Identity theorem for ODEs, auto-parallel graphs and geodesics

Constantin Udrişte, Cristian Ghiu and Ionel Ţevy

Abstract. Looking for the geometrical structures that transform the solutions of a second order ODE into auto-parallel graphs, we need an Identity Theorem for ODEs. It is well-known that a similar theorem holds for polynomial and holomorphic functions. Though our theory is realized on second order ODEs, it can be extended immediately to n-th order ODEs, to PDEs or operator equations.

The main result of Section 1 is the Identity Theorem for ODEs. Section 2 formulates the conditions in which the graphs of the solutions for a given second order ODE are auto-parallel curves, determining the most general connection. Section 3 presents the conditions in which the graphs of the solutions for a given second order ODE are geodesics, finding the most general Riemannian metric. Also this Section gives the equations form of such geodesics. Section 4 interprets the results of the Sections 2 and 3 via isometric manifolds theory.

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1 Identity theorem for ODEs

Second order ODEs on finite dimensional manifolds appear in a wide variety of applications in mathematics, physics and engineering [1] – [10]. They can be classified as: (1) ODEs in differential geometry (the auto-parallel curves of a linear connection, the geodesics of the metric in Riemann and Finsler geometries and the integral curves of the Reeb field on a contact manifold); (2) Euler-Lagrange ODEs in single-time variational calculus; (3) ODEs in Classical Mechanics (Newton equations of motion and the Euler-Lagrange equations of a mechanical Lagrangian); (4) ODEs in general relativity and its variants (the worldlines of free particles); (5) ODEs in classical electrodynamics (the paths of charged particles) etc.

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The theory of identifying two ODEs (or PDEs or operator equations) is very helpful for researchers in Ordinary Differential Equations, Differential Geometry and Operator Theory. But, after our knowledge, here it is the first time when this theory is formulated and presented in details.

Theorem 1.1. *Let $G \subseteq \mathbb{R}^3$ be an open subset whose points are denoted by (x, y, p). Let the functions $f, g : G \to \mathbb{R}$, with g continuous and f a certain function, not necessarily continuous. The functions f and g define the ODEs*

$$f(x, y, y') = 0 \tag{1.1}$$

$$y'' = g(x, y, y') \tag{1.2}$$

(if f is independent of of the variable p, then the relation (1.1) *reduce to $f(x, y) = 0$).*

If any solution of the ODE (1.2) *is also a solution of the equation* (1.1)*, then*

$$f(x, y, p) = 0, \quad \forall (x, y, p) \in G.$$

Proof. Suppose that there exists a point $(x_0, y_0, p_0) \in G$ such that $f(x_0, y_0, p_0) \neq 0$. Since g is continuous, via the Peano Theorem, it follows that there exists $\varepsilon > 0$ and a twice differentiable function $\varphi : (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$, with $\varphi(x_0) = y_0$, $\varphi'(x_0) = p_0$, such that for any $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ one has

$$\Big(x, \varphi(x), \varphi'(x)\Big) \in G, \quad \varphi''(x) = g\Big(x, \varphi(x), \varphi'(x)\Big).$$

That is, the function φ is a solution (not necessarily unique) of the Cauchy problem $\Big\{ (1.2), \quad y(x_0) = y_0, \quad y'(x_0) = p_0 \Big\}$, the continuity hypothesis (of g) ensuring the local existence of the Cauchy problem solution (but not the uniqueness).

According to the hypothesis, φ is also a solution for the equation (1.1), i.e.,

$$f\Big(x, \varphi(x), \varphi'(x)\Big) = 0, \quad \forall x \in (x_0 - \varepsilon, x_0 + \varepsilon),$$

and for $x = x_0$, we obtain $f(x_0, y_0, p_0) = 0$, hence a contradiction. □

Theorem 1.2. (**Identity Theorem**) *Let $G \subseteq \mathbb{R}^3$ be an open set. Let the functions $f, g : G \to \mathbb{R}$, with g continuous and f a certain function, not necessarily continuous. We consider the ODEs*

$$y'' = f(x, y, y') \tag{1.3}$$

$$y'' = g(x, y, y'). \tag{1.4}$$

If any solution of ODE (1.4) *is also a solution of ODE* (1.3)*, then*

$$f(x, y, p) = g(x, y, p), \quad \forall (x, y, p) \in G,$$

hence the ODEs (1.3) *and* (1.4) *coincide (we have in fact a single ODE).*

Proof. Let $\varphi(\cdot)$ be a solution of the ODE (1.4). According to the hypothesis, $\varphi(\cdot)$ is also a solution of ODE (1.3). Taking the difference, we deduce that $\varphi(\cdot)$ is a solution of ODE $f(x, y, y') - g(x, y, y') = 0$. It follows that any solution of the ODE (1.4) is also a solution of this ODE. Hence, we are in the hypotheses of Theorem 1.1, having the function $f - g$ instead of the function f. Applying the Theorem 1.1, it follows that $f(x, y, p) - g(x, y, p) = 0$, $\forall (x, y, p) \in G$. □

Remark 1.1. This theorem provides a very powerful and useful tool to test whether two normal second order ODEs, whose right hand fields coincide on all solutions, are indeed the same ODE.

Theorem 1.3. *Let $D \subseteq \mathbb{R}^2$ be an open set and $g : D \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $n \in \mathbb{N}$. For each $j = \overline{0, n}$, let us consider the functions $c_j : D \to \mathbb{R}$. Assume that the functions $c_j(\cdot, \cdot)$, $j = \overline{0, n}$, and $g(\cdot, \cdot, \cdot)$, define the ODEs*

$$c_n(x, y)(y')^n + \cdots + c_j(x, y)(y')^j + \cdots + c_1(x, y)y' + c_0(x, y) = 0 \tag{1.5}$$

$$y'' = g(x, y, y'). \tag{1.6}$$

If any solution of the ODE (1.6) *is also a solution of the ODE* (1.5)*, then for any index $j = \overline{0, n}$, we have $c_j(x, y) = 0$, $\forall (x, y) \in D$.*

Proof. We are in the hypotheses of Theorem 1.1, with $G = D \times \mathbb{R}$ and $f(x, y, p) = \sum_{j=0}^{n} c_j(x, y)p^j$. Applying the Theorem 1.1, it follows that $\sum_{j=0}^{n} c_j(x, y)p^j = 0$, $\forall (x, y, p) \in D \times \mathbb{R}$. Fixing the point $(x, y) \in D$, the foregoing polynomial function, with respect to p, vanishes for any $p \in \mathbb{R}$, hence the coefficients vanish, i.e., $c_j(x, y) = 0$. □

Theorem 1.4. *Let $D \subseteq \mathbb{R}^2$ be an open set. Let $n \in \mathbb{N}$. For each index $j = \overline{0, n}$, we consider the functions $a_j, b_j : D \to \mathbb{R}$, with b_j continuous. Assume that the functions $a_j(\cdot, \cdot)$, $b_j(\cdot, \cdot)$ define the ODEs*

$$y'' + a_n(x, y)(y')^n + \cdots + a_j(x, y)(y')^j + \cdots + a_1(x, y)y' + a_0(x, y) = 0 \tag{1.7}$$

$$y'' + b_n(x, y)(y')^n + \cdots + b_j(x, y)(y')^j + \cdots + b_1(x, y)y' + b_0(x, y) = 0. \tag{1.8}$$

If any solution of the ODE (1.8) *is also a solution for the ODE* (1.7)*, then for any $j = \overline{0, n}$, we get $a_j(x, y) = b_j(x, y)$, $\forall (x, y) \in D$, hence the ODEs* (1.7) *and* (1.8) *coincide (we have in fact a single ODE).*

Proof. We are in the hypotheses of the Theorem 1.2, with $G = D \times \mathbb{R}$,

$$f(x, y, p) = -\sum_{j=0}^{n} a_j(x, y)p^j, \quad g(x, y, p) = -\sum_{j=0}^{n} b_j(x, y)p^j,$$

the function g being continuous since the functions $b_j(\cdot, \cdot)$ are so.

Applying the Theorem 1.2, it follows

$$\sum_{j=0}^{n} \Big(a_j(x, y) - b_j(x, y)\Big)p^j = 0, \quad \forall (x, y, p) \in D \times \mathbb{R}.$$

Fixing the point $(x, y) \in D$, the foregoing polynomial function, with respect to p, vanishes for any $p \in \mathbb{R}$, hence the coefficients vanish, i.e., $a_j(x, y) - b_j(x, y) = 0$. □

2 Auto-parallel curves as solutions of second order ODEs

Our aim is to find the conditions in which the graphs of the solutions for a given second order ODE are auto-parallel graphs. This is a consequence of our thinking that the important graphs in applications, like Bessel functions, Legendre functions, Hermite functions etc, may be auto-parallel curves in an appropriate geometric structure. But since now, little or nothing about this idea has appeared in the auto-parallel curves literature. This is unusual for such an important topic and it is hoped that this paper can redress the situation. The procedure to solve this problem involves the identification of two ODEs.

