Geometric approach to a class of multidimensional hybrid systems

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Abstract. A class of multidimensional time-invariant hybrid control systems is studied in the geometric approach. The space of the reachable states is characterized as the minimal subspace of the state space which is invariant with respect to the drift matrices and which contains the image of the input matrix. An algorithm is provided which compute this subspace. Some necessary and sufficient conditions of reachability are derived. By duality, it can be shown that the space of the unobservable states is the maximal subspace which is invariant with respect to the drift matrices and is included in the kernel of the output matrix of the system.

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1 Introduction

The Geometric Approach is a trend in Systems and Control Theory developed to achieve a better and neater investigation of the structural properties of linear dynamical systems and to provide elegant solutions of problems of controller synthesis such as decoupling and pole-assignment problems for linear time-invariant multivariable systems. The geometric approach leads to a very clear notion of minimality and to geometric conditions for observability, constructibility, minimality of spectral factors etc. The cornerstone of this approach is the concept of invariance of a subspace with respect to a linear transformation.

In 1969 Basile and Marro [2] introduced and studied the basic geometric tools called controlled and conditioned invariant subspaces which were applied to disturbance rejection or unknown-input observability. In 1970 Wonham and Morse [19] applied a maximal controlled invariant method to decoupling and noninteracting control problems and later on Wonham's book [18] imposed the name of "(A,B)-invariant" instead of "(A,B)-controlled invariant". Basile and Marro, opened the way to new applications by the robust controlled invariant and the emphasis of the duality [2]. The LQ problem was also studied in a geometric framework by Silverman,

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Hautus, Willems. J.C. Willems also developed the theory of almost controlled and almost conditioned invariant subspaces used in high-gain feedback problems. Further contributions are due to numerous researchers among which Anderson, Akashi, Bhattacharyya, Kucera, Malabre, Molinari, Pearson, Francis and Schumacher.

The past three decades have seen a continually growing interest in the theory of two-dimensional (2D) or, more generally, multidimensional (nD) systems, which become a distinct and important branch of the systems theory [1, 3, 9, 15, 17]. The reasons for the interests in this domain are on one side the richness in potential application fields and on the other side the richness and significance of the theoretical approaches. The application fields include circuits, control and signal processing, image processing, computer tomography, gravity and magnetic field mapping, seismology, control of multipass processes, etc.

A quite new field of the nD systems theory is represented by the 2D hybrid models, whose state equation is of differential-difference type [6], [11], [12]. These hybrid models have applications in various areas such as linear repetitive processes [5], [15], pollution modelling [4], long-wall coal cutting and metal rolling [16] or in iterative learning control synthesis [8].

In the present paper, a class of multidimensional hybrid systems described by differential-difference state equations is studied from the point of view of the geometric approach.

In Section 2 the state space representation of these systems is given and the formula of the state is obtained. The considered systems represent extensions to multidimensional hybrid continuous-discrete models of Attasi's 2D discrete-time systems.

Section 3 introduces the notions of controllable and reachable states. A suitable reachability Gramian is constructed for time-invariant multidimensional systems and it is used to obtain conditions for the phase transfer and criteria of reachability.

Section 4 introduces the controllability matrix, which is used to characterize the space of the reachable states as the minimal (A_{ci}, A_{dj})-invariant subspace which contains the columns of the matrix B and to obtain necessary and sufficient conditions of reachability for multidimensional systems.

Section 5 provides an algorithm which computes the minimal $(A_{ci}, A_{dj}; B)$-invariant subspace, which extends the 1D algorithm from [10].

We shall use the following notations: $q \in \mathbb{N}$ and $r \in \mathbb{N}$ being the number of continuous and discrete variables respectively, a function $x(t_1, \ldots, t_q; k_1, \ldots, k_r)$, $t_i \in \mathbb{R}$, $k_i \in \mathbb{Z}$ will be sometimes denoted by $x(t; k)$, where $t = (t_1, \ldots, t_q)$, $k = (k_1, \ldots, k_r)$. By $\overline{m}$ with $m \in \mathbb{N}^*$ we denote the set $\{1, 2, \ldots, m\}$ and by $\mathcal{P}(\overline{m})$ the family of all subsets of $\overline{m}$. By $s \leq t,\ s, t \in \mathbb{R}^q$ we mean $s_i \leq t_i\ \forall i \in \bar{q}$ and a similar signification has $l \leq k,\ l, k \in \mathbb{Z}^r$; $(s; l) < (t; k)$ means $s \leq t,\ (l \leq k)$ and $(s; l) \neq (t; k)$. For $t^0, t^1 \in \mathbb{R}^q$ and $k^0, k^1 \in \mathbb{Z}^r,\ t^0 < t^1,\ k^0 < k^1$ we denote by $[t^0, t^1]$ and $[k^0, k^1]$ respectively the sets $[t^0, t^1] = \prod_{i=1}^q [t_i^0, t_i^1]$ and $[k^0, k^1] = \prod_{i=1}^r \{k_j^0, k_j^0 + 1, \ldots, k_j^1\}$.

