Higher order hyperbolic equations involving a finite set of derivations

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Abstract. The aim of this paper is to solve a higher order hyperbolic equation associated with a finite set of noncommutative derivations in $\mathbb{R}^n$, using first order systems of PDEs generated by a set of commutative derivations on the enlarged space $\mathbb{R}^{n+m}$.

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1 Introduction

Our analysis is focussed on higher order hyperbolic equations generated by a finite set of derivations for which the corresponding first order system of PDEs can be solved using the characteristic system method. The problem (see Problem C) and its solution (see Theorem 4.1.) are presented in Sections 3 and 4, where integral equations and first order systems of PDEs are the main part of this construction. In Lemma 3.1. and Remark 3.2. of Section 3 the algorithm of finding a solution is explained.

The main result (see Theorem 4.1. of Section 4) is related to the paper [6], where only one derivation was considered. It has much in common with the papers [1] and [5] regarding gradient systems associated with an orbit solution. Higher order hyperbolic equations are also treated in [4], [5] and [8] by Y. Shoukaku, Y. H. Zhong and Y.H. Yuan, using continuous deviating arguments and forced oscillations.

The final part of this work, including Comment 4.2. and Theorem 4.3. of Section 4, introduces solutions with bounded jumps satisfying a higher order hyperbolic equation with jumps generated by a single smooth vector field.

Starting with a finite set of noncommutative derivations $\{\vec{X}_1, ..., \vec{X}_m\} \subseteq Der(\mathbb{R}^n)$ we may and do associate the corresponding composition of the global flows $G(p)[\lambda] = G_1(t_1) \circ \cdots \circ G_m(t_m)[\lambda] \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, $p = (t_1, ..., t_m) \in \mathbb{R}^m$, where $\{G_i(t_i)[\lambda] : t_i \in \mathbb{R},\ \lambda \in \mathbb{R}^n\}$ is the flow generated by the complete smooth vector field $X_i \in \left(\mathcal{C}_b^1 \cap \mathcal{C}^m\right)(\mathbb{R}^n; \mathbb{R}^n)$, $i \in \{1, ..., m\}$ (see the index b of $\mathcal{C}_b^1$ as *bounded*). Though the smooth mapping $\{G(p)[\lambda] \in \mathbb{R}^n : p \in \mathbb{R}^m\}$ does not share a group property with

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respect to $p \in \mathbb{R}^m$, there is an inverse mapping $\lambda \doteq \psi(p,x) = [G(p)]^{-1}[x],\ p \in \mathbb{R}^m,\ x \in \mathbb{R}^n$, of $G(p)[\lambda]$ which satisfies a first order system of PDEs

$$\partial_{t_i}\psi(p,x) + [\partial_x\psi(p,x)]Y_i(p_i,x) = 0,\ i \in \{1,...,m\},\ \psi(0,x) = x \in \mathbb{R}^n.$$

Here the smooth vector fields with parameters $Y_i(p_i,x) \in \mathbb{R}^n,\ p_i = (t_1,...,t_{i-1})$, $Y_1(x) = X_1(x),\ i \in \{1,...,m\}$, generate a gradient system (see $\partial_{t_i}Y_j(p_j,x) = [Y_i(p_i),Y_j(p_j)](x),\ 1 \le i \le j-1,\ 1 \le j \le m$) and $\lambda = \psi(p,x)$ stands for a fundamental system of solutions associated with the following first order system of PDEs

$$\partial_{t_i}\varphi(p,x) + \partial_x\varphi(p,x)Y_i(p_i,x) = 0,\ i \in \{1,...,m\},\ x \in \mathbb{R}^n,\ p \in \mathbb{R}^m$$

$$\varphi(0,x) = \varphi_0(x),\ \varphi_0 \in \mathcal{C}^2(\mathbb{R}^n).$$

The last system of PDEs allow us to introduce an extended system of commutative derivations $\{\vec{Z}_1,...,\vec{Z}_m\} \subseteq Der\left(\mathbb{R}^{n+m};\mathbb{R}^{n+m}\right),\ Z_i(z) = col(e_i, Y_i(p_i,x)),\ z = (p,x) \in \mathbb{R}^{n+m}$, such that it can be rewritten as $\vec{Z}_i(\varphi)(z) = 0,\ i \in \{1,...,m\},\ \varphi(0,x) = \varphi_0(x),\ x \in \mathbb{R}^n$. The analysis of higher order hyperbolic equations (see Theorems 4.1. and 4.3. of Section 4) involves the enlarged system of commutative derivations $\{\vec{Z}_1,...,\vec{Z}_m\} \subseteq Der\left(\mathbb{R}^{n+m};\mathbb{R}^{n+m}\right)$ introduced as above. In addition, there is a nice connection between the characteristic system method (usually used for Hamilton-Jacobi equations) and solutions for higher order hyperbolic equations analyzed here.

We mention also that the characteristic system method has been proved useful for treating nonlinear Hamilton-Jacobi equations with nonregular perturbations as it appears in [2] and [3].

2 Background and notation

Let $\mathcal{C}^m(\mathbb{R}^n,\mathbb{R}^n)$ be the space consisting of all nonlinear and smooth vector fields $X(x) : \mathbb{R}^n \to \mathbb{R}^n$, which are continuously differentiable to order $m \ge 2$. A smooth derivation $\vec{X} \in Der(\mathbb{R}^n)$ is defined as a linear mapping

$$\vec{X} : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^m(\mathbb{R}^n),\ \ \vec{X}(\varphi)(x) = \langle\nabla\varphi(x), X(x)\rangle \tag{2.1}$$

with $x \in \mathbb{R}^n,\ \ \nabla = (\partial_1,\ldots,\partial_n)$ - gradient, where the vector field $X \in \mathcal{C}^m(\mathbb{R}^n,\mathbb{R}^n)$ is fixed and $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$. Let us denote a *complete* smooth vector field $X \in \mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^n)$ by $X \in \mathcal{C}_b^1(\mathbb{R}^n,\mathbb{R}^n)$.