Let $D \subseteq \mathbb{R}^2$ be an open subset and ∇ be a linear symmetric connection on D of components Γ^i_{jk}, $i, j, k \in \{1, 2\}$. Since the connections ∇ are introduced as derivation rules on $\mathcal{C}^\infty$ manifolds, and by vector fields we understand $\mathcal{C}^\infty$ vector fields, it follows that the components Γ^i_{jk} are $\mathcal{C}^\infty$ functions. A $\mathcal{C}^2$ curve $\gamma : I \to D$, $\gamma(t) = (x^1(t), x^2(t))$ is auto-parallel with respect to the connection ∇ if and only if

$$\ddot{x}^i + \Gamma^i_{jk}(x^1, x^2)\, \dot{x}^j\, \dot{x}^k = 0,\ \forall i, j, k \in \{1, 2\}.$$

A $\mathcal{C}^2$ curve $\gamma : I \to D$, $\gamma(t) = (t, y(t))$ is an auto-parallel curve with respect to the connection ∇ if and only if the function $y(\cdot) : I \to \mathbb{R}$ is a solution of the differential ODE system

$$\Gamma^1_{22}(t, y)\,(y')^2 + 2\,\Gamma^1_{12}(t, y)\,y' + \Gamma^1_{11}(t, y) = 0 \tag{2.1}$$

$$y'' + \Gamma^2_{22}(t, y)\,(y')^2 + 2\,\Gamma^2_{12}(t, y)\,y' + \Gamma^2_{11}(t, y) = 0. \tag{2.2}$$

Our aim is to prove the following statement: if any solution of the ODE (2.2) is also a solution of the equation (2.1), then $\Gamma^1_{jk} = 0$.

Theorem 2.1. *Let $D \subseteq \mathbb{R}^2$ be an open subset and ∇ be a linear symmetric connection on D. We consider the ODE*

$$y'' + F(x, y, y') = 0, \tag{2.3}$$

where $F : D \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

If, for any solution $y(\cdot) : I \to \mathbb{R}$ of the ODE (2.3), the curve $\gamma : I \to D$, $\gamma(x) = (x, y(x))$ is auto-parallel with respect to the connection ∇, then $\Gamma^1_{jk}(x, y) = 0$, $\forall (x, y) \in D$, $\forall j, k \in \{1, 2\}$ and

$$F(x, y, p) = \Gamma^2_{22}(x, y)\,p^2 + 2\,\Gamma^2_{12}(x, y)\,p + \Gamma^2_{11}(x, y),\ \forall (x, y) \in D,\ \forall p \in \mathbb{R}. \tag{2.4}$$

Proof. Any solution $y(\cdot)$ of ODE (2.3) would be solution for the ODE

$$y'' + \Gamma^2_{22}(x, y)\,(y')^2 + 2\,\Gamma^2_{12}(x, y)\,y' + \Gamma^2_{11}(x, y) = 0. \tag{2.5}$$

Hence we can apply the Theorem 1.2. Here $g = -F$ and $f(x, y, p) = -\Gamma^2_{22}(x, y)\,p^2 - 2\,\Gamma^2_{12}(x, y)\,p - \Gamma^2_{11}(x, y)$. The set G is $G = D \times \mathbb{R}$. Applying the Theorem 1.2, it follows the equality (2.4). Also, any solution $y(\cdot)$ of the ODE (2.3) must be solution for the ODE

$$\Gamma^1_{22}(x,y)\,(y')^2 + 2\,\Gamma^1_{12}(x,y)\,y' + \Gamma^1_{11}(x,y) = 0.$$

We can apply the Theorem 1.3, for $n = 2$, $g = -F$ and $c_2(x,y) = \Gamma^1_{22}(x,y)$, $c_1(x,y) = 2\Gamma^1_{12}(x,y)$, $c_0(x,y) = \Gamma^1_{11}(x,y)$. We find $\Gamma^1_{ij}(x,y) = 0$, $\forall (x,y) \in D$, $\forall j,k \in \{1,2\}$. □

Suppose we have an ODE of type (2.3). We wish to look for the conditions that we must assume in order that for any solution $y(\cdot)$ of the ODE, the curve $\gamma(x) = (x, y(x))$ to be an auto-parallel curve. The Theorem 2.1 says that necessarily the function F must be a polynomial function of degree two in p, having the coefficients as C^∞ functions of variables x, y. In this case, the following Theorem gives the answer to the proposed problem.

Theorem 2.2. *Let $D \subseteq \mathbb{R}^2$ be an open subset and ∇ be a linear symmetric connection on D. Suppose the C^∞ functions $a_j(\cdot,\cdot) : D \to \mathbb{R}$, $j \in \{0,1,2\}$ determine the differential equation*

$$y'' + a_2(x,y)(y')^2 + a_1(x,y)y' + a_0(x,y) = 0. \tag{2.6}$$

The following statements are equivalent:

i) for any solution $y(\cdot) : I \to \mathbb{R}$ of ODE (2.6), the curve $\gamma : I \to D$, $\gamma(x) = (x, y(x))$, $x \in I$, is auto-parallel with respect to ∇.

ii) The connection ∇ has the components

$$\Gamma^1_{11}(x,y) = \Gamma^1_{12}(x,y) = \Gamma^1_{21}(x,y) = \Gamma^1_{22}(x,y) = 0 \tag{2.7}$$

and

$$\Gamma^2_{22}(x,y) = a_2(x,y),\ \Gamma^2_{12}(x,y) = \Gamma^2_{21}(x,y) = \frac{1}{2}a_1(x,y),\ \Gamma^2_{11}(x,y) = a_0(x,y), \tag{2.8}$$

$\forall (x,y) \in D$.

Proof. The implication $i) \Longrightarrow ii)$. The relations (2.7) are obtained from Theorem 2.1.

From the hypothesis one gets that any solution of the ODE (2.6) is also a solution of ODE (2.2). Further, the Theorem 1.4 is applied and one obtains (2.8).

The implication $ii) \Longrightarrow i)$. Since the equalities (2.8) are fulfilled, one obtains that in this case the ODE (2.6) coincides to ODE (2.2). This leads to the fact that any solution of Eq. (2.6) is also a solution of Eq. (2.2).

Since the equalities (2.7) are fulfilled, the equation (2.1) is in fact the equality $0 = 0$.

In conclusion, any solution of the ODE (2.6) is also a solution for both the ODE (2.1) and ODE (2.2); therefore $\gamma(\cdot)$ is an auto-parallel curve. □

Remark 2.1. 1) The connection is uniquely determined by ii).

2) Always still there exist other auto-parallel curves which are not of the form $(t, y(t))$ with $y(\cdot)$ solution of ODE (2.6). Indeed, in the conditions of Theorem 2.2, the curve $\gamma(t) = (x(t) = x^1(t), y(t) = x^2(t))$ is auto-parallel with respect to the foregoing connection ∇ if and only if

$$x''(t) = 0$$

$$y''(t) + a_2(x(t), y(t))(y'(t))^2 + a_1(x(t), y(t))y'(t)x'(t) + a_0(x(t), y(t))(x'(t))^2 = 0.$$

We can select (vertical straight lines) $x(t) = x_0$, $y''(t) + a_2(x(t), y(t))(y'(t))^2 = 0$, the last ODE having always solutions.

Corollary 2.3. *Let I be an open interval in $\mathbb{R}$. Suppose the $\mathcal{C}^\infty$ functions $a_0, a_1 : I \to \mathbb{R}$ determine the linear homogeneous ODE*

$$y'' + a_1(x)y' + a_0(x)y = 0. \tag{2.9}$$

Let ∇ be a linear symmetric connection on $D = I \times \mathbb{R}$. The following statements are equivalent:

i) for any solution $y(\cdot) : J \subseteq I \to \mathbb{R}$ of the ODE (2.9), the curve $\gamma : J \to I \times \mathbb{R}$, $\gamma(x) = (x, y(x)), \forall x \in J$ is auto-parallel with respect to ∇.

ii) The connection ∇ has the components

$$\Gamma^1_{11}(x, y) = \Gamma^1_{12}(x, y) = \Gamma^1_{21}(x, y) = \Gamma^1_{22}(x, y) = 0,$$

$$\Gamma^2_{22}(x, y) = 0,\ \Gamma^2_{12}(x, y) = \Gamma^2_{21}(x, y) = \frac{1}{2}a_1(x),\ \Gamma^2_{11}(x, y) = a_0(x)y, \tag{2.10}$$

$\forall x \in I$, $\forall y \in \mathbb{R}$.

