# Multitime optimal control for linear PDEs with curvilinear cost functional

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*Dedicated to Lawrence C. Evans and Lev S. Pontryagin for their seminal contributions to time optimal control theory*

**Abstract.** In this paper, the multitime optimal control problem consists in devising a control such as to transfer a completely integrable linear PDE system from some given initial state to a specified target (which may be fixed or moving) in an optimal multitime characterized by a minimum mechanical work. For that we use an appropriate curvilinear integral action. This kind of problems are based on Hamiltonian 1-forms depending linearly on the controls. They exhibits additional features which we now discuss. Firstly, we underline some historical data of interest for optimal problems with curvilinear integral cost. Secondly, our original results concentrate on: (1) the existence of multitime optimal controls for problems associated to a curvilinear integral action and a linear $m$-flow type PDE system, (2) some properties of the reachable set, (3) the maximum principle for linear multitime optimal control problems fixed by a curvilinear integral action and an $m$-flow type PDE system, (4) the bang-bang optimal solution, (5) two basic examples: control of a two-time rocket railroad car and of a two-time vibrating spring.

**M.S.C. 2010**: 49J30, 49J20.
**Key words**: linear multitime optimal control; reachable set; multitime maximum principle; bang-bang control.

## 1 Introduction

Let us analyze again a multitime optimal control problem based on a path independent curvilinear integral as cost functional and on $PDE$ constraints of $m$-flow type (see the papers [5], [6], [8]-[29]). The cost functionals of mechanical work type appear in many applications, as for example, a multi-player multitime optimal control problem where we study the effects on a multitime optimal dynamic system of the interaction of

Balkan Journal of Geometry and Its Applications, Vol. 18, No. 1, 2013, pp. 87-100.

several decision makers, having similar interests and choosing cooperative strategies such as to maximize their common payoff. Of course, to describe some $m$-dimensional objects as optimal evolution maps, a deeply understanding of the meaning of evolution is necessary. The main results include generalizations to multitime case of the single-time optimal control in the vision of Lawrence C. Evans and Lev S. Pontryagin. They are complementary to those in the papers [1]-[4], [7], [30], which refer to multiple integral cost functionals.

A multitime optimal control problem where the control variables enter the Hamiltonian linearly, either via the objective functional or the dynamic system or both is called linear. Since the Hamiltonian is linear in the control variables, necessarily the latter are bounded.

Section 1 underlined the interest for the curvilinear cost integral and gives some historical data. Section 2 proves the existence of multitime optimal controls for problems associated to a curvilinear integral action and an $m$-flow type PDE system. Section 3 gives some properties of the reachable set. Section 4 formulates and proves the maximum principle for linear multitime optimal control problems fixed by a curvilinear integral action and an $m$-flow type PDE system, using the control Hamiltonian 1-form. Section 5 proves the existence of bang-bang optimal solution. Section 6 analyzes the control of a two-time rocket railroad car and the control of a two-time vibrating spring.

## 2 Existence of multitime optimal controls in linear problems

Let $\Omega_{0\tau}$ be the parallelepiped determined by the opposite diagonal points $0 = (0, \ldots, 0)$ and $\tau = (\tau^1, \ldots, \tau^m)$ in $R^m_+$, endowed with the product order. We start with a *completely integrable linear first-order multitime dynamic constraints*

$$(PDE) \qquad \frac{\partial x}{\partial s^\alpha}(s) = M_\alpha x(s) + N_\alpha u(s),\ s \in \Omega_{0\tau}$$

and the initial condition $x(0) = x_0$, for given constant matrices $M_\alpha \in \mathcal{M}_{n\times n}(R)$, $N_\alpha \in \mathcal{M}_{n\times k}(R)$, and the control set $U = [-1,1]^k \subset R^k$. Of course, the complete integrability conditions [8], [12]

$$(N_\alpha \delta^\gamma_\beta - N_\beta \delta^\gamma_\alpha)\frac{\partial u}{\partial s^\gamma} = 0, \quad M_\alpha N_\beta = M_\beta N_\alpha,\ M_\alpha M_\beta = M_\beta M_\alpha$$

impose the set $\mathcal{U}$ of admissible controls. To cover more situations, we can enlarge the previous conditions to $x(s) \in H^\infty(\Omega_{0\tau}, R^n)$, $u(s) \in H^\infty(\Omega_{0\tau}, R^k)$.

Let $\phi(t)$ be a differentiable function. Our problem is defined by the cost functional

$$(P) \qquad P(u(\cdot)) = -\int_{\gamma_{0\tau}} d\phi(s) = \phi(0) - \phi(\tau),$$

where $\gamma_{0\tau}$ is an arbitrary $C^1$ increasing curve joining the points $0 = (0, \ldots, 0)$ and $\tau = (\tau^1, \ldots, \tau^m)$, with $\tau = \tau(u(\cdot))$, as a multitime for which the solution $m$-sheet of PDE hits the origin 0 (If the $m$-sheet never hits 0, we set $\tau = \infty$). Of course, the functional $P(u(\cdot))$ is a *path independent curvilinear integral* and always we can assume $\phi(0) = 0$. Also, it is an upper semi-continuous functional.