Consider a finite set of smooth and complete vector fields satisfying

$$\{X_1,\ldots,X_m\} \subseteq \left(\mathcal{C}_b^1 \cap \mathcal{C}^m\right)(\mathbb{R}^n,\mathbb{R}^n). \tag{2.2}$$

Each vector field X_i generates a global flow $\{G_i(t,\lambda) : t \in \mathbb{R},\ \ \lambda \in \mathbb{R}^n\}$, satisfying the following ODEs

$$\frac{d}{dt}[G_i(t,\lambda)] = X_i\left(G_i(t,\lambda)\right),\ \ G_i(0,\lambda) = \lambda,\ \ t \in \mathbb{R},\ \ \lambda \in \mathbb{R}^n. \tag{2.3}$$

The corresponding m-sheet

$$G(p,\lambda) = G_1(t^1) \circ \ldots \circ G_m(t^m)[\lambda],\ \ p = (t^1,\ldots,t^m) \in \mathbb{R}^m,\ \ \lambda \in \mathbb{R}^n \tag{2.4}$$

is generated as a composition of the global flows $\{G_i(t^i,\lambda) : t^i \in \mathbb{R},\ \lambda \in \mathbb{R}^n,\ i \in \{1,\dots,m\}\}$.

Generally, the mapping $\{G(p,\lambda) \in \mathbb{R}^n : p \in \mathbb{R}^m\}$ does not share a group property with respect to $p \in \mathbb{R}^m$, but the inverse mapping $\lambda = \psi(p,x)$ can be computed as a m-sheet verifying

$$G(p,\psi(p,x)) = x,\ \ \psi(p,x) = G_m(-t^m) \circ \ldots \circ G_1(-t^1)[x], \tag{2.5}$$

$$\psi(p, G(p,\lambda)) = \lambda,$$

for any $p \in \mathbb{R}^m$ and each $\lambda, x \in \mathbb{R}^n$.

Remark 2.1. There is a strong connection between a solution of first order systems of PDEs and the m-sheet $y = G(p,\lambda)$ in (2.4), as the solution of the corresponding gradient system. We shall mention two such problems:

Problem A. (overdetermined systems of first order)

Find sufficient conditions on $\{X_1,\dots,X_m\} \subseteq \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^n)$ *(including Lie algebra* $L(X_1,\dots,X_m) \subseteq \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^n)$ *generated by them) such that a non-trivial first integral* $\varphi \in \mathcal{C}^\infty(B(x_0,\rho) \subseteq \mathbb{R}^n, \mathbb{R})$ *exists satisfying*

$$\varphi(G(p,x_0)) = \varphi(x_0),$$

for any $p \in P = \prod_{i=1}^{m}[-a^i, a^i]$*. This problem is solved in* [7] *assuming* $dim L(X_1,\dots,X_m)(x_0) = k < n$ *(see overdetermined first order systems in* [7]*).*

Problem B. (Gradient systems of linear first order PDEs)

For the given $\{X_1,\dots,X_m\} \subseteq (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n,\mathbb{R}^n)$*:*

(a) Find (if possible) a gradient system of vector fields

$$Y_1(x) = X_1(x),\ \ Y_2(p_2,x),\ \ \dots,\ \ Y_m(p_m,x) \tag{2.6}$$

with $p_i = (t^1,\dots,t^{i-1})$, $1 \le i \le m$, $p_0 = 0$, *(satisfying* $\partial_{t^i} Y_j(p_j,x) = [Y_i(p_i), Y_j(p_j)](x)$, $1 \le i \le j-1$, $1 \le j \le m$*) such that* $\{y = G(p,\lambda) : p = (t^1,\dots,t^m) \in \mathbb{R}^m\}$ *given in (2.4) is the unique solution of the corresponding gradient system* $\partial_{t^i} y = Y_i(p_i,y)$, $i = 1,\dots,m$, *with initial condition* $y(0) = \lambda$.

(b) Consider the inverse mapping $\lambda = \psi(p,x)$*, verifying (2.5). Prove that the following system of first order PDEs is fulfilled*

$$\partial_{t^i}\psi(p,x) + \partial_x\psi(p,x) Y_i(p_i,x) = 0,\ \ i = 1,\dots,m,\ \ \psi(0,x) = x \in \mathbb{R}^n, \tag{2.7}$$

where $Y_i(p_i,x)$, $i = 1,\dots,m$, *are the vector fields in (2.6). This problem is solved in [7] (see Frobenius theorem).*

Remark 2.2. The smooth mapping $\psi(p,x) = (\psi_1(p,x),\dots,\psi_n(p,x))$ in Problem B, satisfying (2.7), stands for a fundamental system of solutions associated with the following first order system of PDEs

$$\partial_{t^i}\varphi(p,x) + \partial_x\varphi(p,x) Y_i(p_i,x) = 0,\ \ i = 1,\dots,m,\ \ x \in \mathbb{R}^n,\ \ p \in \mathbb{R}^m \tag{2.8}$$

$$\varphi(0,x) = \varphi_0(x), \ \ \varphi_0 \in \mathcal{C}^1(\mathbb{R}^n)$$

In addition, the solution of the Cauchy problem (2.8) will be represented by

$$\varphi(p,x) = \varphi_0(\psi(p,x)), \ \ p \in \mathbb{R}^m, \ \ x \in \mathbb{R}^n. \tag{2.9}$$