3 Geodesics as solutions of second order ODEs

The aim of this Section is to find the conditions in which the graphs of the solutions for a given second order ODE are geodesics. Since present, little or nothing about this idea has appeared in the geodesics theory. Of course our problem is a special (but important) case of the inverse problem of auto-parallel curves of a linear connection [1], [3], [5]: are auto-parallel graphs the geodesics of some metric?

Proposition 3.1. *Let $D \subseteq \mathbb{R}^2$ be an open, connected and simply connected set, and let $P, Q : D \to \mathbb{R}$ be $\mathcal{C}^1$ ($\mathcal{C}^p$, $p \geq 1$) functions. The following statements are equivalent*

i) For any $(x, y) \in D$, we have

$$\frac{\partial P}{\partial y}(x, y) = \frac{\partial Q}{\partial x}(x, y). \tag{3.1}$$

ii) There exists a differentiable function $u : D \to \mathbb{R}$ (it will be just of class $\mathcal{C}^2$ ($\mathcal{C}^{p+1}$)) such that for any $(x, y) \in D$, we have

$$\frac{\partial u}{\partial x}(x, y) = P(x, y), \quad \frac{\partial u}{\partial y}(x, y) = Q(x, y). \tag{3.2}$$

In these conditions, if $u_0(\cdot, \cdot)$ verifies the PDE system (3.2), on D, then any other solution of this system is of the form $u(x, y) = u_0(x, y) + c$, $\forall (x, y) \in D$, where $c \in \mathbb{R}$ is an arbitrary constant.

Proposition 3.2. *Let $D \subseteq \mathbb{R}^2$ be an open, connected and simply connected set. We consider the functions $u, P, Q : D \to \mathbb{R}$, with P, Q of class $\mathcal{C}^p$, $p \geq 1$, with u differentiable and $u(x,y) \neq 0$, $\forall (x,y) \in D$. The following statements are equivalent*

i) For any $(x,y) \in D$, we have

$$\frac{\partial u}{\partial x}(x,y) = P(x,y)u(x,y), \quad \frac{\partial u}{\partial y}(x,y) = Q(x,y)u(x,y). \tag{3.3}$$

ii) There exists a function $F : D \to \mathbb{R}$, of class $\mathcal{C}^{p+1}$ and a constant $c \in \mathbb{R}$, $c \neq 0$, such that for any $(x,y) \in D$, we have

$$u(x,y) = ce^{F(x,y)}, \quad P(x,y) = \frac{\partial F}{\partial x}(x,y), \quad Q(x,y) = \frac{\partial F}{\partial y}(x,y). \tag{3.4}$$

We remark that in these conditions, u is a $\mathcal{C}^{p+1}$ function.

Proof. *i)* $\Longrightarrow$ *ii)*: u is continuous since it is differentiable. From the relations (3.3) it follows that the first order partial derivatives of u are continuous, hence u is of class $\mathcal{C}^1$. Since P, Q and u are of class $\mathcal{C}^1$, from (3.3), it follows that the first order partial derivatives of u are of class $\mathcal{C}^1$, i.e., u is of class $\mathcal{C}^2$.

From the relations (3.3) and from the Schwarz Theorem, it follows that on the set D we have

$$\frac{\partial}{\partial y}\left(P(x,y)u(x,y)\right) = \frac{\partial}{\partial x}\left(Q(x,y)u(x,y)\right)$$

$$\Longleftrightarrow \frac{\partial P}{\partial y}u(x,y) + P(x,y)\frac{\partial u}{\partial y} = \frac{\partial Q}{\partial x}u(x,y) + Q(x,y)\frac{\partial u}{\partial x}.$$

From the relations (3.3), we replace the partial derivatives of u. We obtain

$$\frac{\partial P}{\partial y}u(x,y) + P(x,y)Q(x,y)u(x,y) = \frac{\partial Q}{\partial x}u(x,y) + Q(x,y)P(x,y)u(x,y),$$

equivalent to $\dfrac{\partial P}{\partial y}u(x,y) = \dfrac{\partial Q}{\partial x}u(x,y)$. Since $u(x,y) \neq 0$, $\forall (x,y) \in D$, it follows that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. From the Proposition 3.1 we deduce the existence of a function $F : D \to \mathbb{R}$, of class $\mathcal{C}^{p+1}$, such that $\dfrac{\partial F}{\partial x} = P$ and $\dfrac{\partial F}{\partial y} = Q$, on D. The relations (3.3) are equivalent to $\dfrac{\partial u}{\partial x} - \dfrac{\partial F}{\partial x}u = 0$, $\dfrac{\partial u}{\partial y} - \dfrac{\partial F}{\partial y}u = 0$, or to $\dfrac{\partial}{\partial x}\left(e^{-F}u\right) = 0$, $\dfrac{\partial}{\partial y}\left(e^{-F}u\right) = 0$. Since the set D is connected, it follows that the function $e^{-F} \cdot u$ is constant on D. There exists $c \in \mathbb{R}$ such that $e^{-F(x,y)}u(x,y) = c$, $\forall (x,y) \in D$, i.e., $u(x,y) = ce^{F(x,y)}$, $\forall (x,y) \in D$. We have $c \neq 0$, since $u(x,y) \neq 0$. We remark that u is of class $\mathcal{C}^{p+1}$ since F is of class $\mathcal{C}^{p+1}$.

The implication *ii)* $\Longrightarrow$ *i)*: the relations (3.3) are easy verified by direct computation, taking into account the equalities (3.4). $\square$

Proposition 3.3. *Let $D \subseteq \mathbb{R}^2$ be an open set and ∇ be a linear symmetric connection on D. Let $g_{ij} : D \to \mathbb{R}$, $i, j \in \{1, 2\}$, with $g_{12} = g_{21}$. Then $\Big(D, g_{ij}\Big)$ is a Riemannian manifold, having ∇ as the Levi-Civita connection if and only if, on the set D we have*

$$\frac{\partial}{\partial x}\begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix} = \begin{pmatrix} 2\Gamma^1_{11} & 2\Gamma^2_{11} & 0 \\ \Gamma^1_{12} & \Gamma^1_{11} + \Gamma^2_{12} & \Gamma^2_{11} \\ 0 & 2\Gamma^1_{12} & 2\Gamma^2_{12} \end{pmatrix}\begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix} \tag{3.5}$$

$$\frac{\partial}{\partial y}\begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix} = \begin{pmatrix} 2\Gamma^1_{12} & 2\Gamma^2_{12} & 0 \\ \Gamma^1_{22} & \Gamma^1_{12} + \Gamma^2_{22} & \Gamma^2_{12} \\ 0 & 2\Gamma^1_{22} & 2\Gamma^2_{22} \end{pmatrix}\begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix} \tag{3.6}$$

$$g_{22} > 0, \quad g_{22}g_{11} - (g_{12})^2 > 0. \tag{3.7}$$

For $\Gamma^1_{ij} = 0$, $g_{22} \neq 0$, let us determine Γ^2_{ij} and g_{ij}, such that (3.5) and (3.6) to be true, on D.

