Non-existence of screen homothetic half lightlike submanifolds of an indefinite Kenmotsu manifold

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Abstract. We study screen homothetic half lightlike submanifolds of indefinite Kenmotsu manifolds. Two natural conditions to impose on this study are that its homothetic factor be either non-zero constant or zero, the latter is equivalent to the screen distribution to be totally geodesic. The purpose of this paper is to prove that there do not exist above two types screen homothetic half lightlike submanifolds of indefinite Kenmotsu manifolds subject to the conditions; (1) the co-screen distribution is parallel and (2) the lightlike transversal connection is flat.

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1 Introduction

In the classical theory of spacetime, while the rest spaces of timelike curves are spacelike subspaces of the tangent spaces, the rest spaces of null curves are lightlike subspaces of the tangent spaces [15]. To investigate this, Hawking and Ellis introduced the notion of so-called screen spaces in section 4.2 of their book [7]. Since for any semi-Riemannian manifold there is a natural existence of lightlike subspaces, Duggal-Bejancu [3] published their work on the general theory of degenerate (lightlike) submanifolds to fill a gap in the study of submanifolds. Since then there has been very active study on lightlike geometry of submanifolds (see up-to date results in two books [5, 6]). The geometry of lightlike submanifolds is used in mathematical physics, in particular, in general relativity since lightlike submanifolds can be models of different types of horizons (event horizons, Cauchy's horizons, Kruskal's horizons). Now we have lightlike version of a large variety of Riemannian submanifolds.

The class of lightlike submanifolds of codimension 2 is compose of two classes by virtue of the rank of its radical distribution, named by half lightlike and coisotropic submanifolds [4]. Half lightlike submanifold is a special case of r-lightlike submanifold [3] such that $r = 1$, and its geometry is more general form than that of coisotrophic

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submanifold. Much of the works on half lightlike submanifolds will be immediately generalized in a formal way to general r-lightlike submanifolds.

The objective of this paper is the study of half lightlike submanifolds M of indefinite Kenmotsu manifolds $\bar{M}$, whose shape operator is homothetic to the shape operator of its screen distribution by some constant φ, which is called the homothetic factor. The motivation for this geometric restriction comes from the classical geometry of non-degenerate submanifolds for which there are only one type of shape operator with its one type of respective second fundamental form. Two natural conditions to impose on this study are that its homothetic factor be either non-zero constant or zero, the latter is equivalent to the screen distribution to be totally geodesic. In this paper, we prove that there do not exist above two types screen homothetic half lightlike submanifolds of indefinite Kenmotsu manifolds subject to the conditions; (1) the co-screen distribution is parallel and (2) the lightlike transversal connection is flat.

2 Half lightlike submanifolds

An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be an *indefinite Kenmotsu manifold* [12, 13, 16] if there exist an almost contract metric structure $(J,\, \zeta,\, \theta,\, \bar{g})$, where J is a $(1,1)$-type tensor field, ζ is a vector field which is called the structure vector field and θ is a 1-form such that

$$J^2X = -X + \theta(X)\zeta, \;\; J\zeta = 0, \;\; \theta \circ J = 0, \;\; \theta(\zeta) = 1, \tag{2.1}$$
$$\theta(X) = \bar{g}(\zeta, X), \quad \bar{g}(JX, JY) = \bar{g}(X, Y) - \theta(X)\theta(Y),$$
$$\bar{\nabla}_X\zeta = -X + \theta(X)\zeta, \tag{2.2}$$
$$(\bar{\nabla}_X J)Y = -\bar{g}(JX, Y)\zeta + \theta(Y)JX, \tag{2.3}$$

for any vector fields X, Y on $\bar{M}$, where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{M}$.

A submanifold (M, g) of a semi-Riemannian manifold $\bar{M}$ of codimension 2 is called a *half lightlike submanifold* if the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of M is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$ of rank 1. Then there exists complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, which are called the *screen* and *co-screen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \;\; TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.4}$$

where the symbol $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Let ξ be a null section on $Rad(TM)$. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit vector field with $\bar{g}(L, L) = \pm 1$. In this paper we may assume that $\bar{g}(L, L) = 1$ without loss of generality. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$. Certainly ξ and L belong to $\Gamma(S(TM)^{\perp})$ and we have

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp},$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$. For any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined null vector field $N \in \Gamma(ltr(TM))$ satisfying

$$\bar{g}(\xi, N) = 1,\ \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0,\quad \forall\, X \in \Gamma(S(TM)).$$