**Remark**. We can take the function $\phi(t)$ as symmetric polynomial in $t^1, ..., t^2$. For example, the function $\phi(t) = t^1 \cdots t^m$ appears in the multitime Itô isometry stochastic theory [10], [29].

**Optimal multitime problem**. Let $\mathcal{U}$ be the set of all admissible controls. Giving the starting point $x_0 \in R^n$, find an optimal control $u^*(\cdot)$ such that

$$P(u^*(\cdot)) = \max_{u(\cdot)\in\mathcal{U}} P(u(\cdot)),$$

using (PDE) evolution as constraint. Since $\phi(\tau^*) = -P(u(\cdot))$, the point $\tau^* = (\tau^{*1}, \dots, \tau^{*m})$ ensures the minimum multitime value $\phi(\tau^*)$ to steer to the origin. The foregoing multitime optimum problem is by no means the only case of Linear Optimal Control, but it is an important one.

**Theorem 2.1.** *(***Existence of multitime optimal control***) For each point $x_0 \in R^n$, there exists an optimal control $u^*(\cdot)$.*

*Proof.* Let $C(t)$ be the *reachable set* for multi-time $t$. Define the set $T_{x_0} = \{t = (t^1, \dots, t^m)|x_0 \in C(t)\}$, and define the point $\tau^* = (\tau^{*1}, \dots, \tau^{*m}) \in \overline{T}_{x_0}$ such that $\phi(\tau^*) = \inf_{t\in T_{x_0}} \phi(t)$.

Let us show that there exists an optimal control $u^*(\cdot)$ steering the point $x_0$ to the point 0 at multitime $\tau^* = (\tau^{*1}, \dots, \tau^{*m})$, i.e., $x_0 \in C(\tau^*)$.

Since $x_0 \in C(\bar{t})$ implies $x_0 \in C(t)$ for all multitimes $t \geq \bar{t}$, we can select a decreasing sequence of multitimes $t_1 \geq t_2 \geq \cdots \geq t_n \geq \cdots$ with $x_0 \in C(t_n)$ and $\lim\limits_{n\to\infty} t_n = \tau^*$. Because $x_0 \in C(t_n)$, there exists a control $u_n(\cdot) \in \mathcal{U}$ satisfying

$$x_0 = -\int_{\gamma_{0t_n}} X^{-1}(s)N_\beta u_n(s)ds^\beta,$$

where $\gamma_{0t_n}$ is an arbitrary $C^1$ increasing curve joining the points 0 and $t_n$ (see the general solution of a linear PDE system, [9], [13]).

If necessary, redefine $u_n(s)$ to be 0 for $t_n \leq s$. According Alaoglu Theorem for a path independent curvilinear integral functional [13], there exists a subsequence $n_k \to \infty$ and a control $u^*(\cdot)$ such that $u_{n_k} \rightharpoonup u^*$ (weak convergence).

Let us prove that $u^*(\cdot)$ is an optimal control. First we remark that $u^*(s) = 0$ for $s \geq \tau^*$. On the other hand,

$$x_0 = -\int_{\gamma_{0t_{n_k}}} X^{-1}(s)N_\beta u_{n_k}(s)ds^\beta = -\int_{\gamma_{0t_1}} X^{-1}(s)N_\beta u_{n_k}(s)ds^\beta$$

because $u_{n_k}(s) = 0$ for $s \geq t_{n_k}$. Taking the limit for $n_k \to \infty$, we obtain

$$x_0 = -\int_{\gamma_{0t_1}} X^{-1}(s)N_\beta u^*(s)ds^\beta = -\int_{\gamma_{0\tau^*}} X^{-1}(s)N_\beta u^*(s)ds^\beta$$

since $u^*(s) = 0$ for $s \geq \tau^*$. Hence $x_0 \in C(\tau^*)$, therefore $u^*(\cdot)$ is optimal. □

**Remark 2.1.** The existence of an optimal control $u^*(\cdot)$ implies the existence of an *optimal bang-bang control* (see Section 5; see also the papers [9], [13]).

# 3 Geometry of the reachable set

Let us show how we compute an optimal control $u^*(\cdot)$. For that we find some properties of the *reachable set*

$$K(t, x_0) = \{x_1 \mid \exists u(\cdot) \in \mathcal{U} \text{ which steers from } x_0 \text{ to } x_1 \text{ at multitime t}\}.$$

Having in mind that the $m$-sheet $x(\cdot)$ is a solution of the (PDE), we can write

$$x_1 \in K(t, x_0) \Leftrightarrow x_1 = X(t)x_0 + X(t) \int_{\gamma_{0t}} X^{-1}(s) N_\beta u(s) ds^\beta = x(t)$$

for some control $u(\cdot) \in \mathcal{U}$.

