Banach Lie algebroids and Dirac structures

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Abstract. We consider the category of anchored Banach vector bundles and we discuss the notion of semispray. Adding on the set of sections of an anchored Banach vector bundle a Lie bracket with some properties one gets the notion of Lie algebroid. We prove that the Lie algebroids form also a category. A Dirac structure on a Banach manifold M is defined as a subbundle of the big tangent bundle $TM \oplus T^*M$ that equals its orthocomplement with respect to the standard neutral metric and is closed with respect to the Courant bracket. Various characterizations of this closeness are provided. We show that with a convenient anchor any Dirac structure becomes a Banach Lie algebroid. Some examples are included.

M.S.C. 2010: 53D17, 58A99.
Key words: Banach Lie algebroids; Dirac structures.

1 Introduction

The notion of Lie algebroid was extended to the category of Banach vector bundles by the present author [2], and independently by F. Pelletier [13]. The next paper studying Banach Lie algebroids is due to P. Cabau and F. Pelletier [4]. Here several well known results in finite dimensional case are extended and some obstructions produced by the new framework are revealed. Moreover, an application to the mechanical systems with constraints is worked out. In a different direction, C. Ida [10] considers the coomology of Banach Lie algebroids and proves that if $(M, \pi_)$ is a Banach Poisson manifold, the Banach Lie algebroid cohomology of $(T^*M, \{.,\}, \sharp_\pi)$ is the Lichnerowicz-Poisson cohomology of (M, π). Next steps are done by M. Anastasiei and A. Sandovici, [3]. They introduced the Dirac structures on Banach manifolds and related them to Lie algebroids.

Dirac structures on finite dimensional manifolds were introduced by T. Courant and A. Weinstein (see [7]) and were systematically studied by T. Courant in [6]. They became an important tool in generalized geometry by studies of I. Vaisman (see [15]. In the geometrical context, it must be pointed out the usefulness of Dirac structures in the study of interconnected systems, [5]. Following the direction opened by S. Vacaru in [14], certain applications of Dirac structures in Theoretical Physics

Balkan Journal of Geometry and Its Applications, Vol. 18, No. 1, 2013, pp. 1-11.

could appear. Dirac structures were used in the study of the mechanical systems described by constraint Hamiltonian systems or implicit Lagrangian systems ([16, 17]).

Another field where Dirac structures are useful is the study of the integrability of the nonlinear evolution equations, [8]. In the monograph [9] I. Dorfman emphasized the necessity of considering Dirac structures on infinite dimensional spaces.

A first study of the concept of a Dirac structure within the framework offered by infinite-dimensional smooth manifolds is due to the present author and A. Sandovici, [3]. Here the well known result that an integrable Dirac structure defines a Lie algebroid is extended to Banach manifold category.

The main difficulty in studying the Lie algebroids and the Dirac structures in the infinite–dimensional vector bundles in the greatest generality is the absence of a property of localization for the sections of a Banach vector bundle. There are cases when the property holds, for instance if the model of fibers is a reflexive Banach space but even in these cases every assertion from the finite dimensional case has to be verified and very often the proof is different from that existing in finite dimensional setting. Our main reference for the geometry of infinite dimensional manifolds is S. Lang's book [11].

This paper contains a survey of author's results on Lie algebroids and Dirac structures in the category of Banach vector bundles. Its content is as follows.

In Section 1 we consider the anchored Banach vector bundles and we show that they form a category. Also, we recall the notion of semispray (see [1]), we mention several examples and exhibit for each one the so called adapted curves. In Section 2 we recall the definition of a Banach Lie algebroid and we show that the set of these algebroids and their morphisms form a category. Section 3 is devoted to the Dirac structures. On the set of sections of the big tangent bundle $TM \oplus T^*M$ one defines the neutral metric and a subbundle of $TM \oplus T^*M$ is called an almost Dirac structure or an almost Dirac bundle if it equals its orthocomplement with respect to the neutral metric. Then one recalls the Courant bracket and one computes its Jacobiator. An almost Dirac structure is called a Dirac structure or a Dirac bundle if it closed with respect to the Courant bracket. Some characterizations of this closeness are provided. Examples of Dirac structures are given. In the end one proves that any Banach Dirac bundle is a Banach Lie algebroid.