If $\tau = \{i_1, \ldots, i_l\}$ is a subset of $\overline{m}$, $|\tau| := l$ and $\tilde{\tau} := \overline{m} \setminus \tau$; for $i \in \overline{m}$, $\tilde{i} := \overline{m} \setminus \{i\}$ and $\tilde{\tilde{i}} := \{i+1, \ldots, m\}$. The notation $(\tau, \delta) \subset (\overline{q}, \overline{r})$ means that τ and δ are subsets of $\overline{q}$ and $\overline{r}$ respectively and $(\tau, \delta) \neq (\overline{q}, \overline{r})$. For $\tau = \{i_1, \ldots, i_l\}$ and $\delta = \{j_1, \ldots, j_h\}$ the operators $\dfrac{\partial}{\partial \tau}$ and σ_δ are defined by

$$\frac{\partial}{\partial \tau} x(t; k) = \frac{\partial^l}{\partial t_{i_1} \ldots \partial t_{i_l}} x(t; k),\ \sigma_\delta x(t; k) = x(t; k + e_\delta)$$

where $e_\delta = e_{j_1} + \cdots + e_{j_h}$, $e_j = (\underbrace{0,\ldots,0}_{j-1},1,0,\ldots,0) \in \mathbb{R}^r$; when $\tau = \overline{q}$ and $\delta = \overline{r}$ we denote $\partial/\partial\tau = \partial/\partial t$ and $\sigma_\delta = \sigma$.

If A_i, $i \in \overline{m}$ is a family of matrices, then $\sum\limits_{i\in\varnothing} A_i = 0$ and $\prod\limits_{i\in\varnothing} A_i = I$.

2 State space representation

The time set of the considered class of multidimensional hybrid systems is $T = (\mathbb{R}^+)^q \times (\mathbb{Z}^+)^r$, $q, r \in \mathbb{N}^*$. The *state space*, the *input space* and the *output space* are respectively $X = \mathbb{R}^n$, $U = \mathbb{R}^m$ and $Y = \mathbb{R}^p$.

Definition 2.1. A (q,r)-D *hybrid system* is a set $\Sigma = (\{A_{ci}|i \in \overline{q}\}, \{A_{dj}|j \in \overline{r}\}, B, C, D)$ with A_{ci}, $i \in \overline{q}$ and A_{dj}, $j \in \overline{r}$ commuting $n \times n$ matrices and B, C, D respectively $n \times m$, $p \times n$ and $p \times m$ real matrices; the state equation is

$$\begin{aligned}\frac{\partial}{\partial t}\sigma x(t;k) = \sum_{(\tau,\delta)\subset(\overline{q},\overline{r})} &(-1)^{q+r-|\tau|-|\delta|-1} \times \\ \times \left(\prod_{i\in\tilde{\tau}} A_{ci}\right)&\left(\prod_{j\in\tilde{\delta}} A_{dj}\right)\frac{\partial}{\partial\tau}\sigma_\delta x(t;k) + Bu(t;k)\end{aligned} \tag{2.1}$$

and the output equation is

$$y(t;k) = Cx(t;k) + Du(t;k), \tag{2.2}$$

where $x(t;k) = x(t_1,\ldots,t_q;k_1,\ldots,k_r) \in X$ is the *state*, $u(t;k) \in U$ is the *input* and $y(t;k) \in Y$ is the *output* of the system Σ.

For any ordered sets $\tau = \{i_1,\ldots,i_l\} \subset \overline{q}$ and $\delta = \{j_1,\ldots,j_h\} \subset \overline{r}$ and for $t_i \in \mathbb{R}^+$, $i \in \tau$, $t_i^0 \in \mathbb{R}^+$, $i \in \overline{\tau}$, $k_j \in \mathbb{Z}^+$, $j \in \delta$, $k_j^0 \in \mathbb{Z}^+$, $j \in \tilde{\delta}$ we use the notation

$$\begin{aligned}x(t_\tau, t^0_{\tilde{\tau}}; k_\delta, k^0_{\tilde{\delta}}) := x(&t_1^0,\ldots,t^0_{i_1-1},t_{i_1},t^0_{i_1+1},\ldots,t^0_{i_l-1},t_{i_l},t^0_{i_l+1},\ldots,t^0_q;\\ &k_1^0,\ldots,k^0_{j_1-1},k_{j_1},k^0_{j_1+1},\ldots,k^0_{j_h-1},k_{j_h},k^0_{j_h+1},\ldots,k^0_{j_r}).\end{aligned}$$

Since the considered system is time-invariant, we shall take $t_i^0 = 0$, $\forall i$ and $k_j^0 = 0$, $\forall j$ and in this case $x(t_\tau, t^0_{\tilde{\tau}}; k_\delta, k^0_{\tilde{\delta}})$ will be denoted $x(t_\tau; k_\delta)$.