Denote $z = (p,x) \in \mathbb{R}^{n+m}$ and consider the extended vector fields

$$\{Z_1,\dots,Z_m\} \subseteq \mathcal{C}^1(\mathbb{R}^{n+m},\mathbb{R}^{n+m})$$

defined by

$$Z_1(z) = col(e_1, X_1(x)), \ \ Z_2(z) = col(e_2, Y_2(p_2,x)),\dots, \tag{2.10}$$
$$Z_m(z) = col(e_m, Y_m(p_m,x)).$$

Here $\{e_1,\dots,e_m\} \subseteq \mathbb{R}^m$ is the canonical basis and relying on

$$\partial_{t^i} Y_j(p_j,y) = [Y_i(p_i), Y_j(p_j)](y), \ \ 1 \le i \le j-1, \ \ j \in \{2,\dots,m\}$$

we get that $[Z_i,Z_j](z) = 0$, $z \in \mathbb{R}^{n+m}$, $i,j \in \{1,\dots,m\}$, i.e. $\{Z_1,\dots,Z_m\}$ mutually commute using Lie bracket.

Remark 2.3. Consider the corresponding derivations

$$\{\vec{Z_1},\dots,\vec{Z_m}\} \subseteq Der(\mathbb{R}^{n+m})$$

and write the first order system of PDE in (2.8) as follows

$$\vec{Z_i}(\varphi)(p,x) = 0, \ \ p \in \mathbb{R}^m, \ \ x \in \mathbb{R}^n, \ \ 1 \le i \le m \tag{2.11}$$
$$\varphi(0,x) = \varphi_0(x)$$

for which $\varphi(p,x) = \varphi_0(\psi(p,x))$, $p \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ is its solution.

Let $\{F_i(t)[z] : t \in \mathbb{R}, \ z \in \mathbb{R}^{n+m}\}$ be the global flow generated by Z_i, for $1 \le i \le m$. It shows that starting with some smooth vector fields

$$\{X_1,\dots,X_m\} \subseteq (\mathcal{C}_b^1 \cap \mathcal{C}^\infty)(\mathbb{R}^n,\mathbb{R}^n)$$

(see (2.2)) and its corresponding m-sheet $\{y = G(p,\lambda) \in \mathbb{R}^n : p \in \mathbb{R}^m, \ \lambda \in \mathbb{R}^n\}$ (see (2.4)) we can associate a sheet in $\mathbb{R}^{n+m}$,

$$F(p,\lambda) = col(p, G(p,\lambda)) = F_1(t^1) \circ \dots \circ F_m(t^m)[z_0(\lambda)] \tag{2.12}$$
$$z_0(\lambda) = (0,\lambda) \in \mathbb{R}^{n+m},$$

such that each solution in (2.11) is a first integral on the sheet

$$\{F(p,\lambda) \in \mathbb{R}^{n+m}\}$$

in (2.12) (see $\varphi(F(p,\lambda)) = \varphi(z_0(\lambda))$). It can be done without imposing any special properties on the Lie algebra $L(X_1,\dots,X_m) \subseteq \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^n)$ (see [7], Frobenius theorem).

In addition, there is an analytic connection between each component $\{Y_i(p_i, x) \in \mathbb{R}^n,\ x \in \mathbb{R}^n\}$ of $Z_i(z) = col(e_i, Y_i(p_i, x)),\ 1 \le i \le m$, and the original vector fields $\{X_1, \ldots, X_m\} \subseteq (\mathcal{C}_b^1 \cap \mathcal{C}^\infty)(\mathbb{R}^n, \mathbb{R}^n)$ expressed by

$$Y_i(p_i, x)) = \sum_{j=1}^{m} \alpha_j^i(p_i) X_j(x),\ \ x \in \mathbb{R}^n,\ \ 1 \le i \le m, \tag{2.13}$$

using some analytic functions $\{\alpha_1^i, \ldots, \alpha_m^i\} \subseteq \mathcal{C}^\omega(\mathbb{R}^m, \mathbb{R})$ and $Y_1(x) = X_1(x)$. Here we assume that $\{X_1, \ldots, X_m\} \subseteq \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ are in involution over $\mathbb{R}$, i.e. any Lie bracket can be expressed as a linear combination

$$[X_i, X_j] = \sum_{k=1}^{m} \alpha_k^{i,j} X_k,$$

using some constants $\alpha_k^{i,j} \in \mathbb{R}$.

The algebraic representation (2.13), under the hypothesis that $\{X_1, \ldots, X_m\}$ are in involution over R, can be deduced from [6] (see gradient systems in a finite dimensional Lie algebra).

