On the Zermelo problem in Riemannian manifolds

Radomír Paláček and Olga Krupková

Abstract. We generalize the Zermelo navigation problem from flat to Riemannian spaces and find the corresponding force representing the action of the wind distribution.

M.S.C. 2010: 53A55, 53B20; 49J53.
Key words: Lagrangian; Euler-Lagrange equations; Zermelo navigation problem.

1 Introduction

In [2] E. Zermelo[1] deals with the following classical control problem:

> *In an unbounded plane where the wind distribution is given by a vector field as a function of position and time, a boat moves with constant velocity relative to the surrounding air mass. How must the boat be directed in order to come from a starting point* 0 *to a destination point* D *in the shortest time?*

Geometrically, the problem is to find the deviation of geodesics under the action of a time–dependent vector field. The aim of this paper is to generalize the Zermelo navigation problem to Riemannian manifolds. We find the solution for this case: we construct a corresponding suitable Lagrangian and specify the properties of the corresponding force. In a different way the problem was treated in [1] where for the case of a "low wind perturbation" (the Riemannian length of the wind vector is ≤ 1 everywhere on M) a new metric corresponding to the deviated geodesics was constructed as a Finsler metric.

2 Notations and preliminaries

Throughout this paper, manifolds and mappings are smooth and the summation convention over repeated indices is assumed.

Let a pair (M, g) be a Riemannian manifold, where

$$g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$$

Balkan Journal of Geometry and Its Applications, Vol.17, No.2, 2012, pp. 77-81.

[1] Ernst Friedrich Ferdinand Zermelo (July 27, 1871 - May 21, 1953)

is a Riemannian metric ((g_{ij}) be non-degenerate, symmetric and positive definite matrix) and M is an m-dimensional manifold with local coordinates (x^a), $1 \leq a \leq m$. We shall consider a fibred manifold $\pi : \mathbb{R} \times M \to \mathbb{R}$, where π is the first canonical projection. On $\mathbb{R} \times M$ we use coordinate charts adapted to the product structure (t, x^a), $1 \leq a \leq m$, where t is the global coordinate on $\mathbb{R}$. A curve $c : \mathbb{R} \to M$, defined in a neighborhood of $0 \in \mathbb{R}$, will be represented by its graph

$$\begin{aligned} \gamma : \quad \mathbb{R} \quad &\to \mathbb{R} \times M, \\ t \quad &\mapsto (t, c(t)), \end{aligned}$$

which is a section of the fibered manifold π. Any section γ of the fibered manifold π can be prolonged to a section $J^1\gamma$ of the fibered manifold $J^1(\mathbb{R} \times M) \approx \mathbb{R} \times TM$, and $J^2\gamma$ of $\mathbb{R} \times T^2M$. Then $J^1\gamma(t) = (t, c(t), \dot{c}(t))$ and $J^2\gamma(t) = (t, c(t), \dot{c}(t), \ddot{c}(t))$.

Let the wind distribution on M be represented by a time-dependent vector field on M, i.e. by a projectable vector field ξ on $\mathbb{R} \times TM$ of the form

$$\xi = \frac{\partial}{\partial t} + \xi^i(t, x^j) \frac{\partial}{\partial x^i}.$$

To analyze the deformations of geodesics consider the variational problem on $\mathbb{R} \times TM$ defined by the kinetic energy in the form

$$\bar{T} = \frac{1}{2} g_{ij} y^i y^j, \tag{2.1}$$

where $y^i = \dot{x}^i + \xi^i$.

3 Euler-Lagrange equations

The Euler-Lagrange equations of the mechanical system (2.1) are expressed in the form

$$\frac{\partial \bar{T}}{\partial x^k} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \bar{T}}{\partial \dot{x}^k} \right) = 0, \qquad 1 \leq k \leq m, \tag{3.1}$$

where

$$\begin{aligned} \bar{T} &= \frac{1}{2} g_{ij} y^i y^j = \frac{1}{2} g_{ij} \left(\dot{x}^i + \xi^i \right) \left(\dot{x}^j + \xi^j \right) = \\ &= \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + g_{ij} \dot{x}^i \xi^j + \frac{1}{2} g_{ij} \xi^i \xi^j. \end{aligned} \tag{3.2}$$

Let us denote

$$V = -g_{ij} \dot{x}^i \xi^j - \frac{1}{2} g_{ij} \xi^i \xi^j, \tag{3.3}$$

then we have

$$\bar{T} = T - V, \tag{3.4}$$

where T is the kinetic energy of the unpertubed problem and V has the meaning of the potential energy caused by the wind.

Computing (3.1) explicitly we obtain

$$F_k - \Gamma_{kij}\dot{x}^i\dot{x}^j - g_{kj}\ddot{x}^j = 0, \tag{3.5}$$

where

$$F_k = \left(\frac{\partial g_{ij}}{\partial x^k}\xi^j + g_{ij}\frac{\partial \xi^j}{\partial x^k}\right)\dot{x}^i + \frac{1}{2}\frac{\partial}{\partial x^k}\left(g_{ij}\xi^i\xi^j\right) - \frac{\partial g_{kj}}{\partial x^i}\dot{x}^i\xi^j - g_{kj}\frac{\partial \xi^j}{\partial t} - g_{kj}\frac{\partial \xi^j}{\partial x^i}\dot{x}^i \tag{3.6}$$

and Γ_{ijk} are standard Christoffel symbols of (g_{ij}),

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right). \tag{3.7}$$

Now, let us introduce the covector $\tilde{\xi}_i = g_{ij}\xi^j$. Then using notation

$$\xi^2 = \xi \cdot \xi = g_{ij}\xi^i\xi^j, \tag{3.8}$$

we obtain the force in the following final form

$$F_k = \left(\frac{\partial \tilde{\xi}_i}{\partial x^k} - \frac{\partial \tilde{\xi}_k}{\partial x^i}\right)\dot{x}^i + \frac{1}{2}\frac{\partial \xi^2}{\partial x^k} - \frac{\partial \tilde{\xi}_k}{\partial t}. \tag{3.9}$$