**Theorem 3.1.** **(Geometry of the reachable set)** *The reachable set $K(t, x_0)$ is convex and closed.*

*Proof.* **Convexity** If $x_1, x_2 \in K(t, x_0)$, then there exists $u_1, u_2 \in \mathcal{U}$ with

$$x_1 = X(t)x_0 + X(t) \int_{\gamma_{0t}} X^{-1}(s) N_\beta u_1(s) ds^\beta$$

$$x_2 = X(t)x_0 + X(t) \int_{\gamma_{0t}} X^{-1}(s) N_\beta u_2(s) ds^\beta.$$

For $0 \leq \lambda \leq 1$, we can write

$$\lambda x_1 + (1-\lambda)x_2 = X(t)x_0 + X(t) \int_{\gamma_{0t}} X^{-1}(s) N_\beta \underbrace{(\lambda u_1(s) + (1-\lambda)u_2(s))}_{\in \mathcal{U}} ds^\beta.$$

Hence $\lambda x_1 + (1-\lambda)x_2 \in K(t, x_0)$.

**Closeness** Suppose $x_n \in K(t, x_0)$, $n = 1, 2, \ldots$ and $x_n \to y$. We need to show $y \in K(t, x_0)$. As $x_n \in K(t, x_0)$, there exists $u_n(\cdot) \in \mathcal{U}$ with

$$x_n = X(t)x_0 + X(t) \int_{\gamma_{0t}} X^{-1}(s) N_\beta u_n(s) ds^\beta.$$

The Theorem of Alaoglu shows the existence of a subsequence $n_j \to \infty$ and $u \in \mathcal{U}$ such that $u_{n_j} \rightharpoonup u$ (weak convergence). Consequently, replacing $n$ with $n_j$ and taking the limit we obtain

$$y = X(t)x_0 + X(t) \int_{\gamma_{0t}} X^{-1}(s) N_\beta u(s) ds^\beta,$$

i.e., $y \in K(t, x_0)$, and hence $K(t, x_0)$ is closed.

Recall $\tau^*$ denotes a multitime corresponding to a maximum $\phi(0) - \phi(\tau)$ it takes to steer to the point 0, using the optimal control $u^*$. Note that then $0 \in \partial K(\tau^*, x_0)$.
□

# 4 The maximum principle for linear multitime optimal control

Let us show how we can find explicitly an optimal control $u^*(\cdot)$ solving a linear multitime optimal control problem.

**Theorem 4.1.** *(***Maximum principle for linear multitime optimal control)** *There exists a nonzero vector $h$ such that*

$$(M_\beta) \qquad h^T X^{-1}(t) N_\beta u^*(t) = \max_{u \in U} \{h^T X^{-1}(t) N_\beta u\}$$

*for each multitime $0 \leq t \leq \tau^*$.*

**Interpretation**. If we know the vector $h$, then the maximization principle $(M_\beta)$ provides us a criterion for computing $u^*(\cdot)$. The papers [7]-[22] show that the assertion $(M_\beta)$ is a special case of a general theory.

*Proof.* **Step 1.** We recall $0 \in \partial K(\tau^*, x_0)$. Since $K(\tau^*, x_0)$ is convex, there exists a supporting plane to $K(\tau^*, x_0)$ at the point 0, i.e., there exists $g \neq 0$ with $gx_1 \leq 0$, $x_1 \in K(\tau^*, x_0)$.

**Step 2.** On the other hand $x_1 \in K(\tau^*, x_0)$ if and only if there exists $u(\cdot) \in \mathcal{U}$ such that

$$x_1 = X(\tau^*) x_0 + X(\tau^*) \int_{\gamma_{0\tau^*}} X^{-1}(s) N_\beta u(s) ds^\beta.$$

Also

$$0 = X(\tau^*) x_0 + X(\tau^*) \int_{\gamma_{0\tau^*}} X^{-1}(s) N_\beta u^*(s) ds^\beta.$$

Since $gx_1 \leq 0$, along an increasing curve $\gamma_{0\tau^*}$, we have

$$g^T \left( X(\tau^*) x_0 + X(\tau^*) \int_{\gamma_{0\tau^*}} X^{-1}(s) N_\beta u(s) ds^\beta \right) \leq 0$$

$$0 = g^T \left( X(\tau^*) x_0 + X(\tau^*) \int_{\gamma_{0\tau^*}} X^{-1}(s) N_\beta u^*(s) ds^\beta \right).$$

Define $h^T = g^T X(\tau^*)$. Then

$$\int_{\gamma_{0\tau^*}} h^T X^{-1}(s) N_\beta u(s) ds^\beta \leq \int_{\gamma_{0\tau^*}} h^T X^{-1}(s) N_\beta u^*(s) ds^\beta$$

and consequently

$$\int_{\gamma_{0\tau^*}} h^T X^{-1}(s) N_\beta (u^*(s) - u(s)) ds^\beta \geq 0, \quad \forall u(\cdot) \in \mathcal{U},$$

(along the increasing curve $\gamma_{0\tau^*}$).

**Step 3.** We claim now that the foregoing inequality implies

$$h^T X^{-1}(s) N_\beta u^*(s) = \max_{u \in \mathcal{U}} \{h^T X^{-1}(s) N_\beta u\}$$

for almost every multitime $s$.