We call N, $ltr(TM)$ and $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to $S(TM)$ respectively. Therefore $T\bar{M}$ is decomposed as

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned} \tag{2.5}$$

In the sequel, we denote by X, Y, Z, $U, \cdots$ the vector fields on M unless otherwise specified. Let P be the projection morphism of TM on $S(TM)$ with respect to the decompositions (2.4). Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \tag{2.6}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.7}$$

$$\bar{\nabla}_X L = -A_L X + \phi(X)N, \tag{2.8}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.9}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.10}$$

where ∇ and ∇^* are induced linear connections on TM and $S(TM)$ respectively, B and D are called the *local second fundamental forms* of M, C is called the *local second fundamental form* on $S(TM)$. A_N, A^*_ξ and A_L are linear operators on TM, which are called the *shape operator*, and τ, ρ and ϕ are 1-forms on TM. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and both B and D are symmetric. From the facts $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ and $D(X, Y) = \bar{g}(\bar{\nabla}_X Y, L)$, we know that B and D are independent of the choice of the screen distribution $S(TM)$ and satisfy

$$B(X, \xi) = 0, \quad D(X, \xi) = -\phi(X). \tag{2.11}$$

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\,\eta(Z) + B(X, Z)\,\eta(Y), \tag{2.12}$$

where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X, N).$$

But the connection ∇^* on $S(TM)$ is metric. The above three local second fundamental forms are related to their shape operators by

$$B(X, Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.13}$$

$$C(X, PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0, \tag{2.14}$$

$$D(X, Y) = g(A_L X, Y) - \phi(X)\eta(Y), \qquad \bar{g}(A_L X, N) = \rho(X). \tag{2.15}$$

Denote by $\bar{R}$, R and R^* the curvature tensors of $\bar{\nabla}$, ∇ and ∇^* respectively. Using the local Gauss-Weingarten formulas (2.6)$\sim$(2.10) for M and $S(TM)$, we have the

Gauss-Codazzi equations for M and $S(TM)$:

$$\begin{aligned}(2.16)\quad \bar{R}(X,Y)Z &= R(X,Y)Z\\ &+ B(X,Z)A_{N}Y - B(Y,Z)A_{N}X + D(X,Z)A_{L}Y - D(Y,Z)A_{L}X\\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\\ &\quad + \phi(X)D(Y,Z) - \phi(Y)D(X,Z)\}N,\\ &+ \{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) + \rho(X)B(Y,Z) - \rho(Y)B(X,Z)\}L,\end{aligned}$$

$$\begin{aligned}(2.17)\quad \bar{R}(X,Y)N &= -\nabla_X(A_{N}Y) + \nabla_Y(A_{N}X) + A_{N}[X,Y]\\ &+ \tau(X)A_{N}Y - \tau(Y)A_{N}X + \rho(X)A_{L}Y - \rho(Y)A_{L}X\\ &+ \{B(Y,A_{N}X) - B(X,A_{N}Y) + 2d\tau(X,Y) + \phi(X)\rho(Y) - \phi(Y)\rho(X)\}N\\ &+ \{D(Y,A_{N}X) - D(X,A_{N}Y) + 2d\rho(X,Y) + \rho(X)\tau(Y) - \rho(Y)\tau(X)\}L,\end{aligned}$$

$$\begin{aligned}(2.18)\quad \bar{R}(X,Y)L &= -\nabla_X(A_{L}Y) + \nabla_Y(A_{L}X) + A_{L}[X,Y]\\ &+ \phi(X)A_{N}Y - \phi(Y)A_{N}X\\ &+ \{B(Y,A_{L}X) - B(X,A_{L}Y) + 2d\phi(X,Y) + \tau(X)\phi(Y) - \tau(Y)\phi(X)\}N,\end{aligned}$$

$$\begin{aligned}(2.19)\quad R(X,Y)PZ &= R^*(X,Y)PZ + C(X,PZ)A^*_\xi Y - C(Y,PZ)A^*_\xi X\\ &+ \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + \tau(Y)C(X,PZ) - \tau(X)C(Y,PZ)\}\xi,\end{aligned}$$

$$\begin{aligned}(2.20)\quad R(X,Y)\xi &= -\nabla^*_X(A^*_\xi Y) + \nabla^*_Y(A^*_\xi X) + A^*_\xi[X,Y] - \tau(X)A^*_\xi Y\\ &+ \tau(Y)A^*_\xi X + \{C(Y,A^*_\xi X) - C(X,A^*_\xi Y) - 2d\tau(X,Y)\}\xi.\end{aligned}$$