2 Anchored vector bundles

Let M be a smooth i.e. C^∞ Banach manifold modeled on Banach space $\mathbb{M}$ and let $\pi : E \to M$ be a Banach vector bundle whose type fiber is a Banach space $\mathbb{E}$. We denote by $\tau : TM \to M$ the tangent bundle of M.

Definition 2.1. We say that the vector bundle $\pi : E \to M$ is an anchored vector bundle if there exists a vector bundle morphism $\rho : E \to TM$. The morphism ρ will be called the anchor map.

Let $\mathcal{F}(M)$ be the ring of smooth real functions on M.

We denote by $\Gamma(E)$ the $\mathcal{F}(M)$-module of smooth sections in the vector bundle (E, π, M) and by $\mathcal{X}(M)$ the module of smooth sections in the tangent bundle of M (vector fields on M).

The vector bundle morphism ρ induces an $\mathcal{F}(M)$-module morphism which will be denoted also by $\rho : \Gamma(E) \to \mathcal{X}(M)$, $\rho(s)(x) = \rho(s(x))$, $x \in M, s \in \Gamma(E)$.

Let $\{(U, \varphi), (V, \psi), \ldots\}$ be an atlas on M. Restricting U, V... if necessary we may choose a vector bundle atlas $\{(\pi^{-1}(U), \overline{\varphi}), (\pi^{-1}(V), \overline{\psi}), \ldots\}$ with $\overline{\varphi} : \pi^{-1}(U) \to U \times \mathbb{E}$ given by $\overline{\varphi}(u) = (\pi(u), \overline{\varphi}_{\pi(u)})$, where $\overline{\varphi}_{\pi(u)} : E_{\pi(u)} \to \mathbb{E}$ is a toplinear isomorphism. Here $E_{\pi(u)}$, is the fiber of (E, π, M) in $x = \pi(u) \in M$. The given atlas on M together with a vector bundle atlas induce a smooth atlas $\{(\pi^{-1}(U), \Phi), (\pi^{-1}(U), \Psi), \ldots\}$ on E such that E becomes a Banach manifold modeled on the Banach space $\mathbb{M} \times \mathbb{E}$. The map $\Phi : \pi^{-1}(U) \to \varphi(U) \times \mathbb{E}$ is given by

$$\Phi(u) = (\varphi(\pi(u)), \overline{\varphi}_{\pi(u)}(u)), \ \ u \in E,$$

and the changes of local charts $(\Psi \circ \Phi^{-1})$ are given by $(\psi \circ \varphi^{-1}, \overline{\psi}_x \circ (\overline{\varphi}_x)^{-1})$, where $M_x := \overline{\psi}_x \circ (\overline{\varphi}_x)^{-1} : \mathbb{E} \mapsto \mathbb{E}$ is a toplinear isomorphism.

For a section $s : U \to \pi^{-1}(U)$, its local representation $\phi \circ s \circ \varphi^{-1} : \varphi(U) \to \varphi(U) \times \mathbb{E}$ given by $(\phi \circ s \circ \varphi^{-1})(\varphi(x)) = (\varphi\pi(s(x)), \overline{\varphi}_{\pi(s(x))}(s(x)) = (\varphi(x), \overline{\varphi}_x(s(x)))$ is completely determined by the map $s_\varphi : \varphi(U) \to \mathbb{E}$ given by $s_\varphi(\varphi(x)) = \overline{\varphi}_x(s(x))$ which will be called the local representative (shortly l.r.) of s. On $U \cap V$ we may speak also of the l.r. s_ψ of a section $s : U \cup V \to \pi^{-1}(U \cap V)$ given by $s_\psi(\psi(x)) = \overline{\psi}_x(s(x))$. It is clear that we have

$$s_\psi(\psi(x)) = \overline{\psi}_x \circ (\overline{\varphi}_x)^{-1}(s_\varphi(\varphi(x))) = M_x(s_\varphi(\varphi(x)), \ \ x \in U \cup V. \tag{2.1}$$

For a vector field $X : U \to \tau^{-1}(U)$ we have a l.r. $X_\varphi : \varphi(U) \to \mathbb{M}$ and on $U \cap V$ we have also a l.r. X_ψ and one holds

$$X_\psi(\psi(x)) = h'(\varphi(x))(X_p(\varphi(x))), \ \ x \in U \cap V, \tag{2.2}$$

where prime means Frechet differentiation and we have set $h = \psi \circ \varphi^{-1}$.