To check, we proceed by reductio to absurdum. Suppose not; then there would exist a subset $E \subset \Omega_{0,\tau^*}$ of positive measure, such that

$$h^T X^{-1}(s) N_\beta u^*(s) < \max_{u \in U} \{h^T X^{-1}(s) N_\beta u\}, \quad s \in E.$$

Introduce a new control

$$\hat{u}(s) = \begin{cases} u^*(s) & \text{for} \quad s \notin E \\ u(s) & \text{for} \quad s \in E, \end{cases}$$

where $u(s)$ is selected by

$$\max_{u \in U} \{h^T X^{-1}(s) N_\beta u\} = h^T X^{-1}(s) N_\beta u(s).$$

Then, along an increasing curve,

$$\int_{\gamma_E} \underbrace{h^T X^{-1}(s) N_\beta (u^*(s) - \hat{u}(s)) ds^\beta}_{<0} \geq 0,$$

in contradiction to Step 2 above. □

Let us change the point of view as in the general theory developed in the papers [8]-[29]. First of all, define the *autonomous control Hamiltonian* 1*-form*

$$H_\beta(x, p, u) = -\frac{\partial \phi}{\partial t^\beta} + p^T (M_\beta x + N_\beta u), \quad x, p \in R^n, \ u \in U.$$

**Theorem 4.2.** *(***Another way to write maximum principle for multi-time optimal control)** *Let $u^*(\cdot)$ be a multitime optimal control and $x^*(\cdot)$ the corresponding response of the evolution system. Then there exists the function $p^*(\cdot) : \Omega_{0\tau^*} \to R^n$ satisfying*

$$(PDE) \qquad \frac{\partial x^*}{\partial t^\alpha}(t) = \frac{\partial H_\alpha}{\partial p^*}(x^*(t), p^*(t), u^*(t))$$

$$(ADJ) \qquad \frac{\partial p^*}{\partial t^\alpha}(t) = -\frac{\partial H_\alpha}{\partial x^*}(x^*(t), p^*(t), u^*(t))$$

*and*

$$(M_\beta) \qquad H_\beta(x^*(t), p^*(t), u^*(t)) = \max_{u \in U} H_\beta(x^*(t), p^*(t), u).$$

The PDEs denoted by (ADJ) are called the *adjoint equations* and $(M_\beta)$ the *maximization principle*. The function $x^*(t)$ is called *optimal state*. The function $p^*(\cdot)$ is called the *optimal costate*.

*Proof.* **Step 1**. We take the vector $h$ like in the Theorem 4.1, and we introduce the Cauchy problem

$$\frac{\partial p^*}{\partial t^\alpha}(t) = -M_\alpha^T p^*(t), \quad p^*(0) = h,$$

associated to a completely integrable PDE system. The solution of this problem is $p^*(t) = e^{-M_\alpha^T t^\alpha} h$. Consequently $p^{*T}(t) = h^T X^{-1}(t)$, because $(e^{-M_\alpha^T t^\alpha})^T = e^{-M_\alpha t^\alpha} = X^{-1}(t)$.

**Step 2.** From Theorem 4.1 and from the conditions $(M_\beta)$, we find

$$h^T X^{-1}(t) N_\beta u^*(t) = \max_{u \in U}\{h^T X^{-1}(t) N_\beta u\}.$$

Since $p^{*T} = h^T X^{-1}(t)$, this means

$$-\phi_\beta(t) + p^{*T}(t)(M_\beta x^*(t) + N_\beta u^*(t)) = \max_{u \in U}\{-\phi_\beta(t) + p^{*T}(t)(M_\beta x^*(t) + N_\beta u)\}.$$

**Step 3.** We remark that the definition of the control Hamiltonian 1-from determines the former (PDE) and (ADJ) for the dynamical equations. □

# 5 Existence of a bang-bang control

The previous multitime optimal control problem is linear since the control variables enter the Hamiltonian 1-form $H_\beta$ linearly affine. The foregoing Hamiltonian 1-form $H_\beta dt^\beta$ can be written as

$$H = H_\beta(x, p, u)dt^\beta = -d\phi(t) + p^T(t) M_{\beta i} x^i dt^\beta + p^T(t) N_{\beta a} u^a dt^\beta,$$

where

$$\frac{\partial p}{\partial t^\alpha}(t) = -M_\alpha^T p(t), \quad p(0) = h,$$

i.e., $p(t) = e^{-M_\alpha^T t^\alpha} h$. The extremum of all components $p^T(t) N_{\beta a} u^a$ exists since the control variables are bounded, i.e., $-1 \leq u^a \leq 1$; for optimum, they must be at the boundary $\partial U$ of the admissible region $U$ (see, linear optimization, simplex method). When the multitime maximum principle is applied to this type of problems, we need the coefficients $Q_{\beta a}(t) = p^T(t) N_{\beta a}$, and then the optimal control $u^{*a}$ must be the function

$$u^{*a} = \operatorname{sgn}(Q_{\beta a}(t)) = \begin{cases} 1 & \text{for } Q_{\beta a}(t) > 0 : \text{bang-bang control} \\ \text{undetermined} & \text{for } Q_{\beta a}(t) = 0 : \text{singular control} \\ -1 & \text{for } Q_{\beta a}(t) < 0 : \text{bang-bang control.} \end{cases}$$