Definition 2.2. The vector $x^0 \in \mathbb{R}^n$ is called an *initial state* of the system Σ if

$$x(t_\tau; k_\delta) = \left(\prod_{i\in\tau} e^{A_{ci}t_i}\right)\left(\prod_{j\in\delta} F_j^{k_j}\right) x^0 \tag{2.3}$$

for any $(\tau,\delta) \subset (\overline{q},\overline{r})$; equalities (2.3) are called the *initial conditions of* Σ.

In [13] one proves

Theorem 2.1. *The state of Σ determined by the control u and the initial state x_0 is*

$$x(t;k) = \left(\prod_{i=1}^{q} e^{A_{ci}t_i}\right)\left(\prod_{j=1}^{r} A_{dj}^{k_j}\right)x^0 + \int_0^{t_1}\cdots\int_0^{t_q}\left(\prod_{i=1}^{q} e^{A_{ci}(t_i-s_i)}\right)\times \tag{2.4}$$
$$\times\sum_{l_1=0}^{k_1-1}\cdots\sum_{l_r=0}^{k_r-1}\left(\prod_{j=1}^{r} A_{dj}^{k_j-l_j-1}\right)Bu(s;l)\mathrm{d}s_1\ldots\mathrm{d}s_q.$$

3 Reachability of multidimensional hybrid systems

In this section the (q,r)-D hybrid system will be represented by the ensemble $\Sigma = (\{A_{ci}|i\in\bar{q}\},\{A_{dj}|j\in\bar{r}\},B)$ since the reachability/controllability topic uses only the state equation (2.1).

A triplet $(t,k,\tilde{x})\in(\mathbb{R}^+)^q\times(\mathbb{Z}^+)^r\times\mathbb{R}^n$ is said to be a *phase* of Σ if $\exists u : T\to\mathbb{R}^m$ and $x^0\in\mathbb{R}^n$ such that $\tilde{x} = x(t;k)$ where $x(t;k)$ is given by (2.4). In this case one says that the control u transfers the phase (t^0,k^0,x^0) to the phase $(t,k,\tilde{x})$.

Definition 3.1. A phase (t,k,x) of Σ is said to be *controllable* if there exist $(t^1,k^1)\in T$, $(t^1,k^1) > (t,k)$ and a control u which transfers the phase (t,k,x) to $(t^1,k^1,0)$.

A phase (t,k,x) is said to be *reachable* if there exist $(t^0,k^0)\in T$, $(t^0,k^0)<(t,k)$ and a control u which transfers the phase $(t^0,k^0,0)$ to (t,k,x).

If for some fixed $(t^0,k^0),(t^1,k^1)\in T$ with $(t^0,k^0)<(t^1,k^1)$ a phase (t^0,k^0,x) $((t^1,k^1,x))$ is controllable (reachable) one says that the state x is *controllable* (*reachable*) *on the multiple interval* $P = [t^0,t^1]\times[k^0,k^1]$. The system Σ is said to be *completely controllable* (*completely reachable*) on P if any state $x\in\mathbb{R}^n$ is controllable (reachable) on P.

In the sequel we shall denote by $\int_{t^0}^{t}$ the multiple integral $\int_{t_1^0}^{t_1}\cdots\int_{t_q^0}^{t_q}$, by $\sum_{l=k^0}^{k-1}$ the sum $\sum_{l_1=k_1^0}^{k_1-1}\cdots\sum_{l_r=k_r^0}^{k_r-1}$ and $\mathrm{d}s = \mathrm{d}s_1\cdots\mathrm{d}s_q$; $t^0=0$ means $t^0=(0,0,\ldots,0)\in\mathbb{R}^q$ and a similar meaning has $k^0=0\in\mathbb{Z}^r$.

Definition 3.2. The *reachability Gramian* of Σ on the multiple interval $P = [0,t]\times[0,k]$ is the matrix

$$\mathcal{R}(t;k) = \int_0^t\sum_{l=0}^{k-1}\exp\left(\sum_{i=1}^{q}A_{ci}(t_i-s_i)\right)\left(\prod_{j=1}^{r}A_{dj}^{k_j-l_j-1}\right)\times \tag{3.1}$$
$$\times BB^T\left(\prod_{j=1}^{r}(A_{dj}^T)^{k_j-l_j-1}\right)\exp\left(\sum_{i=1}^{q}A_{ci}^T(t_i-s_i)\right)ds_1\cdots ds_q.$$

Obviously, $\mathcal{R}=\mathcal{R}(t;k)$ is a symmetrical non-negative definite $n\times n$ matrix.

By adapting [12, Theorem 3.4], we obtain

Proposition 3.1. *The set of the states which are reachable on P is the subspace* $X_r(t,k) = \operatorname{Im}\mathcal{R}(t;k)$.