Locally, ρ reduces to a morphism $U \times \mathbb{E} \to U \times \mathbb{M}$, $(x, v) \to (x, \rho_U(x)v)$ with $\rho_U(x) \in L(\mathbb{E}, \mathbb{M})$. We call $\rho_U(x)$ the l.r. of ρ. On overlaps of local charts one easily gets

$$\rho_V(x) \circ M_x = h'(\varphi(x)) \circ \rho_U(x), \ \ x \in U \cap V. \tag{2.3}$$

Examples.

1. The tangent bundle of M is trivially anchored vector bundle with $\rho = I$ (identity). In (3) we have $M_x = h'(\varphi(x))$.

2. Let A be a tensor field of type $(1, 1)$ on M. It is regarded as a section of the bundle of linear mappings $L(TM, TM) \to M$ and also as a morphism $A : TM \to TM$. In the other words, A may be thought as an anchor map. The l.r A_U is a linear operator on $\mathbb{M}$.

3. Any subbundle of TM is an anchored vector bundle with the anchor the inclusion map in TM.

4. Let now $\zeta : F \to M$, where F is a Banach manifold, be a submersion and ζ_* be the differential (tangent map) of ζ. The union of subspaces $(\zeta_*)^{-1}(u), u \in F$ provides a subbundle of the tangent bundle of F, $\tau_F : TF \mapsto F$, denoted by VF and called the vertical subbundle. As a subbundle, by the Example 3), this is an anchored vector bundle.

5. Let $\tau^* : T^*M \mapsto M$ be the cotangent bundle of M and the Whitney sum $\tau \oplus \tau^* : TM \oplus T^*M \mapsto M$ called sometimes the big tangent bundle. This is an anchored bundle with the anchor given by the projection $pr_1 : TM \oplus T^*M \mapsto TM$.

Theorem 2.1. *The anchored vector bundles form a category.*

Proof. The objects are the pairs (E, π_E, M, ρ_E) with ρ_E the anchor of E and the category morphism $(f, \phi) : (E, \pi_E, M, \rho_E) \to (F, \pi_H, N, \rho_F)$ is a vector bundle morphism $(f, \phi) : E \to F$ which verifies the condition $\rho_F \circ f = \phi_* \circ \rho_E$, where ϕ_* is the tangent map of $\phi : M \mapsto N$.

Now, if (E, π, M, ρ) is an anchored vector bundle, we consider on E a special vector field called semispray. $\square$

Definition 2.2. A vector field S on E, that is a section $S : E \mapsto TE$ will be called a semispray if $\pi_* \circ S = \rho$.

The condition that S is a semispray can be also written in the form

$$\pi_{*,u}(S(u)) = \rho(u) = (\rho \circ \tau_E)(S(u)), \ \ u \in E.$$

Let $c : J \to E$ for $0 \in J \subset \mathbb{R}$ a curve on E. The differential of c is $c_* : J \times \mathbb{R} \to TE$ and using $\mathrm{i} : J \to J \times \mathbf{R}$, $t \to (t, 1)$, $t \in J$ we set $c'(t) = c_* \circ \mathrm{i}$.

It is clear that $\pi \circ c$ is a curve on M and it easily follows that $(\pi \circ c)'(t) = \pi_{*,c(t)} \circ c'(t)$.

Definition 2.3. A curve c on E will be called admissible if $(\pi \circ c)'(t) = \rho(c(t))$, $\forall t \in J$.

Locally, if $c : J \to \varphi(U) \times \mathbb{E}$, $t \to (x(t), w(t))$ then $\pi \circ c : J \to \varphi(U)$ is $t \to x(t)$, $t \in J$.