Suppose the measure of each set $Q_{\beta a}(t) = 0$, $t \in \Omega_{0\tau}$ vanishes. Then the singular control is ruled out and the remaining possibilities are bang-bang controls. This optimal control is discontinuous since each component jumps from a minimum to a maximum and vice versa in response to each change in the sign of each $Q_{\beta a}(t)$. The 1-forms $Q_{\beta a}(t) = p^T(t) N_{\beta a} dt^\beta$ are called the *switching 1-forms.*

# 6 Two-time examples

Let $R^m_+$ endowed with the product order and $t \in R^m_+$. Though a little strange, due to mathematical language, there are many significant multitime evolutions (deformations) $x(t) \in R^n$, from the point $x(0)$ to the point $x(t)$, similar to the image created when we move the cursor on the desktop. Their control is a basic problem in applied sciences.

## 6.1 Two-time rocket railroad car

Let us use the general notations

$$t = (t^1, t^2) \in \Omega_{0\tau},\ x^1(t) = q(t), x^2(t) = v_1(t) = \frac{\partial q}{\partial t^1}, x^3(t) = v_2(t) = \frac{\partial q}{\partial t^2}.$$

Then the *two-time rocket railroad car PDE system* is

$$\frac{\partial}{\partial t^1}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u(t)$$

$$\frac{\partial}{\partial t^2}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}(t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(t),$$

$$U = [-1, 1],\ M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ N_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

This PDE system is (piecewise) completely integrable if $\frac{\partial u}{\partial t^1} = \frac{\partial u}{\partial t^2} = 0$, i.e., $u(t)$ is (piecewise) constant. The maximum principle in Theorem 4.2 shows the existence of a vector $h \neq 0$ such that

$$(M_\beta) \qquad h^T X^{-1}(t) N_\beta u^*(t) = \max_{-1 \le u \le 1}\{h^T X^{-1}(t) N_\beta u\}.$$

We will extract the interesting fact that an optimal control $u^*(t)$ switches at least ones.

We must compute the exponential matrix $e^{M_\alpha t^\alpha}$. To do that, we start with

$$M_1 M_2 = M_2 M_1 = 0;$$

$$M_1^0 = I,\ M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ M_1^2 = 0, \quad \text{and so} \quad M_1^k = 0 \text{ for all } k \ge 2;$$

$$M_2^0 = I,\ M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ M_2^2 = 0, \quad \text{and so} \quad M_2^k = 0 \text{ for all } k \ge 2.$$

Consequently

$$e^{M_\alpha t^\alpha} = I + M_\alpha t^\alpha = \begin{pmatrix} 1 & t^1 & t^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = X(t).$$

Then

$$X^{-1}(t) = \begin{pmatrix} 1 & -t^1 & -t^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X^{-1}(t)N_1 = \begin{pmatrix} -t^1 \\ 1 \\ 0 \end{pmatrix}, \quad X^{-1}(t)N_2 = \begin{pmatrix} -t^2 \\ 0 \\ 1 \end{pmatrix}$$

$$h^T X^{-1}(t)N_1 = (h_1\ h_2\ h_3) \begin{pmatrix} -t^1 \\ 1 \\ 0 \end{pmatrix} = -h_1 t^1 + h_2$$

$$h^T X^{-1}(t)N_2 = (h_1\ h_2\ h_3) \begin{pmatrix} -t^2 \\ 0 \\ 1 \end{pmatrix} = -h_1 t^2 + h_3.$$

According the maximum principle we must have

$$(-h_1 t^1 + h_2)u^*(t) = \max_{|u|\leq 1}\{(-t^1 h_1 + h_2)u\}$$

$$(-h_1 t^2 + h_3)u^*(t) = \max_{|u|\leq 1}\{(-t^2 h_1 + h_3)u\}.$$

Here we have a switching vector function of components $\sigma_1(t) = -h_1 t^1 + h_2$, $\sigma_2(t) = -h_1 t^2 + h_3$. Supposing $\dfrac{h_2}{h_1} < 0$, $\dfrac{h_3}{h_1} < 0$ (see adjoint equations), we have

$$u^*(t) = sign\, h_1 = \begin{cases} 1 & \text{for} \quad h_1 > 0 \\ 0 & \text{for} \quad h_1 = 0 \\ -1 & \text{for} \quad h_1 < 0. \end{cases}$$

In this way the optimal control $u^*(t)$ switches at most once, and so the control is correct. If $h_1 = 0$, then the optimal control $u^*(t)$ is constant.

**Geometric interpretation**.

(i) **Optimal state** For $u = 1$, the optimal evolution is the parametrized surface

$$x^1(t) = x_0^1 + x_0^2 t^1 + x_0^3 t^2 + \frac{1}{2}({t^1}^2 + {t^2}^2),\ x^2(t) = x_0^2 + t^1,\ x^3(t) = x_0^3 + t^2.$$

Consequently, the optimal 2-sheet of deformation is a convex paraboloid (of revolution)

$$\Sigma_1 : x^1 = x_0^1 + (x^2 - x_0^2)x_0^2 + (x^3 - x_0^3)x_0^3 + \frac{1}{2}\left((x^2 - x_0^2)^2 + (x^3 - x_0^3)^2\right),$$

whose axis is parallel to $Ox^1$.