Proposition 3.4. *Let $D \subseteq \mathbb{R}^2$ be an open, connected and simply connected set. Let $\Gamma^2_{11}, \Gamma^2_{12}, \Gamma^2_{22} : D \to \mathbb{R}$, of class $\mathcal{C}^p$, $p \geq 1$; and $g_{11}, g_{12}, g_{22} : D \to \mathbb{R}$, g_{11}, g_{22} of class $\mathcal{C}^1$, g_{12} of class $\mathcal{C}^{p+1}$; with $g_{22}(x, y) \neq 0$, $\forall (x, y) \in D$.*

We consider the PDE system

$$\frac{\partial}{\partial x}\begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix} = \begin{pmatrix} 0 & 2\Gamma^2_{11} & 0 \\ 0 & \Gamma^2_{12} & \Gamma^2_{11} \\ 0 & 0 & 2\Gamma^2_{12} \end{pmatrix}\begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix} \tag{3.8}$$

$$\frac{\partial}{\partial y}\begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix} = \begin{pmatrix} 0 & 2\Gamma^2_{12} & 0 \\ 0 & \Gamma^2_{22} & \Gamma^2_{12} \\ 0 & 0 & 2\Gamma^2_{22} \end{pmatrix}\begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix}. \tag{3.9}$$

Then, the equalities (3.8), (3.9) are true on the set D, if and only if there exists $W : D \to \mathbb{R}$, of class $\mathcal{C}^{p+2}$, and the constant $a, b \in \mathbb{R}$, with $a \neq 0$, such that one has

$$\frac{\partial W}{\partial y} > 0, \quad \Gamma^2_{11} = \frac{\dfrac{\partial^2 W}{\partial x^2}}{\dfrac{\partial W}{\partial y}}, \quad \Gamma^2_{12} = \frac{\dfrac{\partial^2 W}{\partial x \partial y}}{\dfrac{\partial W}{\partial y}}, \quad \Gamma^2_{22} = \frac{\dfrac{\partial^2 W}{\partial y^2}}{\dfrac{\partial W}{\partial y}}, \tag{3.10}$$

$$g_{11} = a\left(\frac{\partial W}{\partial x}\right)^2 + b, \quad g_{12} = a\,\frac{\partial W}{\partial x}\,\frac{\partial W}{\partial y}, \quad g_{22} = a\left(\frac{\partial W}{\partial y}\right)^2. \tag{3.11}$$

In these conditions, the functions g_{11}, g_{12}, g_{22} are of class $\mathcal{C}^{p+1}$.

Proof. Let us consider the last two PDEs from (3.8) and (3.9): $\dfrac{\partial g_{22}}{\partial x} = 2\Gamma^2_{12}g_{22}$, $\dfrac{\partial g_{22}}{\partial y} = 2\Gamma^2_{22}g_{22}$. According to the Proposition 3.2, it follows that the two equalities are true if and only if there exists $F : D \to \mathbb{R}$ of class $\mathcal{C}^{p+1}$ and a constant $a \in \mathbb{R}$, $a \neq 0$, such that

$$g_{22} = ae^F, \quad 2\Gamma^2_{12} = \frac{\partial F}{\partial x}, \quad 2\Gamma^2_{22} = \frac{\partial F}{\partial y}. \tag{3.12}$$

The second PDE from (3.9) becomes $\dfrac{\partial g_{12}}{\partial y} = \dfrac{1}{2}\dfrac{\partial F}{\partial y} g_{12} + \dfrac{1}{2}\dfrac{\partial F}{\partial x} ae^F$. Multiplying by $\dfrac{1}{a}e^{-\frac{F}{2}}$, the PDE is equivalent to $\dfrac{\partial}{\partial y}\left(\dfrac{g_{12}}{a}e^{-\frac{F}{2}}\right) = \dfrac{\partial}{\partial x}\left(e^{\frac{F}{2}}\right)$. We can apply the Proposition 3.1, with $P = \dfrac{g_{12}}{a}e^{-\frac{F}{2}}$ for $Q = e^{\frac{F}{2}}$. Hence there exists $W : D \to \mathbb{R}$, of class $\mathcal{C}^{p+2}$ such that

$$\frac{\partial W}{\partial x} = \frac{g_{12}}{a}e^{-\frac{F}{2}}, \qquad \frac{\partial W}{\partial y} = e^{\frac{F}{2}} > 0. \tag{3.13}$$

From (3.13), we obtain

$$g_{12} = a\frac{\partial W}{\partial x}\frac{\partial W}{\partial y}, \qquad \frac{F}{2} = \ln\frac{\partial W}{\partial y}. \tag{3.14}$$

Using (3.12) and (3.14), we find

$$g_{22} = ae^F = a\left(\frac{\partial W}{\partial y}\right)^2 \tag{3.15}$$

$$\Gamma^2_{12} = \frac{\partial}{\partial x}\left(\frac{F}{2}\right) = \frac{\partial}{\partial x}\left(\ln\frac{\partial W}{\partial y}\right) = \frac{\dfrac{\partial^2 W}{\partial x \partial y}}{\dfrac{\partial W}{\partial y}} \tag{3.16}$$

$$\Gamma^2_{22} = \frac{\partial}{\partial y}\left(\frac{F}{2}\right) = \frac{\partial}{\partial y}\left(\ln\frac{\partial W}{\partial y}\right) = \frac{\dfrac{\partial^2 W}{\partial y^2}}{\dfrac{\partial W}{\partial y}}. \tag{3.17}$$

The second PDE from (3.8) can be written $\dfrac{\partial g_{12}}{\partial x} = \dfrac{1}{2}\dfrac{\partial F}{\partial x} g_{12} + \Gamma^2_{11} ae^F$. We multiply by $\dfrac{1}{a}e^{-\frac{F}{2}}$, and the PDE is equivalent to $\dfrac{\partial}{\partial x}\left(\dfrac{g_{12}}{a}e^{-\frac{F}{2}}\right) = \Gamma^2_{11}e^{\frac{F}{2}}$. The PDEs (3.13) can be written $\dfrac{\partial}{\partial x}\left(\dfrac{\partial W}{\partial x}\right) = \Gamma^2_{11}\dfrac{\partial W}{\partial y}$. Hence

$$\Gamma^2_{11} = \frac{\dfrac{\partial^2 W}{\partial x^2}}{\dfrac{\partial W}{\partial y}}. \tag{3.18}$$

In the first relations from (3.8) and (3.9), we replace Γ^2_{11}, g_{12}, Γ^2_{12} by the values obtained in the formulas (3.18), (3.14), (3.16) and the two mentioned equations become

$$\frac{\partial g_{11}}{\partial x} = 2\Gamma^2_{11} g_{12} = 2a \frac{\partial^2 W}{\partial x^2}\frac{\partial W}{\partial x} = a\frac{\partial}{\partial x}\left(\left(\frac{\partial W}{\partial x}\right)^2\right)$$

$$\frac{\partial g_{11}}{\partial y} = 2\Gamma^2_{12} g_{12} = 2a \frac{\partial^2 W}{\partial x \partial y}\frac{\partial W}{\partial y} = a\frac{\partial}{\partial y}\left(\left(\frac{\partial W}{\partial x}\right)^2\right)$$

hence

$$\frac{\partial}{\partial x}\left(g_{11} - a\left(\frac{\partial W}{\partial x}\right)^2\right) = 0, \quad \frac{\partial}{\partial y}\left(g_{11} - a\left(\frac{\partial W}{\partial x}\right)^2\right) = 0.$$

There exists a constant $b \in \mathbb{R}$, such that

$$g_{11} = a\left(\frac{\partial W}{\partial x}\right)^2 + b. \tag{3.19}$$

Since W is of class $\mathcal{C}^{p+2}$, from the relations (3.14), (3.15) and (3.19) , it follows that g_{11}, g_{12}, g_{22} are of class $\mathcal{C}^{p+1}$, and from (3.10) we deduce that Γ^2_{11}, Γ^2_{12}, Γ^2_{22} are of class $\mathcal{C}^p$.

Conversely, one verifies by direct computation that if (3.10), (3.11) are true, then the equalities (3.8), (3.9) hold. □

Proposition 3.5. *Let $D \subseteq \mathbb{R}^2$ be an open, connected and simply connected set. Let $\Gamma^2_{11}, \Gamma^2_{12}, \Gamma^2_{22} : D \to \mathbb{R}$, be of class $\mathcal{C}^p$, $p \geq 1$. The following statements are equivalent.*

i) On the set D, the relations

$$\frac{\partial \Gamma^2_{12}}{\partial y} = \frac{\partial \Gamma^2_{22}}{\partial x}, \qquad \frac{\partial \Gamma^2_{12}}{\partial x} + \left(\Gamma^2_{12}\right)^2 = \frac{\partial \Gamma^2_{11}}{\partial y} + \Gamma^2_{11}\Gamma^2_{22} \tag{3.20}$$

hold.

ii) There exists $W : D \to \mathbb{R}$, of class $\mathcal{C}^{p+2}$, such that we have the relations (3.10)*, i.e.,*

$$\frac{\partial W}{\partial y} > 0, \quad \Gamma^2_{11} = \frac{\dfrac{\partial^2 W}{\partial x^2}}{\dfrac{\partial W}{\partial y}}, \quad \Gamma^2_{12} = \frac{\dfrac{\partial^2 W}{\partial x \partial y}}{\dfrac{\partial W}{\partial y}}, \quad \Gamma^2_{22} = \frac{\dfrac{\partial^2 W}{\partial y^2}}{\dfrac{\partial W}{\partial y}}.$$

In the equivalent conditions i), ii), if W_0 is a fixed function which verifies (3.10)*, then any other function W, solution of* (3.10)*, is of the form*

$$W(x, y) = cW_0(x, y) + c_1 x + c_2, \quad \forall (x, y) \in D,$$

where c, c_1, c_2 are real constants, and $c > 0$.