It follows that c is an admissible curve if and only if

$$\frac{dx}{dt} = \rho_U(x(t))w(t), \ \ t \in J \tag{2.4}$$

Theorem 2.2. *A vector field S on E is a semispray if and only if all its integral curves are admissible curves.*

For a proof we refer to [2].

In the same paper [2] one proves

Theorem 2.3. *A vector field S on E is a semispray if and only if its l.r. $S_\varphi : \varphi(U) \times \mathbb{E} \to \varphi(U) \times \mathbb{E} \times \mathbb{M} \times \mathbb{E}$ has the form*

$$S_\varphi(x, u) = (x, u, \rho_U(x)u, -2G_\varphi(x, u)),$$

where the functions G_φ satisfy

$$G_\psi(h(x), M(x)u) = M(x)G_\varphi(x, u) - \frac{1}{2}M'(x)(\rho_U(x)u)u$$

on overlaps of local charts.

Remark 2.4. 1. In the Example 1 the admissible curves $(x(t), y(t))$ on TM are those that satisfy $y(t) = \dot{x(t)} := \frac{dx}{dt}$. Such a curve is called the tangent lift of the curve $x(t)$ on M. Thus the system $\dot{x} = y, \dot{y} + 2G_\varphi(x, \dot{x}) = 0$ giving the integral curves of the semispray S has as solutions, by the Theorem 1.2, only the tangent lifts of curves on M. This system is equivalent to the SODE (second order differential equation) $\ddot{x} + 2G_\varphi(x, \dot{x}) = 0$. So a semispray is in this case simply a SODE on M. In the other cases the integral curves of a semispray are solutions of a system of first order differential equations on E.

2. In the Example 2 the admissible curves are solutions of the system $\dot{x} = A(x)w$, $\dot{w} + 2G_\varphi(x, w) = 0$. If $A(x)$ is invertible we can find w as a function of $\dot{x}$ and inserting it in the second equation we get a SODE in the form $\ddot{x}+2\tilde{G}_\varphi(x, \dot{x}) = 0$. This defines a semispray $\tilde{S}$ projectively equivalent with S.

3. In the Example 3 the admissible curves are the tangent lifts $(x(t), \dot{x}(t))$, with the tangent vector $\dot{x}$ in the subbundle (distribution) $F \subset TM$. One says that such a curve is tangent to the distribution F. It follows that the functions G_φ are $\mathbb{F}$-valuated, where $\mathbb{F}$ is the type fibre of the subbunle F.

4. In the Example 4 the admissible curve $(x(t), y(t))$ are given by the equation $\dot{x} = 0$. Thus any curve in the fibre E_{x_0} is an admissible curve. This is also a consequence of the fact that the vertical distribution on E is integrable with the fibres as leaves.

3 Lie algebroids

Let $\pi : E \to M$ be an anchored Banach vector bundle with the anchor $\rho_E : E \to TM$ and the induced morphism $\rho_E : \Gamma(E) \to \mathcal{X}(M)$.

Assume there exists defined a bracket $[,]_E$ on the space $\Gamma(E)$ that provides a structure of real Lie algebra on $\Gamma(E)$.

Definition 3.1. The triplet $(E, \rho_E, [,]_E)$ is called a Banach Lie algebroid if

(i) $\rho : (\Gamma(E), [,]_E) \to (\mathcal{X}(M), [,])$ is a Lie algebra homomorphism and

(ii) $[fs_1, s_2]_E = f[s_1, s_2]_E - \rho_E(s_2)(f)s_1$, for every $f \in \mathcal{F}(M)$ and $s_1, s_2 \in \Gamma(E)$.

Examples:

1. The tangent bundle $\tau : TM \to M$ is a Banach Lie algebroid with the anchor the identity map and the usual Lie bracket of vector fields on M.

2. For any submersion $\zeta : F \to M$,where F is a Banach manifold, the vertical bundle VF over F is an anchored Banach vector bundle. As the Lie bracket of two vertical vector fields is again a vertical vector field it follows that $(VF, i, [,]_{VF})$, where $i : VF \to TF$ is the inclusion map is a Banach Lie algebroid. This applies, in particular, to any Banach vector bundle $\pi : E \to M$.