It results that the system Σ is completely reachable on P if and only if $\operatorname{Im}\mathcal{R}(t;k) = \mathbb{R}^n$, condition which gives the following criterion:

Theorem 3.1. *The system Σ is completely reachable on P if and only if*

$$\operatorname{rank}\mathcal{R}(t;k) = n \tag{3.2}$$

4 $(A_{ci}, A_{dj}; B)$-invariant subspaces and reachability

hspace5mm replacing $x(t;k) = x$ and $x^0 = 0$ in (2.4) we get (see Definition 3.1):

Proposition 4.1. A state $x \in \mathbb{R}^n$ is reachable if and only if there exist $(t;k) \in T$ and a control u such that

$$x = \int_0^t \sum_{l=0}^{k-1} \exp\left(\sum_{i=1}^{q} A_{ci}(t_i - s_i)\right)\left(\prod_{j=1}^{r} A_{dj}^{k_j - l_j - 1}\right) Bu(s;l)ds. \tag{4.1}$$

We shall consider subsets $\gamma \subset \bar{q}$ and $\delta \subset \bar{r}$ and numbers i_α and j_β with $0 \le i_\alpha \le n-1$, $\forall \alpha \in \gamma$ and $0 \le j_\beta \le n-1$, $\forall \beta \in \delta$.

We associate to the system Σ the *controllability matrix*

$$C_\Sigma = \Bigg[B \;\; A_{c1}B \;\; \ldots \;\; A_{c1}^{n-1}B \;\; \ldots \;\; \left(\prod_{\alpha\in\gamma} A_{c\alpha}^{i_\alpha}\right)\left(\prod_{\beta\in\delta} A_{d\beta}^{j_\beta}\right) B \;\; \ldots \\ \ldots \left(\prod_{\alpha=1}^{q} A_{c\alpha}^{n-1}\right)\left(\prod_{\beta=1}^{r} A_{d\beta}^{n-1}\right) B \Bigg] \tag{4.2}$$

(where we consider $\prod_{\alpha\in\phi} A_\alpha = I$).

Theorem 4.1. *The set of all reachable states of the system $\Sigma = (A_{ci}, A_{dj}, B)$ is the subspace of X*

$$X_r = \operatorname{Im} C_\Sigma. \tag{4.3}$$

Proof. For a subspace S of X, the orthogonal complement is the subspace $S^\perp = \{x \in X; x^T s = 0,\ \forall s \in S\}$. We shall prove that $\operatorname{Im}\mathcal{R}(t;k) = \operatorname{Im} C_\Sigma,\ \forall t \in \mathbb{R}^+,\ \forall k \in \mathbb{Z}^+$.

Consider the vector $v \in (\operatorname{Im} C_\Sigma)^\perp$. Since $\operatorname{Im} C_\Sigma$ is the subspace generated by the columns of the matrix C_Σ it follows that $v^T C_\Sigma = 0$, hence

$$v^T \left(\prod_{\alpha\in\gamma} A_{c\alpha}^{i_\alpha}\right)\left(\prod_{\beta\in\delta} A_{d\beta}^{j_\beta}\right) B = 0 \tag{4.4}$$

$\forall\gamma \subset \bar{q}$, $\forall\delta \subset \bar{r}$, $\forall i_\alpha, j_\beta$ with $0 \le i_\alpha \le n-1$ and $0 \le j_\beta \le n-1$. By applying Hamilton-Cayley Theorem to matrices A_{ci} and A_{dj} we can prove that (4.4) is true for any $i_\alpha \ge 0$ and $j_\beta \ge 0$. Then

$$v^T \left(\exp\left(\sum_{i=1}^{q} A_{ci} t_i\right)\right) \left(\prod_{j=1}^{r} A_{dj}^{k_j}\right) B =$$

$$= \sum_{i_1=0}^{\infty} \cdots \sum_{i_q=0}^{\infty} v^T \left(\prod_{\alpha=1}^{q} (i_\alpha!)^{-1} A_{c\alpha}^{i_\alpha} t_\alpha^{i_\alpha}\right) \left(\prod_{j=1}^{r} A_{dj}^{k_j}\right) B = 0$$

for any $t_\alpha \in \mathbb{R}^+$, $\alpha \in \bar{q}$ and $k_j \in \mathbb{Z}^+$, $j \in \bar{r}$. By (3.1) it follows that $v^T \mathcal{R}(t;k) = 0$ $\forall t \in \mathbb{R}^+$, $\forall k \in \mathbb{Z}^+$, hence $v \in (\operatorname{Im} \mathcal{R}(t;k))^\perp$, $\forall t \in \mathbb{R}^+$, $\forall k \in \mathbb{Z}^+$. We have proved that

$$(\operatorname{Im} C_\Sigma)^\perp \subset (\operatorname{Im} \mathcal{R}(t;k))^\perp, \ \forall t \in \mathbb{R}^+, \ \forall k \in \mathbb{Z}^+. \tag{4.5}$$