Proof. *i)* $\Longrightarrow$ *ii)*: From the first equality of (3.20) and from the Proposition 3.1, it follows that there exists $F_1 : D \to \mathbb{R}$, of class $\mathcal{C}^{p+1}$, such that

$$\Gamma^2_{12} = \frac{\partial F_1}{\partial x}, \qquad \Gamma^2_{22} = \frac{\partial F_1}{\partial y}. \tag{3.21}$$

The second equality of (3.20) becomes $\frac{\partial^2}{\partial x^2}\left(e^{F_1}\right) = \frac{\partial}{\partial y}\left(\Gamma^2_{11} e^{F_1}\right)$. We apply the Proposition 3.1, for $P = \Gamma^2_{11} e^{F_1}$, $Q = \frac{\partial}{\partial x}\left(e^{F_1}\right)$. There exists $F_2 : D \to \mathbb{R}$, of class $\mathcal{C}^{p+1}$, such that

$$\frac{\partial F_2}{\partial x} = \Gamma^2_{11} e^{F_1}, \qquad \frac{\partial F_2}{\partial y} = \frac{\partial}{\partial x}\left(e^{F_1}\right). \tag{3.22}$$

We apply again the Proposition 3.1, for the $\mathcal{C}^{p+1}$ functions: $P = F_2$, $Q = e^{F_1}$. There exists $W : D \to \mathbb{R}$, of class $\mathcal{C}^{p+2}$, such that

$$\frac{\partial W}{\partial x} = F_2, \qquad \frac{\partial W}{\partial y} = e^{F_1} > 0. \tag{3.23}$$

Replacing F_2 and e^{F_1} from (3.23) in the first equality of (3.22), we find $\Gamma^2_{11} = \dfrac{\dfrac{\partial^2 W}{\partial x^2}}{\dfrac{\partial W}{\partial y}}$.

From (3.23), it follows $F_1 = \ln\left(\frac{\partial W}{\partial y}\right)$ and replacing in (3.21), one obtains the last two equalities of (3.10).

ii) $\Longrightarrow$ *i)*: We have $\Gamma^2_{12} = \frac{\partial}{\partial x}\left(\ln\left(\frac{\partial W}{\partial y}\right)\right)$, $\Gamma^2_{22} = \frac{\partial}{\partial y}\left(\ln\left(\frac{\partial W}{\partial y}\right)\right)$. It follows that $\frac{\partial \Gamma^2_{12}}{\partial y} = \frac{\partial \Gamma^2_{22}}{\partial x}$. On the other hand $\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial x}\right) = \Gamma^2_{12}\frac{\partial W}{\partial y}$, $\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial x}\right) = \Gamma^2_{11}\frac{\partial W}{\partial y}$. Since W is at least of class $\mathcal{C}^3$, it follows

$$\frac{\partial}{\partial x}\left(\Gamma^2_{12}\frac{\partial W}{\partial y}\right) = \frac{\partial}{\partial y}\left(\Gamma^2_{11}\frac{\partial W}{\partial y}\right) \Longleftrightarrow$$

$$\Longleftrightarrow \frac{\partial \Gamma^2_{12}}{\partial x}\frac{\partial W}{\partial y} + \Gamma^2_{12}\frac{\partial^2 W}{\partial x \partial y} = \frac{\partial \Gamma^2_{11}}{\partial y}\frac{\partial W}{\partial y} + \Gamma^2_{11}\frac{\partial^2 W}{\partial y^2}.$$

We divide by $\frac{\partial W}{\partial y}$; replacing the quotients of partial derivatives by the values of (3.10), we obtain the second equality of (3.20).

Let W_0, fixed, a function of class $\mathcal{C}^{p+2}$ which verifies the relations (3.10). Let us determine all the functions W, of class $\mathcal{C}^{p+2}$, which verify the conditions (3.10). They satisfy

$$\frac{\partial}{\partial x}\left(\ln\frac{\partial W}{\partial y} - \ln\frac{\partial W_0}{\partial y}\right) = \Gamma^2_{12} - \Gamma^2_{12} = 0$$

$$\frac{\partial}{\partial y}\left(\ln\frac{\partial W}{\partial y} - \ln\frac{\partial W_0}{\partial y}\right) = \Gamma^2_{22} - \Gamma^2_{22} = 0.$$

There exists a real constant k, such that $\ln\frac{\partial W}{\partial y} - \ln\frac{\partial W_0}{\partial y} = k$, equivalent to $\frac{\partial W}{\partial y} = c\frac{\partial W_0}{\partial y}$, where $c = e^k > 0$. Hence $\frac{\partial}{\partial y}(W - cW_0) = 0$.

Using the first equality from (3.10), the relation $\frac{\partial^2}{\partial x^2}(W - cW_0) = \frac{\partial^2 W}{\partial x^2} - c\frac{\partial^2 W_0}{\partial x^2}$ is transformed to $\frac{\partial^2}{\partial x^2}(W - cW_0) = \Gamma^2_{11}\frac{\partial W}{\partial y} - \Gamma^2_{11}c\frac{\partial W_0}{\partial y} = 0$. We showed that

$$\frac{\partial^2}{\partial x^2}(W - cW_0) = 0, \qquad \frac{\partial}{\partial y}(W - cW_0) = 0. \tag{3.24}$$

Consequently, $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(W - cW_0)\right) = 0$, $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(W - cW_0)\right) = 0$. Hence there exists a constant $c_1 \in \mathbb{R}$ such that $\frac{\partial}{\partial x}(W - cW_0) = c_1$, which is equivalent to $\frac{\partial}{\partial x}(W - cW_0 - c_1 x) = 0$. On the other hand, from the second equality of (3.24) we have also $\frac{\partial}{\partial y}(W - cW_0 - c_1 x) = 0$. Hence there exists a constant $c_2 \in \mathbb{R}$ such that $W - cW_0 - c_1 x = c_2$ or $W = cW_0 + c_1 x + c_2$.

One remarks that the converse is immediately verified: if W_0 verifies the conditions (3.10) and $W = cW_0 + c_1 x + c_2$, with c, c_1, c_2 constants, $c > 0$, then W verifies the relations (3.10). □

From the Propositions 3.4 and 3.5, it follows the following Theorem.

Theorem 3.6. *Let $D \subseteq \mathbb{R}^2$ be an open, connected and simply connected set. We consider the given functions $\Gamma^2_{11}, \Gamma^2_{12}, \Gamma^2_{22} : D \to \mathbb{R}$, of class $\mathcal{C}^p$, $p \geq 1$.*
a) The following statements are equivalent.

i) On the set D, the relations (3.20) hold, i.e.,

$$\frac{\partial \Gamma^2_{12}}{\partial y} = \frac{\partial \Gamma^2_{22}}{\partial x}, \qquad \frac{\partial \Gamma^2_{12}}{\partial x} + \left(\Gamma^2_{12}\right)^2 = \frac{\partial \Gamma^2_{11}}{\partial y} + \Gamma^2_{11}\Gamma^2_{22}.$$

ii) There exists $W : D \to \mathbb{R}$, of class $\mathcal{C}^{p+2}$, such that on the set D, to have the relations (3.10), i.e.,

$$\frac{\partial W}{\partial y} > 0, \quad \Gamma^2_{11} = \frac{\dfrac{\partial^2 W}{\partial x^2}}{\dfrac{\partial W}{\partial y}}, \quad \Gamma^2_{12} = \frac{\dfrac{\partial^2 W}{\partial x \partial y}}{\dfrac{\partial W}{\partial y}}, \quad \Gamma^2_{22} = \frac{\dfrac{\partial^2 W}{\partial y^2}}{\dfrac{\partial W}{\partial y}}.$$

iii) The PDEs system (3.8), (3.9) has solutions with $g_{11}, g_{12}, g_{22} : D \to \mathbb{R}$, g_{11}, g_{22} of class $\mathcal{C}^1$, g_{12} of class $\mathcal{C}^{p+1}$, and $g_{22}(x, y) \neq 0$, $\forall (x, y) \in D$.
b) Suppose that the equivalent statements i), ii), iii) hold. Let W be a function as at the point ii) (regardless which one). Then all the solutions of the system $\{(3.8), (3.9)\}$, with the conditions of iii), are of the form

$$g_{11} = a\left(\frac{\partial W}{\partial x} + c\right)^2 + b;\ g_{12} = a\left(\frac{\partial W}{\partial x} + c\right)\frac{\partial W}{\partial y};\ g_{22} = a\left(\frac{\partial W}{\partial y}\right)^2 \tag{3.25}$$

with $a \neq 0$, b, c arbitrary real constants.