Let $\Omega^q(E) := \Gamma(L^a_q(E))$ be the $\mathcal{F}(M)-$ module of differential forms of degree q. Its elements are sections of the vector bundle of alternating multilinear forms on E, see [11], p. 61. In particular, $\Omega^q(TM)$ will be denoted by $\Omega^q(M)$. The differential operator $d_E : \Omega^q(E) \to \Omega^{q+1}(E)$ is given by the formula

$$(d_E\omega)(s_0, \ldots, s_q) = \sum_{i=0,\ldots,n} (-1)^i \rho_E(s_i)\omega(s_0, \ldots, \widehat{s}_i, \ldots, s_q)$$
$$+ \sum_{0\leq i<j\leq q} (-1)^{i+j}(\omega([s_i, s_j]_E), s_0, \ldots \widehat{s}_i, \ldots, \widehat{s}_j, \ldots, s_q), \tag{3.1}$$

for $s_1, \ldots, s_q \in \Gamma(E)$, where hat over a symbol shows that symbol must deleted.

Let (E', π', M) be a Banach vector bundle and $(E', \rho_{E'}, [,]_{E'})$ a Banach Lie algebroids based on it.

Definition 3.2. A vector bundle morphism $f : E \to E'$ over $f_0 : M \to M'$ is a morphism of the Banach Lie algebroids $(E, \rho_E, [,]_E$ and $(E', \rho_{E'}, [,]_{E'})$ if the map induced on forms $f^* : \Omega^q(E') \to \Omega^q(E)$ defined by $(f^*\omega')_x(s_1, \ldots, s_q) = \omega'_{f_0(x)}(fs_1, \ldots, fs_q)$, $s_1, \ldots, s_2 \in \Gamma(E)$ commutes with the differential, i.e.,

$$d_E \circ f^* = f^* \circ d_E. \tag{3.2}$$

Using this definition it is easy to prove

Theorem 3.1. *The Banach Lie algebroids with the morphisms defined in the above form a category.*

4 Dirac structures

Let M be a smooth i.e. C^∞ Banach manifold modeled on Banach space $\mathbf{M}$ and the big tangent bundle $TM \oplus T^*M \to M$. On the space of sections $\Gamma(TM \oplus T^*M)$ one defines a symmetric bilinear operation by:

$$\langle (X, \alpha), (Y, \beta)\rangle_+ = \alpha(Y) + \beta(X), \tag{4.1}$$

for all (X, α), $(Y, \beta) \in \Gamma(TM \oplus T^*M)$.

For a subbundle $\mathfrak{B} \subset TM \oplus T^*M$ define the orthogonal complement $\mathfrak{B}^\perp$ by:

$$\mathfrak{B}^\perp = \{(X, \alpha) \in TM \oplus T^*M \,:\, \langle (X, \alpha), (Y, \beta)\rangle_+ = 0, \text{for all}\,(Y, \beta) \in \mathfrak{B}\}.$$

Definition 4.1. An almost Banach Dirac bundle, or an almost Banach Dirac structure, on M is a vector subbundle $\mathfrak{D}$ of $TM \oplus T^*M \to M$ which satisfies the following equality:

$$\mathfrak{D} = \mathfrak{D}^\perp. \tag{4.2}$$

In the case M is a finite–dimensional smooth manifold ($\dim M = n$) the definition of an almost Dirac structure $\mathfrak{D}$ is equivalent to the fact that $\mathfrak{D}$ is maximally isotropic with respect to the pairing $\langle \cdot, \cdot\rangle_+$. More precisely, this means that $\mathfrak{D}$ is a subbundle of rank n and the restriction of $\langle\cdot,\cdot\rangle_+$ to $\mathfrak{D}\times\mathfrak{D}$ vanishes identically; see for instance [6].

Define the Courant bracket on $\Gamma(TM\oplus T^*M)$, that is, the skew–symmetric $\mathbb{R}$–bilinear operation defined by:

$$[(X,\alpha),(Y,\beta)]_C=\left([X,Y],\mathcal{L}_X\beta-\mathcal{L}_Y\alpha+\frac{1}{2}d(\alpha(Y)-\beta(X))\right), \tag{4.3}$$

where $[\cdot,\cdot]$ is the usual Lie bracket of vector fields and $\mathcal{L}_X=d\circ i_X+i_X\circ d$ is the Lie derivation by X.