Problem C. *With the same notations as above, we consider the following higher order hyperbolic equation*

$$A_m(\varphi)(z) = f(z) + \sum_{i=0}^{m-1} a_i(z) A_i(\varphi)(z),\ \ z = (p, x) \in \mathbb{R}^{n+m}, \tag{2.14}$$

where $\{f,\ a_i : i \in \{0, 1, 2, \ldots, m-1\} \subseteq (\mathcal{C}_b \cap \mathcal{C}_b^1)(\mathbb{R}^{n+m}, \mathbb{R})$. *Each linear operator* $A_i : \mathcal{C}^m(\mathbb{R}^{n+m}, \mathbb{R}) \to \mathcal{C}(\mathbb{R}^{n+m}, \mathbb{R})$ *is given by*

$$A_i(\varphi)(z) = (\vec{Z}_i \circ \ldots \circ \vec{Z}_1 \circ \vec{Z}_0)(\varphi)(z),\ \ 0 \le i \le m,\ \ (\vec{Z}_0(\varphi) = \varphi), \tag{2.15}$$

where $\{\vec{Z}_1, \ldots, \vec{Z}_m\}$ *are the derivations generated by* $\{Z_1, \ldots, Z_m\}$ *(see (2.10)).*

Find (if possible) a solution $\varphi \in \mathcal{C}^m(P \times \mathbb{R}^n, \mathbb{R})$ *of (2.14), where*

$$P = \prod_{i=1}^{m} [-a^i, a^i] \subseteq \mathbb{R}^m. \tag{2.16}$$

3 An algorithm for solving Problem C

A solution $\varphi \in \mathcal{C}^m(P \times \mathbb{R}^n, \mathbb{R})$ of (2.14) will be found provided the corresponding first order system of PDEs

$$z_0 = \varphi,\ \ \vec{Z}_i(z_{i-1}) = z_i,\ \ (A_i(\varphi) = z_i),\ \ 1 \le i \le m-1, \tag{3.1}$$

$$\vec{Z}_m(z_{m-1})(p, x) = f(p, x) + \sum_{i=0}^{m-1} a_i(p, x) z_i(p, x),$$

with $p \in P,\ \ x \in \mathbb{R}^n$, is solved. On the other hand, a solution of (3.1) relies on the solution $\hat{z}(p,\lambda) = (\hat{z}_0(p,\lambda),\dots,\hat{z}_{m-1}(p,\lambda))$ (see $\hat{z}_i(p,\lambda) = z_i(p,G(p,\lambda))$) fulfilling the following characteristic system

$$\partial_{t^i}\hat{z}_{i-1}(p,\lambda) = \hat{z}_i(p,\lambda),\ \ 1 \le i \le m-1 \tag{3.2}$$

$$\partial_{t^m}\hat{z}_{m-1}(p,\lambda) = \hat{f}(p,\lambda) + \sum_{i=0}^{m-1}\hat{a}_i(p,\lambda)\hat{z}_i(p,\lambda),\ \ p \in P,\ \ \lambda \in \mathbb{R}^n,$$

where $\hat{f}(p,\lambda) = f(p,G(p,\lambda))$ and $\hat{a}_i(p,\lambda) = a_i(p,G(p,\lambda))$. Here $x = G(p,\lambda)$ is the m-sheet defined in (2.4) and a solution of (3.1) is constructed by

$$z(p,x) = \hat{z}(p,\psi(p,x)),$$

replacing $\lambda = \psi(p,x)$ (see (3.1)) into a solution $\{\hat{z}(p,\lambda) : p \in P,\ \ \lambda \in \mathbb{R}^n\}$ of (3.2). For solving (3.2), associate the following system of integral equations ($1 \le i \le m-1$)

$$\hat{z}_{i-1}(p,\lambda) = \hat{z}^0_{i-1}(p_{i-1},\lambda) + \int_0^{t^i}\hat{z}_i(p_{i-1},\sigma;\lambda)d\sigma,\ \ p = (p_{i-1},t^i), \tag{3.3}$$

$$\hat{z}_{m-1}(p,\lambda) =$$
$$\hat{z}^0_{m-1}(p_{m-1},\lambda) + \int_0^{t^m}[\hat{f}(p_{m-1},\sigma;\lambda) + \sum_{i=0}^{m-1}\hat{a}_i(p_{m-1},\sigma;\lambda)\hat{z}_i(p_{m-1},\sigma;\lambda)]d\sigma$$

where the fixed Cauchy conditions $\hat{z}^0_j(p_j,\lambda),\ \ 0 \le j \le m-1$, are taken in the space $(\mathcal{C}_b \cap \mathcal{C}^1)(\mathbb{R}^{m-1} \times \mathbb{R}^n;\mathbb{R})$.

Lemma 3.1. *Assume that* $\{f,\ \ a_i : 0 \le i \le m-1\} \subseteq (\mathcal{C}_b \cap \mathcal{C}^1_b)(\mathbb{R}^{n+m};\mathbb{R})$ *and consider the integral equations (3.3). Then there exist* $P = \prod_{i=1}^{m}[-a^i,a^i]$ *and a smooth mapping* $\hat{z}(p,\lambda) : P \times \mathbb{R}^n \to \mathbb{R}^m$ *such that*

$$\hat{z}(p,\lambda) = (\hat{z}_0(p,\lambda),\dots,\hat{z}_{m-1}(p,\lambda)) : P \times \mathbb{R}^n \to \mathbb{R}^m \tag{3.4}$$

satisfies (3.3).

Proof. The standard Picard's method will be used and construct a convergent sequence $\{\hat{z}^k(p,\lambda) : p \in P,\ \ \lambda \in \mathbb{R}^n\}_{k\ge 0} \subseteq \mathcal{C}(P \times \mathbb{R}^n;\mathbb{R}^m)$ such that

$$\hat{z}^k_{i-1}(p,\lambda) = \hat{z}^0_{i-1}(p_{i-1},\lambda) + \int_0^{t^i}\hat{z}^{k-1}_i(p_{i-1},\sigma;\lambda)d\sigma, \text{ where}$$
$$p \in P,\ \ 1 \le i \le m-1,\ \ k \ge 1,\ \ \lambda \in \mathbb{R}^n$$

$$\hat{z}^k_{m-1}(p,\lambda) =$$
$$\hat{z}^0_{m-1}(p_{m-1},\lambda) + \int_0^{t^m}[\hat{f}(p_{m-1},\sigma;\lambda) + \sum_{i=0}^{m-1}\hat{a}_i(p_{m-1},\sigma;\lambda)\hat{z}^{k-1}_i(p_{m-1},\sigma;\lambda)]d\sigma$$

and $\hat{z}(p,\lambda) = \lim \hat{z}^k(p,\lambda)$ (for $k \to \infty$) will be a solution of (3.3). □