In these conditions, the functions g_{11}, g_{12}, g_{22} are of class $\mathcal{C}^{p+1}$.

Proof. *a*) The equivalence *i*) $\Longleftrightarrow$ *ii*) is in fact the result obtained in the Proposition 3.5.

ii) $\Longrightarrow$ *iii*): Let W be a function as in *ii*). We choose g_{11}, g_{12}, g_{22} defined by the formulas (3.11) (eventually we choose also $a = 1$ and $b = 0$). According to the Proposition 3.4, the equalities (3.8) and (3.9) are verified.

iii) $\Longrightarrow$ *ii*): There exist (g_{11}, g_{12}, g_{22}), solution of the PDE system (with the specified conditions). From the Proposition 3.4 it follows that there exists W which satisfies the conditions of *ii*).

b) Let W be a function that satisfies the conditions of *ii*). From the Proposition 3.4, we deduce that (g_{11}, g_{12}, g_{22}) is solution for the system, verifying the conditions from *iii*), if and only if there exists $\widetilde{W} : D \to \mathbb{R}$, of class $\mathcal{C}^{p+2}$, and the constants $a_1 \neq 0$, b_1, such that

$$\frac{\partial \widetilde{W}}{\partial y} > 0, \quad \Gamma_{11}^2 = \frac{\dfrac{\partial^2 \widetilde{W}}{\partial x^2}}{\dfrac{\partial \widetilde{W}}{\partial y}}, \quad \Gamma_{12}^2 = \frac{\dfrac{\partial^2 \widetilde{W}}{\partial x \partial y}}{\dfrac{\partial \widetilde{W}}{\partial y}}, \quad \Gamma_{22}^2 = \frac{\dfrac{\partial^2 \widetilde{W}}{\partial y^2}}{\dfrac{\partial \widetilde{W}}{\partial y}},$$

$$g_{11} = a_1 \left(\frac{\partial \widetilde{W}}{\partial x}\right)^2 + b_1;\ g_{12} = a_1 \frac{\partial \widetilde{W}}{\partial x}\frac{\partial \widetilde{W}}{\partial y};\ g_{22} = a_1 \left(\frac{\partial \widetilde{W}}{\partial y}\right)^2. \tag{3.26}$$

Since $\widetilde{W}$ and W verify the relations (3.10), from the Proposition 3.5 we deduce that there exist the constants $k > 0$, c_1, c_2, such that $\widetilde{W} = kW + c_1 x + c_2$. Replacing in (3.26), we get

$$g_{11} = a_1 k^2 \left(\frac{\partial W}{\partial x} + \frac{c_1}{k}\right)^2 + b_1$$

$$g_{12} = a_1 k^2 \left(\frac{\partial W}{\partial x} + \frac{c_1}{k}\right)\left(\frac{\partial W}{\partial y}\right); \quad g_{22} = a_1 k^2 \left(\frac{\partial W}{\partial y}\right)^2.$$

We choose $a = ak^2$, $b = b_1$, $c = \dfrac{c_1}{k}$ and we obtain (3.25). One can verify by direct computation that the functions defined by the formulas (3.25) verify the PDE system. □

We come back in the context of the Proposition 3.3, with $\Gamma_{ij}^1 = 0$. We obtain the conclusion of the Theorem 3.6, with $p = \infty$, but we must moreover to see when the relations (3.7) are true.

The components g_{ij} are given by the formulas (3.25). Since $\dfrac{\partial W}{\partial y} > 0$, we find $g_{22} > 0$ if and only if $a > 0$. On the other hand $g_{11}g_{22} - (g_{12})^2 = ab\left(\dfrac{\partial W}{\partial y}\right)^2$. We have $a > 0$, hence $g_{11}g_{22} - (g_{12})^2 > 0$ if and only if $b > 0$. Consequently, we have obtained the following result.

Theorem 3.7. *Let $D \subseteq \mathbb{R}^2$ be an open, connected and simply connected set and ∇ be a linear symmetric connection on D, with $\Gamma_{ij}^1(x, y) = 0$, $\forall (x, y) \in D$, $\forall i, j \in \{1, 2\}$.*
a) *The following statements are equivalent.*

i) The set D is a Riemannian manifold having ∇ as Levi - Civita connection.
ii) On the set D, the relations (3.20) *hold, i.e.,*

$$\frac{\partial\Gamma^2_{12}}{\partial y} = \frac{\partial\Gamma^2_{22}}{\partial x}, \qquad \frac{\partial\Gamma^2_{12}}{\partial x} + \left(\Gamma^2_{12}\right)^2 = \frac{\partial\Gamma^2_{11}}{\partial y} + \Gamma^2_{11}\Gamma^2_{22}.$$

iii) There exists $W : D \to \mathbb{R}$, of class $\mathcal{C}^\infty$, such that on the set D, to have the relations (3.10)*, i.e.,*

$$\frac{\partial W}{\partial y} > 0, \quad \Gamma^2_{11} = \frac{\dfrac{\partial^2 W}{\partial x^2}}{\dfrac{\partial W}{\partial y}}, \quad \Gamma^2_{12} = \frac{\dfrac{\partial^2 W}{\partial x \partial y}}{\dfrac{\partial W}{\partial y}}, \quad \Gamma^2_{22} = \frac{\dfrac{\partial^2 W}{\partial y^2}}{\dfrac{\partial W}{\partial y}}.$$

b) Suppose that the equivalent statements i), ii), iii) hold. Let W be a function as those in iii) (regardless which one). Then all the metrics $(g_{ij})_{i,j\in\{1,2\}}$, for which $\Big(D, g_{ij}\Big)$ is a Riemannian manifold having ∇ as Levi - Civita connection, are of the form

$$g_{11} = a\left(\frac{\partial W}{\partial x} + c\right)^2 + b;\; g_{12} = a\left(\frac{\partial W}{\partial x} + c\right)\frac{\partial W}{\partial y};\; g_{22} = a\left(\frac{\partial W}{\partial y}\right)^2$$

with $a > 0$, $b > 0$, $c \in \mathbb{R}$.

In the context of the Theorem 3.7, the statements *i*), *ii*), *iii*) are true. We shall determine the geodesics of the manifold $\Big(D, g_{ij}\Big)$. Let W as in the statement *iii*). The geodesics $(x(t), y(t))$ are solutions of the ODE system

$$x''(t) = 0$$

$$y''(t) + \Gamma^2_{22}(x,y)(y'(t))^2 + 2\Gamma^2_{12}(x,y)x'(t)y'(t) + \Gamma^2_{11}(x,y)(x'(t))^2 = 0.$$

In the second ODE, we replace Γ^2_{ij} by the values given in the formulas (3.10). The ODE is changed into

$$\frac{\partial W}{\partial y}(x(t),y(t))y''(t) + \frac{\partial^2 W}{\partial y^2}(x(t),y(t))(y'(t))^2$$
$$+2\frac{\partial^2 W}{\partial x\partial y}(x(t),y(t))x'(t)y'(t) + \frac{\partial^2 W}{\partial x^2}(x(t),y(t))(x'(t))^2 = 0.$$

From $x''(t) = 0$, we obtain $x(t) = pt + q$, p, q constant. The foregoing ODE becomes

$$\frac{\partial W}{\partial y}(pt+q,y(t))y''(t) + \frac{\partial^2 W}{\partial y^2}(pt+q,y(t))(y'(t))^2$$
$$+2\frac{\partial^2 W}{\partial x\partial y}(pt+q,y(t))py'(t) + \frac{\partial^2 W}{\partial x^2}(pt+q,y(t))p^2 = 0.$$

One rewrites as

$$\frac{d^2}{dt^2}\Big(W(pt+q,y(t))\Big) = \frac{\partial W}{\partial y}(pt+q,y(t))y''(t) + \frac{\partial^2 W}{\partial y^2}(pt+q,y(t))(y'(t))^2$$
$$+ 2\frac{\partial^2 W}{\partial x\partial y}(pt+q,y(t))py'(t) + \frac{\partial^2 W}{\partial x^2}(pt+q,y(t))p^2 = 0.$$

Hence there exist the constant c_1, $c_2 \in \mathbb{R}$ such that $W(pt+q, y(t)) = c_1 t + c_2$.

