h$, ∇h being the gradient of h. The Ricci operator Q is a symmetric $(1,1)$-tensor field that is defined by $g(QX, Y) = Ric(X, Y)$, $X, Y \in \mathfrak{X}(M)$, where Ric is the Ricci tensor of the Riemannian manifold. Then we have the following (cf. [7])

Lemma 2.2. *Let (M, g) be a Riemannian manifold and h be a smooth function on M. Then the operator A corresponding to the function h satisfies*

$$\sum_i (\nabla A)(e_i, e_i) = \nabla(\Delta h) + Q(\nabla h),$$

where $\{e_1, ..., e_n\}$ is a local orthonormal frame, Δ is the Laplacian operator on M and $(\nabla A)(X, Y) = \nabla_X AY - A(\nabla_X Y)$, $X, Y \in \mathfrak{X}(M)$.

Using the skew symmetry of the tensor φ and a local orthonormal frame $\{e_1, ..., e_n\}$ on M in equation (2.4), we compute

$$Q(\xi) = -(n-1)\nabla f - \sum_{i=1}^{n} (\nabla\varphi)(e_i, e_i). \tag{2.5}$$

The Lemma 2.1, as φ is skew symmetric, gives that $div\xi = nf$, and consequently, for a conformal vector field ξ on a compact Riemannian manifold (M, g) with potential function f we have

$$\int_M f dV = 0, \tag{2.6}$$

and using minimum principle we arrive at

$$\lambda_1 \int_M f^2 dV \leq \int_M \|\nabla f\|^2 dV, \tag{2.7}$$

where λ_1 is the first nonzero eigenvalue of the Laplacian operator Δ on M.

Lemma 2.3. *Let ξ be a conformal vector field on an n-dimensional compact Riemannian manifold (M,g) with potential function f. Then*

$$\int_M g(\nabla f,\xi)dV = -n\int_M f^2 dV.$$

Proof. Since $div\xi = nf$, it follows that $div(f\xi) = g(\nabla f,\xi) + nf^2$. Integrating this equation we get the Lemma. $\square$

For the tensor φ, we have $\|\varphi\|^2 = \sum \|\varphi e_i\|^2$, where $\{e_1,...,e_n\}$ is a local orthonormal frame on M. Next, we prove the following :

Lemma 2.4. *Let (M,g) be an n-dimensional compact Riemannian manifold and ξ be a conformal vector field on M with potential function f. Then,*

$$\int_M \left\{Ric(\xi,\xi) - n(n-1)f^2 - \|\varphi\|^2\right\} dv = 0.$$

Proof. Using Lemma 2.1, equation (2.5) and skew symmetry of φ together with a point wise constant local orthonormal frame $\{e_1,...,e_n\}$, we compute

$$\begin{aligned} div(\varphi\xi) &= \sum_{i=1}^n g(\nabla_{e_i}\varphi\xi, e_i) = -\sum_{i=1}^n e_i g(\xi,\varphi e_i) \\ &= -\sum_{i=1}^n g(fe_i + \varphi e_i, \varphi e_i) - \sum_{i=1}^n g(\xi, (\nabla\varphi)(e_i,e_i)) \\ &= -\|\varphi\|^2 + (n-1)g(\nabla f,\xi) + Ric(\xi,\xi). \end{aligned}$$

Integrating the above equation and using Lemma 2.3 we get the result. $\square$

As a direct consequence of above Lemma we have the following interesting consequence :

Corollary 2.5. *On a compact Riemannian manifold of negative Ricci curvature there does not exist a nonzero conformal vector field.*

3 Proof of Theorem 1.1

Let (M,g) be an n-dimensional Riemannian manifold. Then the Ricci operator Q satisfies

$$\sum_{i=1}^n (\nabla Q)(e_i,e_i) = \frac{1}{2}\nabla S, \tag{3.1}$$

where $\{e_1,...,e_n\}$ is a local orthonormal frame on M and S is the scalar curvature of M. Let ξ be a conformal vector field on M with potential function f. First we prove the following :

Lemma 3.1. *Let (M,g) be an n-dimensional compact Riemannian manifold of constant scalar curvature S and ξ be a conformal vector field on M with potential function f. Then,*

$$\int_M Ric(\nabla f,\xi)dV = -S\int_M f^2 dV.$$

Proof. We use symmetry of Q and Lemma 2.1 together with equation (3.1) to compute

$$\begin{aligned} div(Q\xi) &= \sum_{i=1}^n g(\nabla_{e_i}Q\xi, e_i) = \sum_{i=1}^n e_i g(\xi, Qe_i) \\ &= \sum_{i=1}^n g(fe_i+\varphi e_i, Qe_i) + \sum_{i=1}^n g(\xi, (\nabla Q)(e_i,e_i)) \\ &= fS + \sum_{i=1}^n g(\varphi e_i, Qe_i). \end{aligned} \tag{3.2}$$