If $u = -1$, the optimal evolution is the surface

$$x^1(t) = x_0^1 + x_0^2 t^1 + x_0^3 t^2 - \frac{1}{2}({t^1}^2 + {t^2}^2),\ x^2(t) = x_0^2 - t^1,\ x^3(t) = x_0^3 - t^2.$$

Eliminating the parameters $t^1$, $t^2$, the optimal 2-sheet of deformation is a concave paraboloid (of revolution)

$$\Sigma_2 :\ x^1 = x_0^1 - (x^2 - x_0^2)x_0^2 - (x^3 - x_0^3)x_0^3 - \frac{1}{2}\left((x^2 - x_0^2)^2 + (x^3 - x_0^3)^2\right),$$

whose axis is parallel to $Ox^1$.

**Conclusion**: to get the origin we must switch our control $u(\cdot)$ back and forth between the values $\pm 1$, causing the 2-sheet to switch between $\Sigma_1$ and $\Sigma_2$.

(ii) **Optimal costate** The Theorem 4.2 shows that the optimal costate is $p^{*T}(t) = h^T X^{-1}(t)$ or

$$p^{*T}(t) = (h_1, h_2, h_3)\begin{pmatrix} 1 & -t^1 & -t^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

## 6.2 Control of two-time vibrating spring

Let us analyze the completely integrable parabolic-hyperbolic PDE system

$$\frac{\partial^2 x}{\partial t^\alpha \partial t^\beta}(t) + x(t) = u(t)\delta_{\alpha\beta},\ t = (t^1, t^2) \in \Omega_{0\tau},\ |u(t)| \leq 1.$$

where we interpret the control $u(t)$ as an exterior force on an oscillating weight (of unit mass) hanging from a spring. The aim is to design an optimal control $u^*(\cdot)$ that bring the two-time motion to a stop in a minimal two-time $\tau$ that provide a minimum $\phi(\tau)$. Suppose $u(t)$ is (piecewise) constant. Then the complete integrability conditions impose $\frac{\partial x}{\partial t^1} = \frac{\partial x}{\partial t^2}$ and hence $x = x(t^1 + t^2)$, i.e., the 2-sheet is reduced to an 1-sheet. In general notations, we can write

$$x^1(t) = x(t), \quad x^2(t) = \frac{\partial x}{\partial t^1}(t) = \frac{\partial x}{\partial t^2}(t)$$

$$\frac{\partial}{\partial t^1}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u(t)$$

$$\frac{\partial}{\partial t^2}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u(t)$$

$$M_1 = M_2 = M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad N_1 = N_2 = N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad U = [-1, 1].$$

**Using the maximum principle**. The maximum principle in Theorem 4.2 asserts the existence of $h \neq 0$ such that

$$(M_\beta) \qquad h^T X^{-1}(t) N_\beta u^*(t) = \max_{u \in U}\{h^T X^{-1}(t) N_\beta u\}.$$

To extract useful information from $(M_\beta)$ we need to compute the fundamental matrix $X(\cdot)$. In this sense we remark that $M$ is a skew symmetric matrix with

$$M^0 = I,\ M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ M^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

and consequently

$$M^k = \begin{cases} I & \text{if} \quad k = 0, 4, 8 \dots \\ M & \text{if} \quad k = 1, 5, 9 \dots \\ -I & \text{if} \quad k = 2, 6 \dots \\ -M & \text{if} \quad k = 3, 7 \dots \end{cases}$$

In this way

$$X(t) = e^{M_\alpha t^\alpha} = e^{M(t^1+t^2)} = I + (t^1 + t^2)M + \frac{(t^1+t^2)^2}{2} M^2 + \dots$$

$$= \left(1 - \frac{(t^1+t^2)^2}{2!} + \frac{(t^1+t^2)^4}{4!} - \dots\right) I + \left((t^1+t^2) - \frac{(t^1+t^2)^3}{3!} + \dots\right) M$$

$$= I\cos(t^1+t^2) + Msin(t^1+t^2) = \begin{pmatrix} \cos(t^1+t^2) & \sin(t^1+t^2) \\ -\sin(t^1+t^2) & \cos(t^1+t^2) \end{pmatrix}$$

and

$$X^{-1}(t) = \begin{pmatrix} \cos(t^1+t^2) & -\sin(t^1+t^2) \\ \sin(t^1+t^2) & \cos(t^1+t^2) \end{pmatrix}.$$