Conversely, let v be a vector $v \in (\operatorname{Im} \mathcal{R}(t;k))^\perp$ for arbitrary $(t:k) \in T$. Then $v^T \mathcal{R}(t;k) = 0$, hence $v^T \mathcal{R}(t;k) v = 0$. We get

$$\int_0^t \sum_{l=0}^{k-1} \left\| B^T \left(\prod_{j=1}^{r} A_{dj}^{k_j - l_j - 1}\right)^T \exp\left(\sum_{i=1}^{q} A_{ci}^T (t_i - s_i)\right) v \right\|^2 ds = 0,$$

hence

$$v^T \left(\exp\left(\sum_{i=1}^{q} A_{ci} t_i\right)\right) \left(\prod_{j=1}^{r} A_{dj}^{k_j}\right) = 0 \quad \text{a.e.}$$

By deriving recurrently this equality with respect to t_i, $i \in \bar{q}$ and by taking $t_\alpha = 0$, $\forall\alpha \in \bar{q}$, one obtains (4.4), hence $v^T C_\Sigma = 0$, i.e. $v \in (C_\Sigma)^\perp$. It follows that $(\operatorname{Im}\mathcal{R}(t;k))^\perp \subset (\operatorname{Im} C_\Sigma)^\perp$ and by (4.5) $(\operatorname{Im} C_\Sigma)^\perp = (\operatorname{Im}\mathcal{R}(t;k))^\perp$, $\forall t \in \mathbb{R}^+$, $\forall k \in \mathbb{Z}^+$. Since the orthogonal complement of a subspace is unique, one obtains

$$\operatorname{Im} C_\Sigma = \operatorname{Im} \mathcal{R}(t;k), \ \forall t \in \mathbb{R}^+, \ \forall k \in \mathbb{Z}^+.$$

By Proposition 3.1, the set of all reachable states of the system Σ is

$$X_r = \bigcup_{t,k \ge 0} (\operatorname{Im} \mathcal{R}(t;k)) = \operatorname{Im} C_\Sigma$$

□

Now, the system Σ is completely reachable if and only if $\operatorname{Im} C_\Sigma = X_r = X = \mathbb{R}^n$, i.e. if and only if $\operatorname{rank} C_\Sigma = n$, hence we have:

Theorem 4.2. *The system $\Sigma = (A_{ci}, A_{dj}, B)$ is completely reachable if and only if*

$$\operatorname{rank} C_\Sigma = n. \tag{4.6}$$

We denote by $[n]_q$ the set $[n]_q = \{\gamma = (\gamma_1, \ldots, \gamma_q) \in \mathbb{N}^q | 0 \le \gamma_i \le n,\ i \in \bar{q}\}$. For $\gamma^0 = (\gamma_1^0, \ldots, \gamma_q^0) \in \mathbb{N}^q$, $\gamma^0 \ge \gamma \ge 0$ means $\gamma_1^0 \ge \gamma_1 \ge 0, \ldots, \gamma_q^0 \ge \gamma_q \ge 0$; $\sum\limits_{\gamma} := \sum\limits_{\gamma_1} \cdots \sum\limits_{\gamma_q}$; for $\gamma = (\gamma_1, \ldots, \gamma_q) \in \mathbb{N}^q$, $\delta = (\delta_1, \ldots, \delta_r) \in \mathbb{N}^r$, $\zeta \in \bar{m}$, $\alpha(\gamma; \delta; \zeta)$ means $\alpha(\gamma_1, \ldots, \gamma_q; \delta_1, \ldots, \delta_r; \zeta); b_\zeta$ stands for the column ζ of the matrix B.

By Theorem 4.2, since the image of a matrix is the subspace of the linear combinations of its columns, one obtains:

Corollary 4.1. The set of all reachable states of Σ is the subspace

$$\begin{aligned} X_r = \mathrm{Im} C_\Sigma = \{x = \sum_{\gamma \in [n-1]} \sum_{\delta \in [n-1]} \sum_{\zeta=1}^{m} \alpha(\gamma; \delta; \zeta) \left(\prod_{i=1}^{q} A_{ci}^{\gamma_i} \right) \left(\prod_{j=1}^{r} A_{dj}^{\delta_j} \right) b_\zeta | \\ \alpha(\gamma; \delta; \zeta) \in \mathbb{R},\ \gamma \in [n-1],\ \delta \in [n-1],\ \zeta \in \bar{m}\}. \end{aligned} \tag{4.7}$$

By Hamilton-Cayley Theorem applied to the matrices A_{ci}, A_{dj} the sums in (4.7) can be extended after the limits $n-1$ and we get

Theorem 4.3. The set of all reachable states of Σ is the subspace of $X = \mathbb{R}^n$

$$\begin{aligned} X_r = \{x = \sum_{0 \le \gamma \le \gamma^0} \sum_{0 \le \delta \le \delta^0} \sum_{\zeta=1}^{m} \alpha(\gamma; \delta; \zeta) \left(\prod_{i=1}^{q} A_{ci}^{\gamma_i} \right) \left(\prod_{j=1}^{r} A_{dj}^{\delta_j} \right) b_\zeta | \\ \alpha(\gamma; \delta; \zeta) \in \mathbb{R},\ 0 \le \gamma \le \gamma^0,\ 0 \le \delta \le \delta^0,\ \zeta \in \bar{m},\ \gamma^0 \in \mathbb{N}^q,\ \delta^0 \in \mathbb{N}^r\}. \end{aligned} \tag{4.8}$$

Definition 4.1. A subspace $\mathcal{V}$ of X is said to be (A_{ci}, A_{dj})-*invariant* if $A_{ci} v \in \mathcal{V},\ \forall v \in \mathcal{V},\ \forall i \in \overline{q},\ A_{dj} v \in \mathcal{V},\ \forall v \in \mathcal{V},\ \forall j \in \overline{r}$.