The bracket $[\cdot,\cdot]_C$ is skew-symmetric but, in general, does not satisfy the Jacobi identity. Denote by (J_1,J_2) the Jacobiator corresponding to $[\cdot,\cdot]_C$, i.e.

$$\begin{aligned}(J_1,J_2) &= [[(X,\alpha),(Y,\beta)]_C\,,(Z,\gamma)]_C\\ &+ [[(Y,\beta),(Z,\gamma)]_C\,,(X,\alpha)]_C\\ &+ [[(Z,\gamma),(X,\alpha)]_C\,,(Y,\beta)]_C\,,\end{aligned}$$

for all (X,α), (Y,β), $(Z,\gamma)\in\Gamma(TM\oplus T^*M)$. Clearly,

$$J_1=[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0.$$

On the other hand for J_2 the following result holds:

Lemma 4.1. *The second component J_2 of the Jacobiator corresponding to the Courant bracket $[\cdot,\cdot]_C$ is given by:*

$$\begin{aligned}J_2 &= \frac{1}{4}d\left(\mathcal{L}_X(\beta(Z)-\gamma(Y))+\mathcal{L}_Y(\gamma(X)-\alpha(Z))+\mathcal{L}_Z(\alpha(Y)-\beta(X))\right)\\ &+ \frac{1}{2}d\left(\gamma([X,Y])+\alpha([Y,Z])+\beta([Z,X])\right),\end{aligned}\tag{4.4}$$

for all (X,α), (Y,β), $(Z,\gamma)\in\Gamma(TM\oplus T^*M)$.

In particular, the restriction of J_2 to an almost Banach Dirac bundle $\mathfrak{D}$ is given by:

$$\begin{aligned}J_2 &= \frac{1}{2}d\left(\mathcal{L}_X(\beta(Z))+\mathcal{L}_Y(\gamma(X))+\mathcal{L}_Z(\alpha(Y))\right)\\ &+ \frac{1}{2}d\left(\gamma([X,Y])+\alpha([Y,Z])+\beta([Z,X])\right).\end{aligned}\tag{4.5}$$

Definition 4.2. Define the map $T:(\Gamma(TM\oplus T^*M))^3\to\mathbb{R}$ by:

$$T((X,\alpha),(Y,\beta),(Z,\gamma))=\langle[(X,\alpha),(Y,\beta)]_C,(Z,\gamma)\rangle_+,$$

where (X,α), (Y,β), $(Z,\gamma)\in\Gamma(TM\oplus T^*M)$.

Theorem 4.2. *Let $\mathfrak{D}$ be an almost Banach Dirac bundle. Then $T_{\mathfrak{D}}$, the restriction of T to $\mathfrak{D}$ satisfies the following identity:*

$$\begin{aligned}T_{\mathfrak{D}}((X,\alpha),(Y,\beta),(Z,\gamma)) &= \alpha([Y,Z])+\beta([Z,X])+\gamma([X,Y])\\ &+\mathcal{L}_X\left(\beta(Z)\right)+\mathcal{L}_Y\left(\gamma(X)\right)+\mathcal{L}_Z\left(\alpha(Y)\right).\end{aligned}\tag{4.6}$$

Moreover, the second component J_2 of the Jacobiator restricted to $\mathfrak{D}$ and the restriction $T_{\mathfrak{D}}$ of T to $\mathfrak{D}$ are related as follows:

$$J_2 \restriction_{\mathfrak{D}} ((X,\alpha),(Y,\beta),(Z,\gamma)) = \frac{1}{2} d\left(T_{\mathfrak{D}}((X,\alpha),(Y,\beta),(Z,\gamma))\right). \tag{4.7}$$

A new useful formula is stated within the next result.