It follows

$$X^{-1}(t)N = \begin{pmatrix} -\sin(t^1+t^2) \\ \cos(t^1+t^2) \end{pmatrix}$$

and

$$h^T X^{-1}(t) N_\beta = (h_1\ h_2) \begin{pmatrix} -\sin(t^1+t^2) \\ \cos(t^1+t^2) \end{pmatrix}$$

$$= -h_1 \sin(t^1+t^2) + h_2 \cos(t^1+t^2).$$

The conditions $(M_\beta)$ show that for each two-time $t = (t^1, t^2)$ we must have

$$(-h_1 \sin(t^1+t^2) + h_2 \cos(t^1+t^2))u^*(t)$$

$$= \max_{|u|\le 1} \{(-h_1 \sin(t^1+t^2) + h_2 \cos(t^1+t^2))u\}.$$

Here we have only one switching function

$$\sigma(t) = -h_1 \sin(t^1+t^2) + h_2 \cos(t^1+t^2).$$

Therefore

$$u^*(t) = sgn\ \sigma(t).$$

**Finding the optimal control**. To simplify, suppose $(h_1)^2 + (h_2)^2 = 1$. Having in mind the identity $\sin(x+y) = \sin x \cos y + \cos x \sin y$, we can choose $-h_1 = \cos\delta,\ h_2 = \sin\delta$ and we write

$$u^*(t) = sgn(\sin(t^1 + t^2 + \delta)).$$

In this way we deduce that $u^*$ switches from $+1$ to $-1$, and vice versa, every $\pi$ units of sum-times.

**Geometric interpretation**.

(i) **Optimal state** For $u = 1$, the evolution PDEs system is

$$\frac{\partial x^1}{\partial t^1} = x^2, \quad \frac{\partial x^2}{\partial t^1} = -x^1 + 1$$
$$\frac{\partial x^1}{\partial t^2} = x^2, \quad \frac{\partial x^2}{\partial t^2} = -x^1 + 1.$$

We remark that

$$\frac{\partial}{\partial t^\alpha}((x^1(t) - 1)^2 + (x^2(t))^2) = 2(x^1(t) - 1)\frac{\partial x^1}{\partial t^\alpha} + 2x^2(t)\frac{\partial x^2}{\partial t^\alpha}$$
$$= 2(x^1(t) - 1)x^2(t) + 2x^2(t)(1 - x^1(t)) = 0.$$

Consequently, the motion satisfies

$$(x^1 - 1)^2 + (x^2)^2 = r_1^2,$$

i.e., the 1-sheet of motion lies on a circle with center $(1, 0)$.

If $u = -1$, the evolution PDEs system is

$$\frac{\partial x^1}{\partial t^1} = x^2, \quad \frac{\partial x^2}{\partial t^1} = -x^1 - 1; \ \frac{\partial x^1}{\partial t^2} = x^2, \quad \frac{\partial x^2}{\partial t^2} = -x^1 - 1$$

and the first integral is

$$(x^1 + 1)^2 + (x^2)^2 = r_2^2.$$

This means that the 1-sheet of motion lies on a circle with center $(-1, 0)$.

**Conclusion**. To get to the origin we must switch our control $u(\cdot)$ back and forth between the values $\pm1$, causing the 1-sheet to switch between lying on circles centered respectively at $(-1, 0)$, $(+1, 0)$. The switches occurs each $\pi$ units of time.

(ii) **Optimal costate** The Theorem 4.2 shows that the optimal costate is $p^{*T}(t) = h^T X^{-1}(t)$ or

$$p^{*T}(t) = (h_1, h_2)\begin{pmatrix} \cos(t^1 + t^2) & -\sin(t^1 + t^2) \\ \sin(t^1 + t^2) & \cos(t^1 + t^2) \end{pmatrix}.$$

# 7 Conclusion and further development

Our recent endeavor is dedicated to finding appropriate responses to problems of multi-temporal optimal control based on curvilinear integral functionals (mechanical works) and multi-temporal evolutions.

Issues addressed in this paper show that sometimes firstly we can find the optimal control and then determine the optimal state and the optimal costate. It also appears quite clear that, in treating multi-temporal optimal control problems, we need to a sense for optimizing differential 1-forms. Our work in these topics are well known and they will be enriched soon with multi-temporal optimal control theory.

# References


[1] F. Giannessi, *Constrained Optimization and Image Space Analysis*, Springer, 2005.

[2] T. Malakorn, *Multidimensional linear systems and robust control*, PhD Thesis, Virginia Polytechnic Institute and State University, 2003.

[3] S. Pickenhain, M. Wagner, *Pontryagin principle for state-constrained control problems governed by a first-order PDE system*, J. Optim. Theory Appl., 107, 2 (2000), 297-330.

[4] S. Pickenhain, M. Wagner, *Piecewise continuous controls in Dieudonné-Rashevsky type problems*, J. Optim. Theory Appl., 127, 1 (2005), 145-163.

[5] A. Pitea, C. Udrişte, Ş. Mititelu, *PDI & PDE-constrained optimization problems with curvilinear functional quotients as objective vectors*, Balkan J. Geom. Appl., 14, 2 (2009), 75-88.

[6] A. Pitea, *A study of some general problems of Dieudonné-Rashevski type*, Abstr. Appl. Anal., ID: 592804 (2012)

[7] V. Prepeliţă, *Minimal realization algorithm for (q,r)-D hybrid systems*, WSEAS Trans. Syst., 8, 1 (2009), 22-33.