The geodesics $(x(t), y(t))$ are given by the equations

$$x(t) = pt + q, \qquad W(pt+q, y(t)) = c_1 t + c_2. \tag{3.27}$$

We remark that $y(t)$ is perfectly determined (locally) by the implicit equation $W(pt+q, y(t)) - c_1 t - c_2 = 0$, since $\dfrac{\partial W}{\partial y} > 0$.

Let us consider the case $p \neq 0$. If $p = 1$, $q = 0$, then $x = t$, and hence the geodesic is $(x, y(x))$, where $y(x)$ is solution of the ODE

$$y''(x) + \Gamma^2_{22}(x, y(x))(y'(x))^2 + 2\Gamma^2_{12}(x, y(x))y'(x) + \Gamma^2_{11}(x, y(x)) = 0. \tag{3.28}$$

This ODE has solutions given by the implicit equation $W(x, y(x)) = c_1 x + c_2$ (where c_1 and c_2 are arbitrary constants). We remark immediately that all geodesics (3.27), with $p \neq 0$, are in fact affine reparameterizations of the curves $(x, y(x))$, with $y(\cdot)$ solution of the equation (3.28) (the parameter change is $x = pt + q$).

Let us consider the case $p = 0$. These geodesics are given by $x(t) = q$, $W(q, y(t)) = c_1 t + c_2$, with $c_1 \neq 0$ (contrary the geodesic reduces to a point). Obviously these geodesics reduces to portions of vertical straight lines.

If $c_1 = 1$, $c_2 = 0$, then we have $W(q, y(t)) = t$. Let us denote with $\psi_q(\cdot)$, the function $y(\cdot)$ (locally) obtained in this case. One obtains the geodesic $x = q$, $y = \psi_q(t)$, where $W(q, \psi_q(t)) = t$. One observes immediately that all geodesics, with $x = q$, $\forall t$, are in fact affine reparameterizations of the foregoing curves: $(q, \psi_q(t))$; the change of the parameter being $t = c_1 s + c_2$.

Obviously that the geodesics with $p = 0$ differ from those with $p \neq 0$, since when $p \neq 0$, x is no longer constant, the image of the curve is no longer portion of vertical straight line, but it intersects a vertical straight line in at most one point.

We have obtained the following result

Theorem 3.8. *Suppose that we are in the conditions of the Theorem 3.7 and that the equivalent statements i), ii), iii) are true. Let W as in the statement iii) of the Theorem 3.7.*

Then the geodesics of the manifold D are only of two types:

a) of the form $(x, y(x))$, with $y(\cdot)$ solution of the ODE (3.28)*:*

$$y''(x) + \Gamma^2_{22}(x, y(x))(y'(x))^2 + 2\Gamma^2_{12}(x, y(x))y'(x) + \Gamma^2_{11}(x, y(x)) = 0,$$

i.e., $y(\cdot)$ is given implicitly by $W(x, y) = c_1 x + c_2$, c_1, c_2 constants (fixed). Furthermore, we still have the affine reparameterizations of the curves $(x, y(x))$, with $y(\cdot)$ solution for (3.28)*.*

b) of the form $(q, \psi_q(t))$, where $\psi_q(\cdot)$ is given implicitly by $W(q, \psi_q(t)) = t$; as well as the affine reparameterizations of these.

Remark 3.1. Let $D \subseteq \mathbb{R}^2$ be an open, connected and simple connected set. Let $\Gamma^2_{11}, \Gamma^2_{12}, \Gamma^2_{22} : D \to \mathbb{R}$, of class $\mathcal{C}^p$, $p \geq 1$ such that on the set D are satisfied the conditions (3.20).

From the proof of the Propositions 3.5, implication $i) \implies ii)$, we obtain the following algorithm of finding a function W, that satisfies the relations (3.10):

– one determines F_1, such that $\dfrac{\partial F_1}{\partial x} = \Gamma^2_{12}$, $\dfrac{\partial F_1}{\partial y} = \Gamma^2_{22}$, on D;
– one determines F_2, such that $\dfrac{\partial F_2}{\partial x} = \Gamma^2_{11} e^{F_1}$, $\dfrac{\partial F_2}{\partial y} = \dfrac{\partial}{\partial x}\left(e^{F_1}\right)$, on D;
– one determines W, such that $\dfrac{\partial W}{\partial x} = F_2$, $\dfrac{\partial W}{\partial y} = e^{F_1}$, on D.

As a rule we need only one function W. But, if we need all, they are described in the Proposition 3.5.

Let us determine the geodesics in the case in which the ODE (3.28) is linear. We have seen (Theorem 2.2) that in this case $D = I \times \mathbb{R}$, with I open interval, and

$$\Gamma^2_{11}(x,y) = a_0(x)y, \quad \Gamma^2_{12}(x,y) = \frac{a_1(x)}{2}, \quad \Gamma^2_{22}(x,y) = 0, \quad \forall (x,y) \in I \times \mathbb{R}.$$

The first relation (3.20) is verified for any $a_0(\cdot)$, $a_1(\cdot)$, and the second becomes $\dfrac{a_1'(x)}{2} + \left(\dfrac{a_1(x)}{2}\right)^2 = a_0(x)$. The equation (3.28) can be written now in the form

$$y''(x) + a_1(x)y'(x) + \left(\frac{a_1'(x)}{2} + \left(\frac{a_1(x)}{2}\right)^2\right) y(x) = 0$$

equivalent to

$$\frac{d^2}{dx^2}\left(y(x)e^{\frac{A_1(x)}{2}}\right) = 0,$$

where $A_1(\cdot)$ is a fixed primitive on I of the function $a_1(\cdot)$.

We apply the algorithm described in the Remark 3.1 for determining a function W. One gets

$$F_1(x,y) = \frac{A_1(x)}{2}, \quad F_2(x,y) = y \cdot \frac{\partial}{\partial x}\left(e^{\frac{A_1(x)}{2}}\right), \quad W(x,y) = ye^{\frac{A_1(x)}{2}}.$$

We apply the Theorem 3.8. One obtains the geodesics of the form $(x, y(x))$, with $y(x) = (c_1 x + c_2)e^{-\frac{A_1(x)}{2}}$, together their affine reparameterizations. Furthermore, one finds also the geodesics (with $p = 0$) of the form $x = q$, $y(t)e^{\frac{A_1(q)}{2}} = c_1 t + c_2$ or, if we re-denote the constants, $x = q$, $y(t) = \widetilde{c}_1 t + \widetilde{c}_2$; i.e., the vertical straight lines $x = q, y = t$ and their affine reparameterizations.

One obtains the following Theorem which completes the Corollary 2.3:

Theorem 3.9. *Let $I \subseteq \mathbb{R}$ be an open interval and $a_0, a_1 : I \to \mathbb{R}$ be functions of class $\mathcal{C}^\infty$. On $I \times \mathbb{R}$, we consider the linear and symmetric connection ∇, with*

$$\Gamma^1_{11}(x,y) = \Gamma^1_{12}(x,y) = \Gamma^1_{22}(x,y) = 0, \quad \forall (x,y) \in I \times \mathbb{R},$$

$$\Gamma^2_{11}(x,y) = a_0(x)y,\ \Gamma^2_{12}(x,y) = \frac{a_1(x)}{2},\ \Gamma^2_{22}(x,y) = 0,\ \forall (x,y) \in I \times \mathbb{R}.$$

Let $A_1(\cdot)$ be a primitive of the function $a_1(\cdot)$.

a) $I \times \mathbb{R}$ is a Riemannian manifold having ∇ as Levi - Civita connection if and only if

$$a_0(x) = \frac{a_1'(x)}{2} + \left(\frac{a_1(x)}{2}\right)^2, \quad \forall x \in I. \tag{3.29}$$

b) If the relation (3.29) *is satisfied, then all the metrics (g_{ij}) are of the form*

$$g_{11}(x,y) = a\left(\frac{ya_1(x)}{2}e^{\frac{A_1(x)}{2}} + c\right)^2 + b,$$

$$g_{12}(x,y) = a\left(\frac{ya_1(x)}{2}e^{\frac{A_1(x)}{2}} + c\right)e^{\frac{A_1(x)}{2}}, \quad g_{22}(x,y) = ae^{A_1(x)},$$

with $a > 0$, $b > 0$, $c \in \mathbb{R}$.

c) If the relation (3.29) *is fulfilled, then the geodesics of the Riemannian manifold $\left(I \times \mathbb{R},\, g_{ij}\right)$ are*

– of graph type: $(x, y(x))$, $y(x) = (c_1x + c_2)e^{-\frac{A_1(x)}{2}}$, $x \in I$ (c_1, c_2 arbitrary real constants); as well as their affine reparameterizations;

– vertical straight lines: $x = q$, $y = t$, $t \in \mathbb{R}$ (with $q \in I$ constant); as well as their affine reparameterizations.

d) If $I = \mathbb{R}$ and the relation (3.29) *is fulfilled, the manifold $\left(\mathbb{R}^2,\, g_{ij}\right)$ is complete.*

Remark 3.2. If the relation (3.29) is satisfied, then the ODE (2.9) is equivalent to the ODE

$$\frac{d^2}{dx^2}\left(y(x)e^{\frac{A_1(x)}{2}}\right) = 0,$$

where $A_1(\cdot)$ is an arbitrary primitive on I of the function $a_1(\cdot)$.