Lemma 4.3. *Let $\mathfrak{D}$ be an almost Banach Dirac bundle. Then:*

$$\begin{aligned} T_{\mathfrak{D}}((X,\alpha),(Y,\beta),(Z,\gamma)) = & -d\alpha(Y,Z) - d\beta(Z,X) - d\gamma(X,Y) \\ & + \mathcal{L}_X\left(\gamma(Y)\right) + \mathcal{L}_Y\left(\alpha(Z)\right) + \mathcal{L}_Z\left(\beta(X)\right). \end{aligned} \tag{4.8}$$

Define on the total space of sections of $TM \oplus T^*M$ the following skew–symmetric bilinear form:

$$\langle (X,\alpha),(Y,\beta)\rangle_- = \alpha(Y) - \beta(X), \tag{4.9}$$

for all (X,α), $(Y,\beta) \in \Gamma(TM \oplus T^*M)$.

Furthermore, define the map $\mathbf{U} : \left(\Gamma(TM \oplus T^*M)\right)^3 \to \mathbb{R}$ by:

$$\begin{aligned} \mathbf{U}((X,\alpha),(Y,\beta),(Z,\gamma)) = & -d\alpha(Y,Z) - d\beta(Z,X) - d\gamma(X,Y) \\ & -\frac{1}{2}\mathcal{L}_X\left(\langle (Y,\beta),(Z,\gamma)\rangle_-\right) - \frac{1}{2}\mathcal{L}_X\left(\langle (Z,\gamma),(X,\alpha)\rangle_-\right) \\ & -\frac{1}{2}\mathcal{L}_Z\left(\langle (X,\alpha),(Y,\beta)\rangle_-\right), \end{aligned} \tag{4.10}$$

where (X,α), (Y,β), $(Z,\gamma) \in \Gamma(TM \oplus T^*M)$.

It is easily seen that $\mathbf{U}$ is a totally skew–symmetric form. Moreover, it can be shown that the maps T and $\mathbf{U}$ do not coincide on the whole $\left(\Gamma(TM \oplus T^*M)\right)^3$. However, a special case is stated within the next results.

Lemma 4.4. *Assume that M is a smooth Banach manifold and that $\mathfrak{D}$ is an almost Banach Dirac bundle on M. Then $\mathbf{U}_{\mathfrak{D}} = T_{\mathfrak{D}}$.*

Thus we have

Theorem 4.5. *Assume that M is a smooth Banach manifold and that $\mathfrak{D}$ is an almost Banach Dirac bundle on M. Then $T_{\mathfrak{D}}$ is a $(0,3)$–tensor field.*

Now we define the notions of Banach Dirac bundles and Banach Dirac manifolds, respectively.

Definition 4.3. A Banach Dirac bundle, or a Banach Dirac structure on M is an almost Banach Dirac bundle $\mathfrak{D}$ which is integrable in the sense that the space of all sections of $\mathfrak{D}$ is closed under the Courant bracket. In this case, the pair $(M,\mathfrak{D})$ is called a Banach Dirac manifold.

In what follows two basic examples of Banach Dirac manifolds are presented.

Examples:

1. **Integrable Banach subbundles.** Let $\mathcal{E}$ be a subbundle of the tangent bundle TM. One says that $\mathcal{E}$ is integrable in a point $p \in M$ if there exists a sub–manifold M' of M containing p such that the tangent map of the inclusion $M' \hookrightarrow M$ induces a vector bundle isomorphism of TM' with the subbundle $\mathcal{E}$ restricted to M'. By the Frobenius Theorem, $\mathcal{E}$ is integrable if and only if the set of its sections $\Gamma(\mathcal{E})$ is closed under the Lie bracket of vector fields on M. Then, it can be easily seen that
$$\mathfrak{D} = \{(X, \alpha) \, : \, X \in \mathcal{E}, \, \alpha \in ann\mathcal{E}\}$$
is a Banach Dirac bundle if and only if the vector subbundle $\mathcal{E}$ is integrable.

2. **Presimplectic Banach 2–forms.** Any 2-form ω on M induces a mapping $\omega : TM \mapsto T^*M$ by $\omega(X) = i_X\omega$. The 2-form ω is called symplectic if the map ω is non–singular and the 2-form ω is closed. Furthermore, one says that the 2-form ω is presymplectic if the map ω is singular and the 2–form ω is closed. It is easy to check that for a presymplectic 2-form the graph of the map ω is a Banach Dirac bundle.