[8] C. Udrişte, *Multi-time maximum principle*, Short Communication, International Congress of Mathematicians, Madrid, August 22-30, ICM Abstracts, 2006, p.47; Plenary Lecture at 6-th WSEAS Int. Conf. Circuits, Systems, Electronics, Control & Signal Processing (CSECS'07), pp. 10-11, and 12-th WSEAS Int. Conf. Appl. Math., Cairo, Egypt, December 29-31, 2007, p. ii.

[9] C. Udrişte, *Controllability and observability of multitime linear PDE systems*, Proceedings of The Sixth Congress of Romanian Mathematicians, Bucharest, Romania, June 28 - July 4, vol. 1 (2007), pp. 313-319.

[10] C. Udrişte, *Multitime stochastic control theory*, Selected Topics on Circuits, Systems, Electronics, Control & Signal Processing, Proc. 6-th WSEAS Int. Conf. Circuits, Systems, Electronics, Control & Signal Processing (CSECS'07), pp. 171-176; Cairo, Egypt, December 29-31, 2007.

[11] C. Udrişte, I. Ţevy, *Multitime Euler-Lagrange-Hamilton theory*, WSEAS Trans. Math., 6, 6 (2007), pp. 701-709.

[12] C. Udrişte, I. Ţevy, *Multitime Euler-Lagrange dynamics*, Proc. 7th WSEAS Int. Conf. Systems Theory Sci. Comput. (ISTASC'07), Vouliagmeni Beach, Athens, Greece, August 24-26, (2007), pp. 66-71.

[13] C. Udrişte, *Multitime controllability, observability and bang-bang principle*, J. Optim. Theory Appl., 139, 1 (2008), 141-157.

[14] C. Udrişte, L. Matei, *Lagrange-Hamilton Theories* (in Romanian), Monographs and Textbooks 8, Geometry Balkan Press, Bucharest (2008).

[15] C. Udrişte, *Simplified multitime maximum principle*, Balkan J. Geom. Appl., 14, 1 (2009), 102-119.

[16] C. Udrişte, *Nonholonomic approach of multitime maximum principle*, Balkan J. Geom. Appl., 14, 2 (2009), 111-126.

[17] C. Udrişte, I. Ţevy, *Multitime linear-quadratic regulator problem based on curvilinear integral*, Balkan J. Geom. Appl., 14, 2 (2009), 127-137.

[18] C. Udrişte, I. Ţevy, *Multitime dynamic programming for curvilinear integral actions*, J. Optim. Theory Appl., 146, 1 (2010), 189-207.

[19] C. Udrişte, L. Matei, I. Duca, *Multitime Hamilton-Jacobi theory*, Proc. of 8-th WSEAS Int. Conf. Appl. Comput. Sci. (ACACOS-09), 509-513, Hangzhou, China, May 20-22, 2009; Proc. 9-th WSEAS Int. Conf. Systems Theory Sci. Comput. (ISTASC-09), pp. 129-133, Moscow, Russia, August 20-22, 2009.

[20] M. Pîrvan, C. Udrişte, *Optimal control of electromagnetic energy*, Balkan J. Geom. Appl., 15, 1 (2010), 131-141.

[21] C. Udrişte, *Equivalence of multitime optimal control problems*, Balkan J. Geom. Appl., 15, 1 (2010), 155-162.

[22] C. Udrişte, O. Dogaru, I. Ţevy, D. Bala, *Elementary work, Newton law and Euler-Lagrange equations*, Balkan J. Geom. Appl., 15, 2 (2010), 92-99.

[23] C. Udrişte, V. Arsinte, C. Cipu, *Von Neumann analysis of linearized discrete Tzitzeica PDE*, Balkan J. Geom. Appl., 15, 2 (2010), 100-112.

[24] C. Udrişte, I. Ţevy, V. Arsinte, *Minimal surfaces between two points*, J. Adv. Math. Stud., 3, 2 (2010), 105-116.

[25] C. Udrişte, *Multitime maximum principle for curvilinear integral cost*, Balkan J. Geom. Appl., 16, 1 (2011), 128-149.

[26] C. Udrişte, *Multitime maximum principle approach of minimal submanifolds and harmonic maps*, arXiv:1110.4745v1 [math.DG] 21 Oct 2011.

[27] C. Udrişte, A. Bejenaru, *Multitime optimal control with area integral costs on boundary*, Balkan J. Geom. Appl., 16, 2 (2011), 138-154.

[28] C. Udrişte, I. Ţevy, *Multitime dynamic programming for multiple integral actions*, J. Glob. Optim., 51, 2 (2011), 345-360.

[29] C. Udrişte, V. Damian, *Multitime stochastic maximum principle on curvilinear integral actions*, arXiv:1112.0865v1 [math.OC] 5 Dec. 2011.

[30] M. Wagner, *Pontryagin's maximum principle for Dieudonné-Rashevsky type problems involving Lipschitz functions*, Optimization 46 (1999), 165 - 184.

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