Remark 3.2. A smooth solution satisfying (3.1) is defined as follows

$$z(p,x) = \hat{z}(p, \psi(p,x)), \tag{3.5}$$

where $\{\hat{z}(p,\lambda)\}$ fulfils (3.3) and $\lambda = \psi(p,x)$ is given in (2.5). By definition, $\psi(p, G(p,\lambda)) = \lambda,\ \ p \in P$, which implies that the following equations

$$z_i(p, G(p,\lambda)) = \hat{z}_i(p,\lambda),\ \ 0 \le i \le m-1,\ \ p \in P,\ \ \lambda \in \mathbb{R}^n \tag{3.6}$$

are valid. By a direct computation, from (3.2) and (3.6), we get

$$\vec{Z}_i(z_{i-1})(p, G(p,\lambda)) = \partial_{t^i} \hat{z}_{i-1}(p,\lambda) = \hat{z}_i(p,\lambda),\ \ 1 \le i \le m-1, \tag{3.7}$$

$$\vec{Z}_m(z_{m-1})(p, G(p,\lambda)) = \partial_{t^m} \hat{z}_{m-1}(p,\lambda) = \hat{f}(p,\lambda) + \sum_{i=0}^{m-1} \hat{a}_i(p,\lambda)\hat{z}_i(p,\lambda)$$

for any $p \in P = \prod_{i=1}^{m} [-a^i, a^i] \subseteq \mathbb{R}^m$ and $\lambda \in \mathbb{R}^n$, where $\hat{f}(p,\lambda) = f(p, G(p,\lambda))$ and $\hat{a}_i(p,\lambda) = a_i(p, G(p,\lambda)),\ \ 0 \le i \le m-1$. Particularly, take $\lambda = \psi(p,x)$ in (3.7) and notice that

$$\vec{Z}_i(z_{i-1})(p,x) = z_i(p,x),\ \ 1 \le i \le m-1, \tag{3.8}$$

$$\vec{Z}_m(z_{m-1})(p,x) = f(p,x) + \sum_{i=0}^{m-1} a_i(p,x) z_i(p,x),$$

for any $p \in P$ and $x \in \mathbb{R}^n$.

The last equation leads us to the conclusion that the smooth mapping $z(p,x) = (z_0(p,x), \ldots, z_{m-1}(p,x)) \in \mathbb{R}^m,\ \ p \in P,\ \ x \in \mathbb{R}^n$, defined in (3.5), is a solution of the first order system of PDEs given in (3.1). In addition, $\{z_0(p,x) : p \in P,\ \ x \in \mathbb{R}^n\}$ is a solution of the higher order hyperbolic equation (2.14), i.e. $\varphi(p,x) = z_0(p,x)$, $p \in P$, $x \in \mathbb{R}^n$, fulfils the following higher order equation

$$A_m(\varphi)(p,x) = f(p,x) + \sum_{i=0}^{m-1} a_i(p,x) A_i(\varphi)(p,x),\ \ p \in P,\ \ x \in \mathbb{R}^n, \tag{3.9}$$

where $A_i(\varphi)(p,x) = (\vec{Z}_i \circ \ldots \circ \vec{Z}_1 \circ \vec{Z}_0)(\varphi)(p,x),\ \ 0 \le i \le m,\ \ \vec{Z}_0(\varphi) = \varphi$. Here $\{Z_1, \ldots, Z_m\} \subseteq \mathcal{C}^1(\mathbb{R}^{n+m}, \mathbb{R}^{n+m})$ are given in (2.10).

4 Solution for Problem C

Under the same notations and definitions as above, the analysis performed in Lemma 3.1. and Remark 2.3. is the main ingredient for solving both the first order system of PDE's (3.1) and its higher order hyperbolic equation (2.14).

Theorem 4.1. *Consider a finite set of complete vector fields*

$$\{X_1, \ldots, X_m\} \subseteq (\mathcal{C}_b^1 \cap \mathcal{C}^m)(\mathbb{R}^n, \mathbb{R}^n),\ \ m \ge 2,$$

and associate the m-sheet $\{y = G(p,\lambda) \in \mathbb{R}^n : p \in \mathbb{R}^m,\ \ \lambda \in \mathbb{R}^n\}$ *defined in (2.4). There are given* $\{f,\ a_i : 0 \le i \le m-1\} \subseteq (\mathcal{C}_b \cap \mathcal{C}_b^1)(\mathbb{R}^{n+m}, \mathbb{R})$ *and consider the corresponding higher order hyperbolic equation (2.14) associated with its first order system of PDEs in (3.1). Let*

$$\{\hat{z}(p,\lambda) = (\hat{z}_0(p,\lambda), \dots, \hat{z}_{m-1}(p,\lambda)) : p \in P,\ \ \lambda \in \mathbb{R}^n\}$$

be the solution of integral equations (3.3) given in Lemma 3.1. and define

$$z(p,x) = \hat{z}(p, \psi(p,x)),\ \ p \in P,\ \ x \in \mathbb{R}^n, \tag{4.1}$$

where $\lambda = \psi(p,x)$ *is given in (2.5) and fulfils PDEs (2.7). Then*

$$\{z(p,x) = (z_0(p,x), \dots, z_{m-1}(p,x)) : p \in P,\ \ x \in \mathbb{R}^n\}$$

is a solution of the system (3.1) and $\{\varphi(p,x) = z_0(p,x) : p \in P,\ \ x \in \mathbb{R}^n\}$ *verifies the higher order hyperbolic equation (2.14).*