Let $\nabla^\ell_X N = \pi(\bar{\nabla}_X N)$, where π is the projection morphism of $T\bar{M}$ on $ltr(TM)$ with respect to the decomposition (2.5). Then ∇^ℓ is a linear connection on the lightlike transversal vector bundle $ltr(TM)$ of M. We say that ∇^ℓ is the *lightlike transversal connection* of M. We define the curvature tensor R^ℓ on $ltr(TM)$ by

$$(2.21)\qquad R^\ell(X,Y)N = \nabla^\ell_X\nabla^\ell_Y N - \nabla^\ell_Y\nabla^\ell_X N - \nabla^\ell_{[X,Y]}N.$$

If R^ℓ vanishes identically, then the transversal connection ∇^ℓ is said to be *flat*. This definition comes from the definition of *flat normal connection* [2] in the theory of classical geometry of non-degenerate submanifolds.

From (2.7) and the definition of ∇^ℓ, we get $\nabla^\ell_X N = \tau(X)N$ for all $X \in \Gamma(TM)$. Substituting this equation into the right side of (2.21), we get

$$R^\ell(X,Y)N = 2d\tau(X,Y)N.$$

From this equation, we deduce the following result ([11]):

Theorem 2.1. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. Then the lightlike transversal connection of M is flat if and only if the 1-form τ is closed, i.e., $d\tau = 0$, on a coordinate neighborhood $\mathcal{U} \subset M$.*

Remark 2.1. We know that $d\tau$ is independent of the choice of the section ξ on $Rad(TM)$. In fact, if we take $\tilde{\xi} = \gamma\xi$ and $\tilde{\tau}(X) = \bar{g}(\bar{\nabla}_X \tilde{N}, \tilde{\xi})$, it follows that $\tau(X) = \tilde{\tau}(X) + X(\text{In}\,\gamma)$, where τ is given by $\tau(X) = \bar{g}(\bar{\nabla}_X N, \xi)$. If we take the exterior derivative d on the last equation, then we have $d\tau = d\tilde{\tau}$ [3].

In case $d\tau = 0$, by the cohomology theory there exist a smooth function l such that $\tau = dl$. Thus $\tau(X) = X(l)$. If we take $\tilde{\xi} = \gamma\xi$, then $\tau(X) = \tilde{\tau}(X) + X(\text{In}\,\gamma)$. Setting $\gamma = \exp(l)$ in this equation, we get $\tilde{\tau}(X) = 0$. We call the pair $\{\xi, N\}$ such that the corresponding 1-form τ vanishes the *canonical null pair* of M. Although $S(TM)$ is not unique and the lightlike geometry depends on its choice but it is canonically isomorphic to the factor vector bundle $S(TM)^\sharp = TM/Rad(TM)$ due to Kupeli [14]. Thus all $S(TM)$ are mutually isomorphic. In the sequel, we deal with only half lightlike submanifolds M equipped with the canonical null pair $\{\xi, N\}$.

3 Non-existence theorem and its corollaries

Definition 3.1. A half lightlike submanifold M of an indefinite Kenmotsu manifold $\bar{M}$ is *screen homothetic* [8, 9] if the shape operators A_N and A^*_ξ of M and $S(TM)$ respectively are related by $A_N = \varphi A^*_\xi$, or equivalently,

$$C(X, PY) = \varphi B(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM), \tag{3.1}$$

where φ is a constant on a coordinate neighborhood $\mathcal{U}$ in M. In particular, if $\varphi = 0$, i.e., $C = 0$ on $\mathcal{U}$, then we say that $S(TM)$ is *totally geodesic* [10] (in M), and if $\varphi \neq 0$ on $\mathcal{U}$, then we say that M is *proper screen homothetic*.

From (2.9), we show that the screen distribution $S(TM)$ is totally geodesic if and only if $S(TM)$ is a parallel distribution on M, i.e.,

$$\nabla_X Y \in \Gamma(S(TM)), \quad \forall\, X \in \Gamma(TM) \text{ and } Y \in \Gamma(S(TM)).$$

For the rest of this paper, by saying that M is screen homothetic we shall mean not only M is proper screen homothetic but also $S(TM)$ is totally geodesic in M.