A subspace $\mathcal{V}$ of X is said to be $(A_{ci}, A_{dj}; B)$-*invariant* if $\mathcal{V}$ is (A_{ci}, A_{dj})-*invariant* and $\mathrm{Im}\, B \subset \mathcal{V}$.

Theorem 4.4. X_r is the minimal $(A_{ci}, A_{dj}; B)$-invariant subspace of $\mathbb{R}^n$.

Proof. Let $x \in X_r$ be an arbitrary vector and $l \in \overline{q}$ an arbitrary index. By (4.8), x has the form $x = \sum\limits_{0 \le \gamma \le \gamma^0} \sum\limits_{0 \le \delta \le \delta^0} \sum\limits_{\zeta=1}^{m} \alpha(\gamma; \delta; \zeta) \left(\prod\limits_{i=1}^{q} A_{ci}^{\gamma_i} \right) \left(\prod\limits_{j=1}^{r} A_{dj}^{\delta_j} \right) b_\zeta$. Then

$$A_{cl} x = \sum_{0 \le \gamma \le \gamma^0} \sum_{0 \le \delta \le \delta^0} \sum_{\zeta=1}^{m} \alpha(\gamma; \delta; \zeta) \left(\prod_{i=1}^{q} A_{ci}^{\overline{\gamma}_i} \right) \left(\prod_{j=1}^{r} A_{dj}^{\delta_j} \right) b_\zeta$$

where $\overline{\gamma}_i = \gamma_i$ for $i \in \overline{q},\ i \ne l$ and $\overline{\gamma}_l = \gamma_l + 1$, hence by (4.8) $A_{cl} x \in X_r$ and similarly $A_{dj} x \in X_r\ \forall j \in \overline{r}$, i.e. X_r is (A_{ci}, A_{dj})-invariant. Obviously, by (4.8) one obtains $b_\zeta \in X_r$, $\zeta \in \bar{m}$, hence $\mathrm{Im}\, B \subset X_r$. Therefore X_r is an $(A_{ci}, A_{dj}; B)$-invariant subspace.

Assume that $\mathcal{V}$ is any subspace of X which is (A_{ci}, A_{dj})-invariant and contains the columns b_ζ (i.e. $\mathrm{Im}\, B \subset \mathcal{V}$). Then we have $\left(\prod\limits_{i=1}^{q} A_{ci}^{\gamma_i} \right) \left(\prod\limits_{j=1}^{r} A_{dj}^{\delta_j} \right) b_\zeta \in \mathcal{V},\ \forall \gamma_i \ge 0,$

$\delta_j \geq 0$, $i \in \bar{q}$, $j \in \bar{r}$, $\zeta \in \bar{m}$. Since $\mathcal{V}$ is a subspace of $\mathbb{R}^n$, any linear combination of these vectors belongs to $\mathcal{V}$, hence by (4.8), $X_r \subset \mathcal{V}$, i.e. X_r is minimal. □

An immediate consequence of Theorem 4.4 is the following

Theorem 4.5. *The system Σ is completely reachable if and only if $X = \mathbb{R}^n$ is its minimal $(A_{ci}, A_{dj}; B)$-invariant subspace .*

5 Computation of the minimal $(A_{ci}, A_{dj}; B)$-invariant subspace

For the sake of clarity we shall consider the case of two-dimensional (2D) hybrid continuous-discrete linear systems, hence $q = r = 1$; the time set is $T = \mathbb{R} \times \mathbb{Z}$, the commutative drift matrices are A_1 and A_2 (instead of A_{c1} and A_{d1}, respectively).

A 2D hybrid system is the quintuplet

$$\Sigma = (A_1, A_2, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$$

with the state and the output equations

$$\dot{x}(t, k+1) = A_1 x(t, k+1) + A_2 \dot{x}(t, k) - A_1 A_2 x(t, k) + Bu(t, k)$$

$$y(t, k) = Cx(t, k) + Du(t, k)$$

The controllability matrix is

$$C_\Sigma = [\,B\ A_1B\ \ldots\ A_1^{n-1}B\ A_2B\ A_1A_2B\ \ldots\ A_1^{n-1}A_2B\ \ldots\ A_2^{n-1}B \quad A_1A_2^{n-1}B\ \ldots$$