Proof. By hypothesis, the conclusions of Lemma 3.1. and Remark 2.3. are valid. Using the unique smooth solution

$$\{\hat{z}(p,\lambda) = (\hat{z}_0(p,\lambda), \dots, \hat{z}_{m-1}(p,\lambda)) : p \in P,\ \ \lambda \in \mathbb{R}^n\}$$

of integral equations (3.3) given in Lemma 3.1., define

$$z_i(p,x) = \hat{z}_i(p, \psi(p,x)),\ \ 0 \le i \le m-1,\ \ p \in P,\ \ x \in \mathbb{R}^n, \tag{4.2}$$

where $\lambda = \psi(p,x) \in \mathbb{R}^n$ fulfils (2.5) and (2.7). The computations presented in Remark 3.2. (see (3.6)-(3.9)) show us that the smooth mapping defined in (4.2) (see (4.1)) is a solution of the first order system of PDEs in (3.1) and

$$\{z_0(p,x) : p \in P,\ \ x \in \mathbb{R}^n\}$$

fulfils the higher order hyperbolic equation (2.14). The proof is complete. □

Comment 4.2. A higher order hyperbolic equation generated by one vector field $Z \in (\mathcal{C}_b^1 \cap \mathcal{C}^m)(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ can be described as follows

$$(\vec{Z})^m(\varphi)(z) = f(z) + \sum_{i=0}^{m-1} a_i(z)(\vec{Z})^i(\varphi)(z),\ \ z = (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{4.3}$$

where $\{f,\ a_i : 0 \le i \le m-1\} \subseteq (\mathcal{C}_b \cap \mathcal{C}_b^1)(\mathbb{R}^{n+1}, \mathbb{R})$. Here the derivation (liniar application) $\vec{Z} \in Der(\mathbb{R}^{n+1})$ associated to the vector field Z is defined by $\vec{Z}(\varphi)(z) = \langle \partial_z \varphi(z), Z(z) \rangle,\ \ \varphi \in \mathcal{C}^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ and any power $(\vec{Z})^i$, for $1 \le i \le m$, makes sense as a linear application.

For the sake of simplicity, consider that the vector field Z has the following structure $Z(z) = col(1, X(x)),\ z = (t,x) \in \mathbb{R} \times \mathbb{R}^n$, where $X \in (\mathcal{C}_b^1 \cap \mathcal{C}^m)(\mathbb{R}^n, \mathbb{R}^n)$. Let $\{G(t,\lambda) : t \in \mathbb{R},\ \ \lambda \in \mathbb{R}^n\}$ be the solution of the following ODEs

$$\frac{dx}{dt}(t) = X(x(t)),\ \ t \in \mathbb{R}, \tag{4.4}$$

with initial condition $x(0) = \lambda \in \mathbb{R}^n$. Then, the flow equation, $G(t,\lambda) = x$, has a global solution $\{\lambda = \psi(t,x) = G(-t,x) : t \in \mathbb{R},\ \ x \in \mathbb{R}^n\}$ satisfying $\psi(0,x) = x$ and

$$\psi(t, G(t,\lambda)) = \lambda,\ \ G(t, \psi(t,x)) = x,\ \ t \in \mathbb{R},\ \ \lambda, x \in \mathbb{R}^n, \tag{4.5}$$

$$\vec{Z}(\psi)(t,x) = \partial_t\psi(t,x) + [\partial_x\psi(t,x)]X(x) = 0, \;\; t \in \mathbb{R}, \;\; x \in \mathbb{R}^n.$$

A solution of the higher order hyperbolic equation (4.3) is constructed using the same algorithm as in Theorem 4.1. In this respect, associate the corresponding system of linear first order PDEs

(4.6)

$$\varphi = y_0, \;\; \vec{Z}(y_0) = y_1, \;\; \ldots, \;\; \vec{Z}(y_{m-2}) = y_{m-1}, \;\; \left((\vec{Z})^i(y_0) = y_i, \;\; 0 \le i \le m-1\right)$$

$$\vec{Z}(y_{m-1}) = f(z) + \sum_{i=0}^{m-1} a_i(z)y_i(z), \;\; z = (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

A Cauchy problem for (4.6) means to find a solution of (4.6) satisfying the initial condition $y(0,x) = y^0(x)$, $x \in \mathbb{R}^n$, where $y(t,x) = (y_0(t,x),\ldots,y_{m-1}(t,x))$ and $y^0 \in (\mathcal{C}_b \cap \mathcal{C}_b^1)(\mathbb{R}^n,\mathbb{R}^m)$ are given. A Cauchy problem solution for (4.6) is found relying on the corresponding characteristic system of ODE written for $\{\hat{y}(t,\lambda) = y\,(t,G(t,\lambda)) \in \mathbb{R}^m : t \in \mathbb{R}, \;\; \lambda \in \mathbb{R}^n\}$. We get (see [6])

$$\frac{d\hat{y}}{dt}(t,\lambda) = A_0\hat{y}(t,\lambda) + \sum_{i=0}^{m-1} \hat{a}_i(t,\lambda)B_i\hat{y}(t,\lambda) + \hat{f}(t,\lambda)e_m, \;\; t \in \mathbb{R}, \tag{4.7}$$