Theorem 3.1. *Let M be a half lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ with parallel co-screen distribution. If the lightlike transversal connection is flat, then neither M is proper screen homothetic nor $S(TM)$ is totally geodesic in M.*

Proof. From the decomposition (2.5) of $T\bar{M}$, ζ is decomposed as follow:

$$\zeta = W + mN + nL, \tag{3.2}$$

where W is a smooth vector field on M and $m = \theta(\xi)$ and $n = \theta(L)$ are smooth functions. Substituting (3.2) in (2.2) and using (2.6)∼(2.8), we have

$$\nabla_X W = -X + \theta(X)W + mA_N X + nA_L X, \tag{3.3}$$

$$Xm + m\tau(X) + n\phi(X) + B(X, W) = m\theta(X), \tag{3.4}$$

$$Xn + m\rho(X) + D(X, W) = n\theta(X). \tag{3.5}$$

Substituting (3.4) and (3.5) into the following two equations

$$[X, Y]m = X(Ym) - Y(Xm), \quad [X, Y]n = X(Yn) - Y(Xn),$$

and using (2.16), (2.17), (2.18), (3.2), (3.4), (3.5), we have respectively

$$2m\, d\theta(X,Y) = \bar{g}(\bar{R}(X,Y)\zeta,\, \xi), \quad 2n\, d\theta(X,Y) = \bar{g}(\bar{R}(X,Y)\zeta,\, L). \tag{3.6}$$

Substituting (3.3) into $R(X,Y)W = \nabla_X\nabla_Y W - \nabla_Y\nabla_X W - \nabla_{[X,\,Y]}W$ and using (2.16)$\sim$(2.18), (3.3)$\sim$(3.6) and the fact ∇ is torsion-free, we have

$$\bar{R}(X,Y)\zeta = \theta(X)Y - \theta(Y)X + 2d\theta(X,Y)\zeta. \tag{3.7}$$

Taking the scalar product with ζ to (3.7) and using (2.1), we show that the structure 1-form θ is closed, i.e., $d\theta = 0$ on TM.

Călin [1] proved that if ζ is tangent to M, then it belongs to $S(TM)$ which we assume in this case. Replacing Y by ζ to (2.6) and using (2.2), we have

$$\nabla_X\zeta = -X + \theta(X)\zeta, \quad B(X,\zeta) = D(X,\zeta) = 0.$$

Taking the scalar product with N to the first equation and using (2.9), we have

$$C(X,\zeta) = -\,\eta(X).$$

In case either M is proper screen homothetic or $S(TM)$ is totally geodesic in M. Using above equations and the equation (3.1), we get

$$-\,\eta(X) = C(X,\zeta) = \varphi B(X,\zeta) = 0, \quad \forall\, X \in \Gamma(TM).$$

It is a contradiction as $\eta(\xi) = 1$. Thus the structure vector field ζ of $\bar{M}$ is not tangent to M. Consequently we show that $(m,\, n) \neq (0,\, 0)$.

As $S(TM^{\perp})$ is a parallel distribution, we have $A_{_L} = \phi = 0$ due to (2.8). By (2.15), we also have $D = \rho = 0$. Assume that $l = \theta(N) = 0$. Applying $\bar{\nabla}_X$ to $\bar{g}(\zeta, N) = 0$ and using (2.2) and (2.7), we have $\bar{g}(A_{_N}X, \zeta) = -\eta(X)$. Replacing X by ξ to this and using (3.1) and the fact $A^*_\xi\xi = 0$, we have $0 = \varphi\bar{g}(A^*_\xi\xi, \zeta) = -\eta(\xi) = -1$. It is a contradiction. Thus $l = \theta(N)$ is non-vanishing smooth function. Substituting (3.2) into (3.7) with $d\theta = 0$ and using (2.16)$\sim$ (2.18) and (3.6), we have

$$\begin{aligned} R(X,Y)W &= \theta(X)Y - \theta(Y)X + B(Y,W)A_{_N}X - B(X,W)A_{_N}Y \\ &\quad + m\{\nabla_X(A_{_N}Y) - \nabla_Y(A_{_N}X) - A_{_N}[X,\, Y] + \tau(Y)A_{_N}X - \tau(X)A_{_N}Y\}. \end{aligned} \tag{3.8}$$