The next theorems offer characterizations for the integrability of almost Banach Dirac bundles. For proofs we refer to [3].

Theorem 4.6. *Assume that M is a smooth Banach manifold and that $\mathfrak{D}$ is an almost Banach Dirac bundle on M. Then, $\mathfrak{D}$ is a Banach Dirac bundle if and only if $T_{\mathfrak{D}} = 0$.*

Theorem 4.7. *Assume that M is a smooth Banach manifold and that $\mathfrak{D}$ is an almost Banach Dirac bundle on M. Then $\mathfrak{D}$ is integrable if and only if*
$$(\mathcal{L}_X\beta)(Z) + (\mathcal{L}_Y\gamma)(X) + (\mathcal{L}_Z\alpha)(Y) = 0, \tag{4.11}$$
for all (X, α), (Y, β), $(Z, \gamma) \in \Gamma(\mathfrak{D})$.

Remark 4.4. It should be emphasized that Theorems 4.5 and 4.6 show that, in fact, the integrability of an almost Banach Dirac bundle is determined by the vanishing of a $(0, 3)$–tensor field on $\mathfrak{D}$.

Remark 4.5. In the algebraic general setting from [9] (see especially [9, Theorem 2.1]) the condition (4.11) was chosen as the definition of closeness (or integrability) of almost Dirac structures.

The next result obtained by a straightforward calculus provides some preparatory ingredients.

Lemma 4.8. *Let (X, α) and $(Y, \beta) \in \Gamma(TM \oplus T^*M)$ and $f \in C^\infty(M)$. Then*
$$\begin{aligned} [f \cdot (X, \alpha), (Y, \beta)]_C &= f \cdot [(X, \alpha), (Y, \beta)]_C - Y(f) \cdot (X, \alpha) \\ &+ \left(0, \frac{1}{2}\langle (X, \alpha), (Y, \beta)\rangle_+ \cdot df\right). \end{aligned} \tag{4.12}$$

In particular, if (X, α) and $(Y, \beta) \in \Gamma(\mathfrak{D})$ where $\mathfrak{D}$ is an almost Banach Dirac bundle, then
$$[f \cdot (X, \alpha), (Y, \beta)]_C = f \cdot [(X, \alpha), (Y, \beta)]_C - Y(f) \cdot (X, \alpha). \tag{4.13}$$

Furthermore, we recall that the bundle $TM \oplus T^*M$ is an anchored Banach vector bundle with the anchor say ρ, cf. Example 1. On $\Gamma(TM \oplus T^*M)$ consider the Courant bracket. It is called an **almost Banach Lie bracket** with respect to the anchor ρ if it satisfies the following Leibniz rule:

$$[f \cdot (X, \alpha), (Y, \beta)]_C = f \cdot [(X, \alpha), Y, \beta)]_C - (\rho(Y, \beta))\,(f) \cdot (X, \alpha)$$

for any function $f \in C^\infty(M)$.

It follows from Lemma 4.8 that this condition holds if, for instance (X, α) and (Y, β) belong to an almost Dirac structure $\mathfrak{D}$.

Thus, the pair $(\mathfrak{D}, \rho\restriction_{\mathfrak{D}})$ is an anchored Banach bundle for which the Courant bracket is an almost Banach Lie bracket with respect to $\rho\restriction_{\mathfrak{D}}$. An almost Lie bracket with respect to the anchor ρ is called a Lie bracket if it satisfies the Jacobi identity. In such a case the mapping induced by the anchor on sections becomes a Lie algebra morphism and $(\mathfrak{D}, \rho\restriction_{\mathfrak{D}}, [\cdot,\cdot]_C)$ is a Banach Lie algebroid. More precisely, the following result holds.

Theorem 4.9. *Assume that M is a smooth Banach manifold. An almost Banach Dirac bundle $\mathfrak{D}$ on M is a Banach Dirac bundle if and only if $(\mathfrak{D}, \rho\restriction_{\mathfrak{D}}, [\cdot,\cdot]_C)$ is a Banach Lie algebroid.*

Acknowledgements. The author was partially supported by a grant of the Romanian National Authority for Scientific Research, ANCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0256.

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