Choosing a local orthonormal frame that diagonalizes the symmetric operator Q and using the skew symmetry of φ, we conclude that

$$\sum_{i=1}^n g(\varphi e_i, Qe_i) = 0. \tag{3.3}$$

Using $div(fQ\xi) = Ric(\nabla f,\xi) + f div(Q\xi)$ and equations (3.2), (3.3) we get the Lemma. □

Let ξ be the conformal vector field on an n-dimensional compact Riemannian manifold of constant scalar curvature S with potential function f. Then for the function f, using $div\xi = nf$, we have,

$$div(\Delta f\xi) = g(\nabla\Delta f,\xi) + nf\Delta f = g(\nabla\Delta f,\xi) + \frac{n}{2}\Delta f^2 - n\|\nabla f\|^2,$$

which on integration gives

$$\int_M g(\nabla\Delta f,\xi)dV = n\int_M \|\nabla . f\|^2 dV. \tag{3.4}$$

Using the operator $A(X) = \nabla_X\nabla f$ together with Lemma 2.1 and Lemma 2.2, we compute

$$\begin{aligned} div(A\xi) &= f\Delta f + g(\xi,\nabla\Delta f) + Ric(\nabla f,\xi) \\ &= \frac{1}{2}\Delta f^2 - \|\nabla f\|^2 + g(\xi,\nabla\Delta f) + Ric(\nabla f,\xi), \end{aligned} \tag{3.5}$$

where we used $\Delta f = \sum g(Ae_i,e_i)$ and $\sum g(\varphi e_i, Ae_i) = 0$ which follows by choosing a local orthonormal frame that diagonalizes the symmetric operator A and the skew symmetry of φ. Integrating (3.5) and using equation (3.4) together with Lemma 3.1, we conclude

$$(n-1)\int_M \|\nabla f\|^2 dV = S\int_M f^2 dV. \tag{3.6}$$

Since ξ is nontrivial, f is a nonzero function, from equation (3.6) it follows that the constant $S > 0$. Combining equations (2.7) and (3.6) we conclude

$$\lambda_1(n-1) \leq S,$$

and the equality holds if and only if equality in (2.7) holds and the equality in (2.7) holds if and only if $\Delta f = -\lambda_1 f$ (cf. [4]). However, with the assumption $S \leq \lambda_1(n-1)$ in the statement, we get the equality and consequently

$$\Delta f = -\lambda_1 f \quad \text{and} \quad S = \lambda_1(n-1). \tag{3.7}$$

As ξ is closed, we have $d\eta = 0$ and $\varphi = 0$ consequently, Lemma 2.1 gives

$$\nabla_X \xi = fX, \quad X \in \mathfrak{X}(M). \tag{3.8}$$

Choosing $S = n(n-1)c$, we have

$$Ric(\nabla f + c\xi, \nabla f + c\xi) = Ric(\nabla f, \nabla f) + c^2 Ric(\xi, \xi) + 2cRic(\nabla f, \xi). \tag{3.9}$$

Since the equality $\Delta f = -\lambda_1 f = -ncf$ holds, we have

$$\int_M \|\nabla f\|^2 \, dV = nc \int_M f^2 dV. \tag{3.10}$$

Using Lemma 2.2 to compute $div(A\nabla f)$, it is straight forward to derive

$$\int_M \left\{ Ric(\nabla f, \nabla f) + g(\nabla f, \nabla \Delta f) + \|A\|^2 \right\} dV = 0, \tag{3.11}$$

which together with $\Delta f = -ncf$ and the Schwartz inequality $\|A\|^2 \geq \frac{1}{n}(\Delta f)^2 = nc^2 f^2$ gives

$$\begin{aligned} \int_M Ric(\nabla f, \nabla f) dV &= \int_M \left\{ \lambda_1 \|\nabla f\|^2 - \|A\|^2 \right\} dV \\ &\leq n(n-1)c^2 \int_M f^2 dV. \end{aligned} \tag{3.12}$$

Integrating equation (3.9) and using inequality (3.12) together with Lemmas 2.4 and 3.1, we arrive at

$$\int_M Ric(\nabla f + c\xi, \nabla f + c\xi) dV \leq 0.$$

As the Ricci curvature $Ric(\nabla f + c\xi, \nabla f + c\xi)$ is nonnegative the above inequality gives

$$Ric(\nabla f + c\xi, \nabla f + c\xi) = 0,$$

which together with equations (3.9) and Lemmas 2.4 and 3.1gives

$$\int_M \left\{ Ric(\nabla f, \nabla f) - n(n-1)c^2 f^2 \right\} dV = 0.$$

Using the above equation and the equation (3.10) in the equation (3.11), we arrive at

$$\int_M \left\{ \|A\|^2 - nc^2 f^2 \right\} dV = 0. \tag{3.13}$$

However, the Schwartz's inequality implies $\|A\|^2 - \frac{1}{n}(trA)^2 = \|A\|^2 - nc^2f^2 \geq 0$, with equality holding if and only if $A = -cfI$. Thus the equation (3.13) confirms that $A = -cfI$, that is

$$\nabla_X \nabla f = -cfX, \quad X \in \mathfrak{X}(M),$$

which is Obata's differential equation on M with non-constant f (as ξ is nontrivial conformal vector field), and this proves that M is isometric to $S^n(c)$.