Assume that M is proper screen homothetic or $S(TM)$ is totally geodesic in M. Taking the scalar product with N to (3.8) and using $(2.14)_2$, we get

$$g(R(X,Y)W, N) = \theta(X)\eta(Y) - \theta(Y)\eta(X). \tag{3.9}$$

As the lightlike transversal connection is flat, we have $d\tau = 0$. Therefore we can take a canonical null pair $\{\xi, N\}$ such that $\tau = 0$ due to Remark 2.2. Substituting $W = PW + l\xi$ into (3.9) and using (2.13), (2.19), (2.20) and (3.1), we have

$$(\nabla_X C)(Y, PW) - (\nabla_Y C)(X, PW) = \theta(X)\eta(Y) - \theta(Y)\eta(X), \tag{3.10}$$

for any $X,\, Y \in \Gamma(TM)$. Applying ∇_X to $C(Y, PW) = \varphi B(Y, W)$, we get

$$(\nabla_X C)(Y, PW) = \varphi(\nabla_X B)(Y, W).$$

Substituting this equation into (3.10) and using $(3.6)_1$ with $d\tau = 0$, we have

$$\theta(X)\eta(Y) - \theta(Y)\eta(X) = 0, \quad \forall\, X,\, Y \in \Gamma(TM).$$

Replacing Y by ξ to this equation, we have $g(X, W) = 0$ for all $X \in \Gamma(TM)$. This implies $W = l\xi$. Consequently the structure vector field ζ is decomposed as

$$\zeta = l\xi + mN + nL. \tag{3.11}$$

Applying $\bar{\nabla}_X$ to (3.11) and using (2.2), (2.7), (2.8), (2.10) and (3.1), we have

$$\begin{aligned} &-X + l\theta(X)\xi + m\theta(X)N + n\theta(X)L \\ &\quad = -(l + m\varphi)A^*_\xi X + X[l]\xi + X[m]N + X[n]L. \end{aligned}$$

Taking the scalar product with ξ, N and L to this result by turns, we obtain

$$X[m] = m^2\eta(X), \quad X[l] = (lm - 1)\eta(X), \quad X[n] = mn\eta(X), \tag{3.12}$$

respectively, and we get $(l + m\varphi)A^*_\xi X = PX$. From this result we show that $l + m\varphi$ is non-vanishing smooth function. Putting $\alpha = (l + m\varphi)^{-1}$, we have

$$A^*_\xi X = \alpha PX, \quad B(X, Y) = \alpha g(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.13}$$

Applying ∇_X to $\alpha = (l + m\varphi)^{-1}$ and $l\alpha$, and then, using $(3.12)_{1,\,2}$, we have

$$X[\alpha] = \alpha(\alpha - m)\eta(X), \quad X[l\alpha] = \alpha(l\alpha - 1)\eta(X), \quad \forall\, X \in \Gamma(TM). \tag{3.14}$$

It is known [11] that, for any half lightlike submanifold of an indefinite almost contact metric manifold $\bar{M}$, all of the distributions $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^\perp))$ are vector subbundles of $S(TM)$, of rank 1. Applying $\bar{\nabla}_X$ to $\bar{g}(JN, L) = 0$ and using (2.1), (2.3), (2.7), (2.8), (3.1) and (3.13), we have

$$ng(X, JN) + (\alpha\varphi - l)g(X, JL) = 0, \quad \forall\, X \in \Gamma(TM).$$

Replacing X by $J\xi$ and JL to this by turns and using $(2.1)_6$, we have

$$n(1 - m\alpha\varphi) = 0, \quad (1 - n^2)\alpha\varphi = l, \tag{3.15}$$

respectively. As $m\alpha\varphi = 1 - l\alpha$ we have $ln\alpha = 0$ by $(3.15)_1$. Applying $\bar{\nabla}_X$ to $\bar{g}(J\xi, N) = 0$ and using (2.1), (2.3), (2.7), (2.10), (3.1) and (3.13), we have

$$(l - \alpha\varphi)g(X, J\xi) + (\alpha - m)g(X, JN) = 0, \quad \forall\, X \in \Gamma(TM). \tag{3.16}$$

Replacing X by JL to this and using $(2.1)_6$, we get $n = 2\,ln\alpha$ as $m\alpha\varphi = 1 - l\alpha$. Since $ln\alpha = 0$, we have $n = 0$ and $\alpha\varphi = l$ due to $(3.15)_2$.