$$\ldots A_1^{n-1}A_2^{n-1}B].$$

From Section 4 and [13] one obtains

Theorem 5.1. *The following statements are equivalent:*
(i) $\Sigma = (A_1, A_2, B)$ *is completely reachable* ;
(ii) $\mathrm{rank} C_\Sigma = n$;
(iii) *the condition: there exist* $v \in \mathbb{R}^n$ *and* $(t_0, k_0) \in T, (t, k) \in T, (t_0, k_0) < (t, k)$ *such that* $v^T e^{A_1 t} A_2^j B = 0$ *a.e. for* $(s, j) \in I'$ *implies* $v = 0$;
(iv) $\forall (t_0, k_0) \in T, (t, k) \in T, (t_0, k_0) < (t, k)$, *the reachability Gramian* $R(t_0, t; k_0, k)$ *is positive definite*;
(v) X *is the smallest subspace of* X *which is* (A_1, A_2)*-invariant and contains the columns of* B;
(vi) Σ *is not isomorphic to a system* $\tilde{\Sigma} = (\tilde{A}_1, \tilde{A}_2, \tilde{B})$ *of the form*

$$\tilde{A}_1 = \begin{bmatrix} A_{111} & A_{121} \\ 0 & A_{221} \end{bmatrix}, \ \tilde{A}_2 = \begin{bmatrix} A_{112} & A_{122} \\ 0 & A_{222} \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

with $A_{111}, A_{112} \in \mathbb{R}^{q\times q}, B_1 \in \mathbb{R}^{q\times m}, q < n$;
(vii) *there is no common left eigenvector of matrices* A_1 *and* A_2*, orthogonal on the columns of* B*;*
(viii) *for any* $\lambda_1, \lambda_2 \in \mathbb{C}$, $\mathrm{rank}\begin{bmatrix} B & \lambda_1 I - A_1 & \lambda_2 I - A_2 \end{bmatrix} = n$.

Theorem 4.4 can be rewritten as

Theorem 5.2. *The set X_r of all reachable states of Σ is the minimal subspace of X which is $(A_1, A_2; B)$-invariant (i.e. is (A_1, A_2)-invariant and contains the columns of B).*

Let the system $\Sigma = (A_1, A_2, B)$ be given, with A_1 and A_2 commutative matrices. We denote by $I(A_1, A_2; B)$ the minimal (A_1, A_2)-invariant subspace of X which contains the columns of B and $\mathcal{B} = \mathrm{Im}B$.

We propose the following algorithm for the computation of the subspace $X_r = I(A_1, A_2; B)$:

Algorithm 5.1.

Stage 1. Determine the controllability matrix C_Σ and $r = \text{rank}\, C_\Sigma$. If $r = n$, $I(A_1, A_2; B) = \mathbb{R}^n$. If $r < n$, go to Stage 2.

Stage 2. Construct the double-indexed sequence of subspaces $(S_{ij})_{0 \le i,j \le n-1}$ of the space $X = \mathbb{R}^n$:

$$S_{0,0} = \mathcal{B};$$

$$S_{0,j} = \mathcal{B} + A_2 S_{0,j-1}, j = 1, ..., n-1; \tag{5.1}$$

$$S_{i,0} = \mathcal{B} + A_1 S_{i-1,0}, i = 1, ..., n-1;$$

$$S_{i,j} = \mathcal{B} + A_1 S_{i-1,j} + A_2 S_{i,j-1}, i, j = 1, ..., n-1.$$

Stage3. Determine j, the first index $0 \le j \le n-2$ which verifies

$$S_{0,j+1} = S_{0,j}. \tag{5.2}$$

Stage 4. Determine i , the first index $0 \le i \le n-2$ which verifies

$$S_{i+1,j} = S_{i,j}. \tag{5.3}$$

Then $I(A_1, A_2; B) = S_{i,j}$.

Proof. Using (5.1) we can prove by induction the following formula:

$$S_{i,j} = \left(\sum_{k=0}^{i} A_1^k \right) \left(\sum_{l=0}^{j} A_2^k \right) \mathcal{B}, \tag{5.4}$$

where $A_1^0 \mathcal{B} = \mathcal{B}$.

Since $\sum_{k=0}^{i-1} A_l^k \subseteq \sum_{k=0}^{i} A_l^k,\ l = 1, 2$, by (5.4) one obtains

$$S_{i,j} \supseteq S_{i-1,j} \text{ and } S_{i,j} \supseteq S_{i,j-1} \ \ \forall i, j = 0, 1, ..., n-1. \tag{5.5}$$

In the chain of subspaces

$$\{0\} \subset S_{0,0} \subseteq S_{0,1} \subseteq ... \subseteq S_{0,k-1} \subseteq S_{0,k} \subseteq ... \subseteq S_{0,n-1} \subset \mathbb{R}^n$$

$\dim S_{0,0} = \text{rank}\, B \ge 1$ and $\dim S_{0,n-1} < n$ since $\text{rank}\, C_\Sigma < n$, hence the 1D system (A_2, B) is not completely controllable. Therefore there exists the at least one index