4 Proof of Theorem 1.2

Let (M,g) be an n-dimensional Riemannian manifolds and $F : M \to M$ be a conformal transformation with $F^*(g) = g' = e^{-2f}g$, for a smooth function f on M.

Lemma 4.1. *Let dV_g and $dV_{g'}$ be the volume elements of the orientable Riemannian manifolds (M,g) and $(M,g^{'})$, $g^{'} = e^{-2f}g$. Then*

$$dV_{g'} = e^{-nf}dV_g.$$

Proof. Let $\{\omega^1, ..., \omega^n\}$ be the basis of smooth 1-forms dual to $\{e_1, ..., e_n\}$ on (M,g) and $\{\overline{\omega}^1, ..., \overline{\omega}^n\}$ be that of $\{e^f e_1, ..., e^f e_n\}$ on (M,g') respectively. Then we have for $X \in \mathfrak{X}(M)$

$$\omega^i(X) = g(X,e_i) = e^{2f}g'(X,e_i) = e^f g^{'}(X, e^f e_i) = e^f \overline{\omega}^i(X),$$

which gives $\omega^i = e^f \overline{\omega}^i$. Consequently we have

$$\begin{aligned} dV_g &= \omega^1 \Lambda ... \Lambda \omega^n \\ &= e^{nf}\overline{\omega}^1 \Lambda ... \Lambda \overline{\omega}^n = e^{nf}dV_{g'}, \end{aligned}$$

and this proves the Lemma. □

Finally we prove the Theorem 1.2. Let h be the eigenfunction of the Laplacian operator Δ corresponding to the nonzero eigenvalue λ, that is $\Delta h = -\lambda h$, $\lambda > 0$. Then by the hypothesis of the theorem we have $\Delta' h = -\mu h$ for a constant μ. Note that the constant $\mu > 0$, for other wise $\Delta' h = 0$ on the compact Riemannian manifold (M,g') would imply h is a constant which is against the assumption $\Delta h = -\lambda h$, $\lambda > 0$. Thus we have

$$\int_M h dV_{g'} = -\frac{1}{\mu}\int_M \Delta' h dV_{g'} = 0,$$

which together with Lemma 4.1 gives

$$\int_M h e^{-nf} dV_g = 0.$$

Since h is arbitrary eigenfunction corresponding to nonzero eigenvalue of Δ, above integral suggests that the function e^{-nf} is orthogonal to each eigenfunction corresponding to nonzero eigenvalue on the Riemannian manifold (M,g). This proves e^{-nf} is a constant function and consequently that F is a homothety.

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References

[1] H. Alodan, *Hypersurfaces of a Riemannian manifold with Killing vector field*, Differ. Geom. Dyn. Syst. 13 (2011), 19-26.

[2] K. Amur and V. S. Hedge, *Conformality of Riemannian manifolds to spheres*, J. Diff. Geom. 9 (1974), 571-576.

[3] A. L. Besse, *Einstein Manifolds*, Springer Verlag, 1987.

[4] I. Chavel, *Eigenvalues in Riemannian geometry*, Academic press 1984.

[5] S. Deshmukh and A. Al-Eid, *Curvature bounds for the spectrum of a compact Riemannian manifold of constant scalar curvature*, J. Geom. Anal. 15, 4 (2005), 589-606.

[6] S. Deshmukh, H. Alodan and H. Al-Sodais, *A note on Ricci solitons*, Balkan J. Geom. Appl. 16, 1 (2011), 48-55.

[7] S. Deshmukh and F. R. Al-Solamy, *Conformal gradient vector fields on a compact Riemannian manifold*, Colloquium Math. 112, 1 (2008), 157-161.

[8] M. Obata, *Conformal transformations of Riemannian manifolds*, J. Diff. Geom. 4 (1970), 311-333.

[9] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan 14 (1962), 333-340.

[10] U. Simon, *Curvature bounds for the spectrum of a closed Einstein space, Canada*, J. Math. XXX, 4 (1978), 1087-1091.

[11] S. Tanno and W. Weber, *Closed conformal vector fields*, J. Diff. Geom. 3 (1969), 361-366.

[12] Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. 117 (1965), 251-275.

[13] Xingwang Xu, *Prescribing a Ricci tensor in a conformal class of Riemannian metrics*, Proc. Amer. Math. Soc. 115, 2 (1992), 455-459.

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