Case 1. Assume that $S(TM)$ is totally geodesic in M. Then we have $\varphi = 0$. From the facts $\alpha\varphi = l$, we get $l = 0$. It is a contradiction as $l \neq 0$. Thus $S(TM)$ is not totally geodesic in M.

Case 2. Assume that M is proper screen homothetic. As $(m, n) \neq (0, 0)$ and $n = 0$, we have $m \neq 0$. Consequently we get

$$\zeta = l\xi + mN, \quad 2ml = 1, \quad J\xi = -2m^2 JN.$$

Replacing X by $J\xi$ to (3.16) satisfying $\alpha\varphi = l$ and using $2lm = 1$, we have $\alpha = m$. From this, (3.12) and (3.14), we have $m^2\eta(X) = 0$ for all $X \in \Gamma(TM)$. Thus we have $m = 0$. It is a contradiction as $m \neq 0$. Thus M is not proper screen homothetic. This completes the proof of the theorem. $\square$

Corollary 3.2. *Let M be a half lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ with parallel co-screen distribution. If M is locally symmetric, then $S(TM)$ is not totally geodesic in M.*

Proof. Assume that $S(TM)$ is totally geodesic in M. (3.8) reduce to

$$R(X,Y)W = \theta(X)Y - \theta(Y)X, \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.17}$$

Applying $\bar{\nabla}_X$ to $\theta(Y) = g(Y,\zeta)$ and using (2.2) and (2.5), we have

$$(\nabla_X\theta)(Y) = lB(X,Y) - g(X,Y) + \theta(X)\theta(Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.18}$$

Applying ∇_Z to (3.17) and using (3.3), (3.18) and the fact $\nabla_Z R = 0$, we have

$$R(X,Y)Z = \{g(X,Z) - lB(X,Z)\}Y - \{g(Y,Z) - lB(Y,Z)\}X, \tag{3.19}$$

for all X, $Y \in \Gamma(TM)$. Replacing Z by ξ to (3.19) and using $(2.11)_1$, we have

$$R(X,Y)\xi = 0, \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.20}$$

Comparing the $Rad(TM)$-components of (2.20) and (3.20) and using $C = 0$, we have $d\tau = 0$. This implies that the lightlike transversal connection is flat. Thus, by Theorem 3.1, we show that $S(TM)$ is not totally geodesic in M. $\square$

Definition 3.2. An indefinite Kenmotsu manifold $\bar{M}$ is called an *indefinite Kenmotsu space form*, denoted by $\bar{M}(\bar{c})$, if it has the constant J-sectional curvature c [13]. The curvature tensor $\bar{R}$ of this space form $\bar{M}(\bar{c})$ is given by

$$\begin{aligned} 4\bar{R}(X,Y)Z &= (\bar{c}-3)\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\ &+ (\bar{c}+1)\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X + \bar{g}(X,Z)\theta(Y)\zeta - \bar{g}(Y,Z)\theta(X)\zeta \\ &+ \bar{g}(JY,Z)JX + \bar{g}(JZ,X)JY - 2\bar{g}(JX,Y)JZ\}, \ \ \forall\, X,\, Y,\, Z \in \Gamma(T\bar{M}). \end{aligned}$$

It is well known [13] that if an indefinite Kenmotsu manifold $\bar{M}$ is a space form, then it is Einstein and $\bar{c} = -1$, i.e., $\bar{R}$ is given by

$$\bar{R}(X,Y)Z = \bar{g}(X,Z)Y - \bar{g}(Y,Z)X, \quad \forall\, X,\, Y,\, Z \in \Gamma(T\bar{M}). \tag{3.21}$$

Corollary 3.3. *Let M be a half lightlike submanifold of an indefinite Kenmotsu space form $\bar{M}(c)$ with parallel co-screen distribution. Then neither M is proper screen homothetic nor $S(TM)$ is totally geodesic.*

Proof. Assume that M is screen homothetic. As $S(TM^{\perp})$ is parallel distribution, we get $A_{_L} = \phi = 0$. Taking the scalar product with ξ to (2.16) and (3.21) and then, comparing the resulting two equations, we have

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) = 0. \tag{3.22}$$

Comparing $(3.6)_1$ with $d\theta = 0$ and (3.22) with $Z = W$ and then, using (3.1) and the fact $m \neq 0$, we have $d\tau = 0$. Thus, by Theorem 2.1, we show that the transversal connection is flat. Thus, by Theorem 3.1, we show that Then neither M is proper screen homothetic nor $S(TM)$ is totally geodesic. $\square$

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