If M is 3–dimensional we can write

$$\vec{F} = \operatorname{rot}\tilde{\xi} \times \vec{v} + \vec{E}, \tag{3.10}$$

where

$$E_k = \frac{1}{2}\frac{\partial \xi^2}{\partial x^k} - \frac{\partial \tilde{\xi}^k}{\partial t}. \tag{3.11}$$

The force F is a deformation force, arising due to the wind distribution ξ, giving rise to a deformed family of geodesics compared to the original ones (describing the "free particle" on M).

Notice an interesting relation with electrodynamics: equations (3.10) have the same form as the equations for a charged particle moving in an electromagnetic field with the electromagnetic potentials

$$\vec{B} = -\operatorname{rot}\tilde{\xi} \quad \textit{and} \quad \vec{E}. \tag{3.12}$$

4 Simulation of a 2-dimensional situation

As an example we provide a solution of the problem for the case $\dim(\mathbb{R} \times M) = 2$.

We choose

$$\begin{aligned} g &= x\,\mathrm{d}x, \\ \xi &= \frac{\partial}{\partial t} + x\frac{\partial}{\partial x}. \end{aligned} \tag{4.1}$$

Equation (3.5) takes the form

$$\frac{3}{2}x^2 - \frac{1}{2}\dot{x}^2 - x\ddot{x} = 0 \tag{4.2}$$

and $F = \frac{3}{2}x^2$.

Bellow the solution is simulated with help of Wolfram Mathematica.

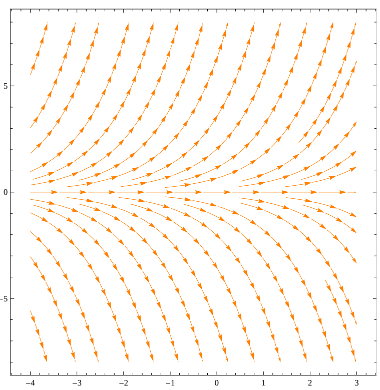

Fig. 1

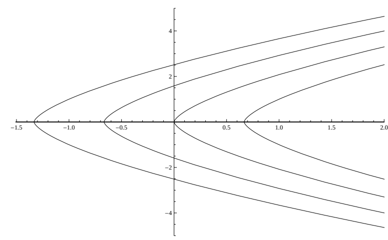

Fig. 2

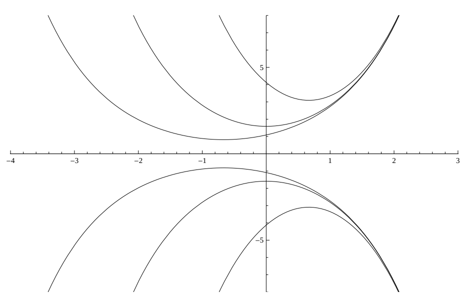

Fig. 3

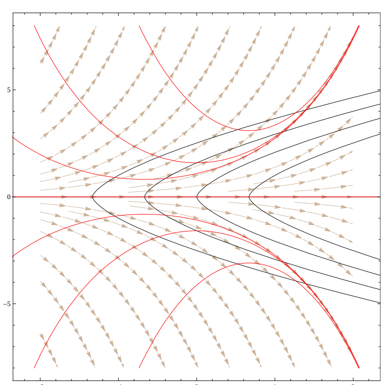

Fig. 4

On Figure 1 the vector field ξ that was chosen for our simulation is modeled. Curves on the Figure 2 represent curves before the "wind" deformation

$$x = c_1 \sqrt[3]{(3t - 2c_2)^2}, \quad c_1, c_2 \;\; \textit{are arbitrary},$$

and next Figure 3 shows curves after "wind" deformation

$$x = c_1 e^{-t} \sqrt[3]{(e^{3t} + e^{2c_2})^2}, \quad c_1, c_2 \;\; \textit{are arbitrary}.$$

Last Figure 4 demonstrates the whole situation where we can see changes on the curves caused by the force F.

Acknowledgement. Research was supported by grants GACR 201/09/0981 of the Czech Science Foundation and SGS 18/PřF/2010 of the University of Ostrava.

References

[1] D. Bao, C. Robles and Z. Shen, *Zermelo navigation on Riemannian manifolds*, J. Diff. Geom. 66 (2004), 391-449.

[2] E. Zermelo, *Über das Navigationsproblem bei ruhender oder veränderlicher Windverteilung*, Ztschr. f. angew. Math. und Mech.(1931), Band 11, Heft 2, 114–124.

Authors' addresses:

Radomír Paláček
VŠB - Technical University of Ostrava,
Department of Mathematics and Descriptive Geometry,
17. listopadu 15/2172,
708 33 Ostrava, Czech Republic.
E-mail: radomir.palacek@vsb.cz

Olga Krupková
Department of Mathematics, Faculty of Science,
The University of Ostrava, 30. dubna 22,
701 03 Ostrava, Czech Republic
and
Department of Mathematics and Statistics, La Trobe University,
Melbourne, Victoria 3086, Australia.
E-mail: olga.krupkova@osu.cz