$$\hat{y}(0,\lambda) = y^0(\lambda), \;\; \lambda \in \mathbb{R}^n.$$

Here

$$\hat{a}_i(t,\lambda) = a_i\,(t,G(t,\lambda))\,, \;\; \hat{f}\,(t,\lambda) = f\,(t,G(t,\lambda))\,,$$

for $e_m = col(0,0,\ldots,0,1) \in \mathbb{R}^m$ and the $(m \times m)$ constant matrices A_0, B_i have a special structure including commutative properties of B_i,

$$A_0 = [0 \;\; e_0 \;\; \ldots \;\; e_{m-2}], \;\; B_i = [0 \;\; 0 \;\; 0 \;\; \ldots \;\; e_i \;\; \ldots \;\; 0 \;\; 0 \;\; 0], \;\; 0 \le i \le m-1, \tag{4.8}$$

for $\{e_0,\ldots,e_{m-1}\} \subseteq R^m$ and $[B_i,B_j] = B_jB_i - B_iB_j = 0$ (null matrix), $i,j \in \{0,1,\ldots,m-1\}$. Using $\{B_0,\ldots,B_{m-1}\}$, define an $(m \times m)$ nonsingular matrix

$$M(\tau) = [exp\;\tau_0B_0]\ldots[exp\;\tau_{m-1}B_{m-1}], \;\; \tau = (\tau_0,\ldots,\tau_{m-1}) \in \mathbb{R}^m \tag{4.9}$$

and a constant variation type formula lead us to the following representation of $\{\hat{y}(t,\lambda)\}$ in (4.7)

$$\hat{y}(t,\lambda) = M(\tau(t,\lambda))b(t,\lambda), \;\; t \in \mathbb{R}, \;\; \lambda \in \mathbb{R}^n. \tag{4.10}$$

Here $\tau_i(t,\lambda) = \int_0^t \hat{a}_i(s,\lambda)ds$, $0 \le i \le m-1$, are the components of $\tau(t,\lambda)$ and the vector function $\{b(t,\lambda) \in \mathbb{R}^m : t \in \mathbb{R}, \;\; \lambda \in \mathbb{R}^n\}$ is the initial value solution of ODE

$$\frac{db}{dt}(t,\lambda) = [M(\tau(t,\lambda))]^{-1}A_0[M(\tau(t,\lambda))]b(t,\lambda)+ \tag{4.11}$$

$$+\hat{f}(t,\lambda)[M(\tau(t,\lambda))]^{-1}e_m$$

$$b(0,\lambda) = y^0(\lambda), \;\; \lambda \in \mathbb{R}^n, \;\; t \in \mathbb{R}.$$

Finally, a solution for the Cauchy problem with initial value $y(0,x) = y^0(x)$, $x \in \mathbb{R}^n$, is obtained as a composition

$$y(t,x) = \hat{y}\,(t, \psi(t,x))\,, \tag{4.12}$$

for $\psi(t,x) = G(-t,x)$, where $\hat{y}(t,\lambda) \in \mathbb{R}^m$ is represented in (4.10).

In addition, $y(t,x) = (y_0(t,x), \ldots, y_{m-1}(t,x))$ satisfies (4.6) and the first component $\{y_0(t,x) : t \in \mathbb{R},\ x \in \mathbb{R}^n\}$ is a solution of the higher hyperbolic equation (4.3). In particular, when the coefficients $a_i,\ 0 \le i \le m-1$, in (4.3), are scalar functions depending only on $t \in \mathbb{R}$, the integral representation (4.10) and (4.12) change accordingly and we get

$$(y_0(t,x), \ldots, y_{m-1}(t,x)) = y(t,x) = M\,(\tau(t))b(t, \psi(t,x))\,, \tag{4.13}$$

where $\tau(t) = (\tau_0(t), \ldots, \tau_{m-1}(t))\,,\ \ \tau_i(t) = \displaystyle\int_0^t a_i(s)ds,\ \ 0 \le i \le m-1$, and the smooth mapping $b(t,\lambda) \in \mathbb{R}^m$ verifies ODE (4.11). Starting with the integral representation (4.13), notice that not only smooth scalar functions but piecewise smooth scalar functions $\{\tau_i(t) : t \in [0,T],\ 0 \le i \le m-1\}$, can be used also leading us to a bounded solution $\{y(t,x) \in \mathbb{R}^m : t \in [0,T]\}$ with jumps. Consider a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the fixed interval $[0,T]$ and define bounded piecewise smooth scalar functions $\tilde{\tau}_i(t) : [0,T] \to \mathbb{R},\ \ 0 \le i \le m-1$ such that

$$\frac{d\tilde{\tau}_i}{dt}(t) = a_i^k(t),\ \ t \in [t_k, t_{k+1}), \tag{4.14}$$

for some $a_i^k \in C\,([t_k, t_{k+1}], R)\,,\ \ k = 0, 1, \ldots, N-1$ and $0 \le i \le m-1$. Consider the corresponding bounded and piecewise smooth mapping (see (4.13) and $\tilde{\tau}(0) = 0$)

$$\tilde{y}(t,x) = M\,(\tilde{\tau}(t))\,b\,(t, \psi(t,x))\,,\ \ t \in [0,T],\ \ x \in \mathbb{R}^n. \tag{4.15}$$

Then $\tilde{y}(t,x) = (\tilde{y}_0(t,x), \ldots, \tilde{y}_{m-1}(t,x))\,,\ \ t \in [t_k, t_{k+1}),\ \ x \in \mathbb{R}^n$ is a smooth solution of PDEs (4.6) and $\{\tilde{y}_0(t,x) : t \in [t_k, t_{k+1}),\ \ x \in \mathbb{R}^n\}$ is a smooth solution of the hyperbolic equation (4.3), for each $k \in \{0, 1, \ldots, N-1\}$. The above given remarks and computations will be restated as a Problem D and its solution.