4 Interpretation via isometric manifolds

Suppose we are in the context of the Theorem 3.7, with true statements *i*), *ii*), *iii*). We try to embed the manifold $\left(D, g_{ij}\right)$ in the Euclidean space $(\mathbb{R}^3, \delta_{ij})$. In fact we want to look at $\left(D, g_{ij}\right)$ as an usual surface.

According to the Theorem 3.7, we have

$$g_{11} = a\left(\frac{\partial W}{\partial x} + c\right)^2 + b;\ g_{12} = a\left(\frac{\partial W}{\partial x} + c\right)\frac{\partial W}{\partial y};\ g_{22} = a\left(\frac{\partial W}{\partial y}\right)^2$$

with $a > 0$, $b > 0$, $c \in \mathbb{R}$. Let $\widetilde{h} = (\widetilde{h}_1, \widetilde{h}_2, \widetilde{h}_3) : D \to \mathbb{R}^3$, $\widetilde{h}_1(x,y) = \sqrt{b}\, x$, $\widetilde{h}_2(x,y) = \sqrt{a}\Big(W(x,y) + cx\Big)$, $\widetilde{h}_3(x,y) = 0$. We find

$$\frac{\partial \widetilde{h}}{\partial x} = \left(\sqrt{b}, \sqrt{a}\left(\frac{\partial W}{\partial x} + c\right), 0\right), \quad \frac{\partial \widetilde{h}}{\partial y} = \left(0, \sqrt{a}\,\frac{\partial W}{\partial y}, 0\right).$$

Denote $S = \widetilde{h}(D)$. One remarks that the map $\widetilde{h}$ is injective and has the image S, hence it is a parameterization of the surface S.

Denote by $\langle \cdot, \cdot \rangle$ the usual scalar product δ_{ij} on $\mathbb{R}^3$. We have

$$\left\langle \frac{\partial \widetilde{h}}{\partial x}, \frac{\partial \widetilde{h}}{\partial x} \right\rangle = a\left(\frac{\partial W}{\partial x} + c\right)^2 + b = g_{11},$$

$$\left\langle \frac{\partial \widetilde{h}}{\partial x}, \frac{\partial \widetilde{h}}{\partial y} \right\rangle = a\left(\frac{\partial W}{\partial x} + c\right)\frac{\partial W}{\partial y} = g_{12},$$

$$\left\langle \frac{\partial \widetilde{h}}{\partial y}, \frac{\partial \widetilde{h}}{\partial y} \right\rangle = a\left(\frac{\partial W}{\partial y}\right)^2 = g_{22}.$$

Hence the scalar products on D and S coincide. The two manifolds (D, δ_{ij}) and (S, g_{ij}) are diffeomorphic and isometric. In fact, S is a planar surface. Summarizing we obtain

Proposition 4.1. *Let $D \subseteq \mathbb{R}^2$ be an open, connected and simply connected set, which has the following property: for any two points $A = (x_A, y_A)$, $B = (x_B, y_B)$, $x_A = x_B$, in D, it follows that the segment $[A, B]$ is included in D.*

Suppose we are in the context of the Theorem 3.7, with true statements i), ii), iii). Let

$$h = (h_1, h_2) : D \to \mathbb{R}^2, \quad h_1(x,y) = \sqrt{b}\; x, \quad h_2(x,y) = \sqrt{a}\Big(W(x,y) + cx\Big)$$

and $D_0 := h(D)$. Then

i) D_0 is open, connected and simply connected.

ii) h realizes a diffeomorphism of class $\mathcal{C}^\infty$ between D and D_0.

iii) We endow D_0 with the metric $(\delta_{ij})_{i,j\in\{1,2\}}$ and denote by $\langle \cdot, \cdot \rangle$ the usual scalar product on $\mathbb{R}^2$, hence also on D_0. The manifolds $\Big(D, g_{ij}\Big)$ and $\Big(D_0, \delta_{ij}\Big)$ are isometric.

Hence in fact the manifold $\Big(D, g_{ij}\Big)$ is similar to a part of the Euclidean plane, with all consequences that follows from here. For example, the geodesics identify to parts of straight lines.

Proof. *i)* Since h is continuous and D is connected and simple connected, it follows that $h(D) = D_0$ is also connected and simple connected.

Let us show that h is injective. Let $(x_1, y_1), (x_2, y_2) \in D$, such that $h_1(x_1, y_1) = h_1(x_2, y_2)$, $h_2(x_1, y_1) = h_2(x_2, y_2)$. From $\sqrt{b}\, x_1 = \sqrt{b}\, x_2$, we have $x_1 = x_2$. From $h_2(x_1, y_1) = h_2(x_1, y_2)$, we deduce that $W(x_1, y_1) = W(x_1, y_2)$. Let us assume that

$y_1 \neq y_2$, for example $y_1 < y_2$. Since $(x_1, y_1), (x_1, y_2) \in D$, it follows that $(x_1, y) \in D$, $\forall y \in [y_1, y_2]$; consequently we can define the function

$$\varphi : [y_1, y_2] \to \mathbb{R}, \quad \varphi(y) = W(x_1, y), \quad \forall y \in [y_1, y_2].$$

But $\varphi'(y) = \dfrac{\partial W}{\partial y}(x_1, y) > 0$, hence the function φ is strictly increasing. It follows $\varphi(y_1) < \varphi(y_2)$, i.e., $W(x_1, y_1) < W(x_1, y_2)$. We have obtained a contradiction. Hence $y_1 = y_2$. In this way, the map h is injective.

Since h is continuous, injective and D is open, it follows that $h(D) = D_0$ is open.

ii) At *i*) we showed that h is injective. Hence $h : D \to h(D) = D_0$ is bijective. Since we have

$$\frac{\partial h}{\partial x} = \left(\sqrt{b}, \sqrt{a}\left(\frac{\partial W}{\partial x} + c\right)\right), \quad \frac{\partial h}{\partial y} = \left(0, \sqrt{a}\,\frac{\partial W}{\partial y}\right),$$

the Jacobian of h is $\sqrt{ab} \cdot \dfrac{\partial W}{\partial y} > 0$, hence nonzero. From the inverse function Theorem, it follows that $h^{-1} : D_0 \to D$ is of class $\mathcal{C}^\infty$ (since h is of class $\mathcal{C}^\infty$).

iii) From the point *ii*) we deduce that h realizes a reparameterization (changing of coordinates) of the manifold D_0 (changing with respect to the Cartesian coordinates). Since

$$\left\langle \frac{\partial h}{\partial x}, \frac{\partial h}{\partial x} \right\rangle = a\left(\frac{\partial W}{\partial x} + c\right)^2 + b = g_{11},$$

$$\left\langle \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right\rangle = a\left(\frac{\partial W}{\partial x} + c\right)\frac{\partial W}{\partial y} = g_{12},$$

$$\left\langle \frac{\partial h}{\partial y}, \frac{\partial h}{\partial y} \right\rangle = a\left(\frac{\partial W}{\partial y}\right)^2 = g_{22},$$

i.e., the scalar product is conserved, we deduce that h induces an isometry between the tangent spaces of the manifolds D and D_0. $\square$

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Authors' addresses:

Constantin Udrişte and Ionel Ţevy
University Politehnica of Bucharest, Faculty of Applied Sciences,
Department of Mathematics-Informatics,
Splaiul Independenţei 313, 060042 Bucharest, Romania.
E-mail: udriste@mathem.pub.ro, anet.udri@yahoo.com, vascatevy@yahoo.fr

Cristian Ghiu
University Politehnica of Bucharest, Faculty of Applied Sciences,
Department of Mathematical Methods and Models,
Splaiul Independenţei 313, 060042 Bucharest, Romania.
E-mail: crisghiu@yahoo.com