$0 \le j \le n-2$ such that $S_{0,j-1} = S_{0,j}$. Then, by (5.1) $S_{0,j+2} = \mathcal{B} + A_2 S_{0,j+1} = \mathcal{B} + A_2 S_{0,j} = S_{0,j+1} = S_{0,j}$ and in the same manner we obtain $S_{0,k} = S_{0,j}$ $\forall k \ge j+1$. By (5.1) again it follows:

$$S_{1,j+1} = \mathcal{B} + A_1 S_{0,j+1} = \mathcal{B} + A_1 S_{0,j} = S_{1,j}$$

and similarly we can prove by induction that $S_{k,j+1} = S_{k,j}$ and $S_{k,l} = S_{k,j}$ $\forall k \ge 0$ and $l \ge j$.

By considering the chain of subspaces

$$\{0\} \subset S_{0,j} \subseteq S_{1,j} \subseteq ... \subseteq S_{k-1,j} \subseteq S_{k,j} \subseteq ... \subseteq S_{n-1,j} \subset \mathbb{R}^n$$

and the fact that Σ is not completely controllable one finds the first index i such that $S_{i+1,j} = S_{i,j}$ and we get

$$S_{k,l} = S_{i,j} \ \ \forall k = i, ..., n-1, \ \forall l = j, ..., n-1. \tag{5.6}$$

By Theorem 4.3 and by (5.6) it follows that $S_{i,j} = S_{n-1,n-1}$ is (A_1, A_2)-invariant and it contains the columns of B.

Let $\mathcal{V}$ be any subspace of X which is (A_1, A_2)-invariant and which contains the columns of B. Obviously $\mathcal{V} \supseteq \mathcal{B} = S_{0,0}$. Assume that $\mathcal{V} \supseteq S_{0,l-1}$. Then, since $\mathcal{V}$ is A_2-invariant, we obtain by (5.1):

$$\mathcal{V} \supseteq \mathcal{B} + A_2\mathcal{V} \supseteq \mathcal{B} + A_2 S_{0,l-1} = S_{0,l},$$

hence $\mathcal{V} \supseteq S_{0,l}$, $\forall l$.

Similarly, if $\mathcal{V} \supseteq S_{k-1,l}$, then ($\mathcal{V}$ being A_1-invariant)

$$\mathcal{V} \supseteq \mathcal{B} + A_1\mathcal{V} \supseteq \mathcal{B} + A_1 S_{k-1,l} = S_{k,l}.$$

We proved by induction that $\mathcal{V} \supseteq S_{k,l}$ $\forall k, l$, hence $\mathcal{V} \supseteq S_{i,j}$. Therefore $S_{i,j}$ is minimal, i.e.

$$I(A_1, A_2; B) = S_{i,j}.$$

$\square$

The *Matlab* program presented below illustrates the above detailed bi-dimensional case, but it can be easily seen that it can be rewritten with no major difficulty for any other dimension by increasing the number of loops and even for an arbitrary number of matrices, by including the main loop in a "larger" one that takes into consideration the given state matrices one at a time. The formulae used in the program are $S_{0,j} = S_{0,j-1} + A_2 S_{0,j-1}$ for the first loop and $S_{i,j} = S_{i-1,j} + A_1 S_{i-1,j}$ for the second loop, both following easily from (5.4).

The instructions make use of the *m*-functions *ima* and *sums* included in the Geometric Approach toolbox published by G. Marro and G. Basile at

http://www3.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm

This GA toolbox works with with Matlab 5, Matlab 6 and Matlab 7 and the Control System Toolbox.

Given the matrices $A1$, $A2$ that commute and the matrix B, the following commands will compute and display the dimension of a basis and an orthonormal one in the space $S = I(A_1, A_2; B)$. For the sake of brevity the "error-detecting" instructions and the corresponding error messages (non-commuting matrices, non-matching dimensions) are being omitted.

```
% begin m-file
S = ima(B, 0); [n, dimInv] = size(S);
for j= 0:n-2       % first loop
  S = sums(S, A2*S); [n, m1] = size(S);
  if m1 == dimInv  break;    else  dimInv = m1;   end
end
for i= 0:n-2      % second loop
  S = sums(S, A1*S);[n, m1] = size(S);
  if dimInv == m1  break;    else dimInv = m1;    end
end
disp(['The minimal invariant space has the dimension ', ...
num2str(dimInv)])
disp('An orthonormal basis for the invariant space is:')
disp(S)
% end m-file
```

Conclusion. This paper studies a class of multidimensional hybrid linear systems from the point of view of reachability. In the case of time-varying systems, necessary and sufficient conditions are expressed by introducing a suitable reachability Gramian. The geometric characterization of the subspace of reachable states is given. By duality, it can be shown that the space of the unobservable states is the maximal subspace which is invariant with respect to the drift matrices and which is included in the kernel of the output matrix of the system. This maximal subspace can be used to obtain criteria of observability for multidimensional hybrid systems.

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