Consider a vector field $Z \in (\mathcal{C}_b^1 \cap \mathcal{C}^m)(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ satisfying

$$Z(z) = col(1, X(x)),\ \ z = (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$

for $X \in (\mathcal{C}_b^1 \cap \mathcal{C}^m)(\mathbb{R}^n, \mathbb{R}^n)$. There are given right continuous scalar functions of bounded variation $\{\tilde{\tau}_i(t),\ \ \tilde{f}(t) : 0 \le i \le m-1,\ \ t \in [0,T]\}$ verifying

$$\frac{d\tilde{\tau}_i}{dt}(t) = a_i^k(t),\ \ \frac{d\tilde{f}}{dt}(t) = f^k(t),\ \ t \in [t_k, t_{k+1}), \tag{4.16}$$

for some $\{a_i^k, f^k\} \subseteq \mathcal{C}([t_k, t_{k+1}], \mathbb{R})$, for $0 \le k \le N-1$, where
$0 = t_0 < t_1 < \ldots < t_N = T$ is a partition of $[0,T]$. Consider the higher order hyperbolic equation

$$(\vec{Z})^m(\varphi)(t,x) = f^k(t) + \sum_{i=0}^{m-1} a_i^k(t)(\vec{Z})^i(\varphi)(t,x),\ \ t \in [t_k, t_{k+1}),\ \ x \in \mathbb{R}^n, \tag{4.17}$$

for $0 \le i \le m-1$, $0 \le k \le N-1$, where $\{a_i^k, f^k\}$ are given in (4.16), with Cauchy condition $(\vec{Z})^i(\varphi)(0,x) = y_i^0(x)$, $0 \le i \le m-1$, and bounded jumps $\triangle y_i(t_k,x) = (\vec{Z})^i(\varphi)(t_k,x) - (\vec{Z})^i(\varphi)(t_k-,x)$, $0 \le k \le N$, where

$$\{y^0(x) = (y_0^0(x), \dots, y_{m-1}^0(x)) : x \in \mathbb{R}^n\} \subseteq (\mathcal{C}_b \cap \mathcal{C}_b^1)(\mathbb{R}^n, \mathbb{R}^m).$$

Problem D. *a) Find (if possible) a piecewise smooth and bounded solution* $\varphi(t,x) : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *satisfying (4.17);*

b) Find a piecewise smooth and bounded solution

$$\tilde{y}(t,x) = (\tilde{y}_0(t,x), \dots, \tilde{y}_{m-1}(t,x))$$

on $[0,T] \times \mathbb{R}^n$*, satisfying the first order system of PDEs with bounded jumps*

$$\varphi = \tilde{y}_0, \tag{4.18}$$

$$\vec{Z}(\tilde{y}_0)(t,x) = \tilde{y}_1(t,x), \ \dots, \ \vec{Z}(\tilde{y}_{m-2})(t,x) = \tilde{y}_{m-1}(t,x), \ \left((\vec{Z})^i(\tilde{y}_0) = \tilde{y}_i\right)$$

$$\vec{Z}(\tilde{y}_{m-1})(t,x) = f^k(t) + \sum_{i=0}^{m-1} a_i^k(t)\tilde{y}_i(t,x), \ t \in [t_k, t_{k+1}), \ x \in \mathbb{R}^n, \ 0 \le k \le N-1,$$

verifying the initial conditions $\tilde{y}(0,x) = y^0(x)$, $x \in \mathbb{R}^n$*;*

c) Notice that the first component $\tilde{y}_0(t,x) : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *of the solution in (4.18) stands for a solution in (4.17);*

Theorem 4.3. **(Solution for Problem D.)** *Consider the higher order hyperbolic equation (4.17), where* $\vec{Z}(\varphi)(t,x) = \partial_t\varphi(t,x) + \langle \partial_x\varphi(t,x), X(x)\rangle$ *for* $X \in (\mathcal{C}_b^1 \cap \mathcal{C}^m)(\mathbb{R}^n, \mathbb{R}^n)$ *and* $\{a_i^k, f^k\} \subseteq \mathcal{C}([t_k, t_{k+1}], \mathbb{R})$ *are used in (4.16).*
Associate the first order system of PDEs with bounded jumps in (4.18). Then there exists a piecewise smooth and bounded solution,

$$\tilde{y}(t,x) = (\tilde{y}_0(t,x), \dots, \tilde{y}_{m-1}(t,x)) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m,$$

satisfying the first order system of PDEs with jumps (4.18) such that the first component $\tilde{y}_0(t,x) : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *stands for a solution of the hyperbolic equation (4.17). In addition,* $\{\tilde{y}(t,x) \in \mathbb{R}^m : t \in [0,T], \ x \in \mathbb{R}^n\}$ *can be represented by (see (4.15))*

$$\tilde{y}(t,x) = M(\tilde{\tau}(t))\, b(t, \psi(t,x)), \ t \in [0,T], \ x \in \mathbb{R}^n, \tag{4.19}$$

where $\tilde{\tau}(t) = (\tilde{\tau}_0(t), \dots, \tilde{\tau}_m(t))$, $t \in [0,T]$*, is given in (4.16), the* $(m \times m)$ *matrix* $M(\tau)$ *is defined in (4.9) and* $b(t,\lambda)$, $t \in [t_k, t_{k+1}]$, $\lambda \in \mathbb{R}^n$ *satisfies (4.11),* $0 \le k \le N-1$.

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