Ricci flow and the manifold of Riemannian metrics

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Abstract. R.Hamilton defined Ricci flow as a weak parabolic partial differential equation, in spite of weakness he could prove the existence and uniqueness in the short time, while later DeTurck found a shorter proof. On the other hand the space of Riemannian metrics on a compact manifold had been proved to be an infinite dimensional manifold which is a projective limit of Banach manifolds. In this paper we consider the Ricci flow as an integral curve of certain vector fields on the manifold of Riemannian metrics and in spite of being infinite dimensional, we prove the existence and uniqueness for the short time, and moreover we find further results on the behavior of these curves.

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1 Introduction

The Ricci flow was first defined by Hamilton in the early 1980 [14]. Following works of Eells and Sampson[8], he introduced an evolution equation for a family of Riemannian metrics as follows:

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2Rc(g(t)) \\ g(0) = g_0, \end{cases}$$

where $Rc(g(t))$ denotes the Ricci curvature of the metric $g(t)$, and by rescaling the space and time we obtain its cousin, the normalized Ricci flow, as:

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2Rc(g(t)) + \frac{2}{n} \frac{\int_M R d\mu}{\int_M d\mu} g(t) \\ g(0) = g_0, \end{cases}$$

where R denotes the scalar curvature of the metric $g(t)$ and $d\mu$ is the volume element of $g(t)$. The Ricci flow is an evolution equation, considered as a partial differential equation.

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A fundamental problem for any system of partial differential equation is the existence and uniqueness of solution for the short time. Since Ricci flow is a weak parabolic partial differential equation, standard parabolic theory does not work for it. Hamilton proved the existence and uniqueness of solution for the Ricci flow in the short time by using Nash and Moser implicit function theorem [13, 14]. Later D.M.DeTurck provided an easier proof using linearizing of differential operators [6]. Ebin studied the space of Riemannian metrics on a compact manifold M in his paper from geometric point of view[9]. He enlarged this space to a space of a certain type of infinite dimensional manifolds. The space of Riemannian metrics on a compact, oriented, smooth n-manifold M is an infinite dimensional manifold. It is an open subset of a Fréchet space, and is a Fréchet manifold. Since some basic theorems of differential geometry such as implicit function theorem and theory of ordinary differential equation do not hold in the infinite dimensional manifolds such as Fréechet space, authors have focussed on the special type of Fréchet manifolds namely the projective limits. Geometry of those Fréchet manifolds which can be obtained as projective limit Banach manifolds has been studied widely([1, 7, 11, 17, 18]). The space of Riemannian metrics is a type of projective limit manifolds. In this paper we use this viewpoint and give a new approach for the proof of short time existence theorem for the Ricci flow, moreover we find the properties of the Ricci flow as a curve.

2 The space of Riemannian metrics

Let M be a compact, oriented, smooth n-manifold, without boundary. Consider the collection $Met(M)$ of all smooth Riemannian metrics on M. In fact $Met(M)$ is the subset of all sections in S^2T^*M of symmetric rank-2 covariant tensor fields which are positive definite on each T_p^*M for $p \in M$, moreover $Met(M)$ is an open convex positive cone in $\Gamma(S^2T^*M)$. For convenience, we will abbreviate $Met(M)$ as $\mathfrak{M}$. The space of $\Gamma(S^2T^*M)$ is an infinite-dimensional Fréchet space [13], therefore $\mathfrak{M}$ is also infinite-dimensional Fréchet manifold. A Fréchet space is a complete Hausdorff metrizable locally convex topological vector space. A Fréchet manifold is a Hausdorff topological space with an atlas of coordinate charts taking their values in Fréchet spaces, such that the coordinate transition functions are all smooth maps between Fréchet spaces. Since $\mathfrak{M}$ is an open subset of the vector space $\Gamma(S^2T^*M)$, the tangent space $T_g\mathfrak{M}$ for any $g \in \mathfrak{M}$ is $\Gamma(S^2T^*M)$ itself. Geometry of $\mathfrak{M}$ at first has been studied by Ebin [9]. Later Freed and Groisser gave a decomposition of this space to $Vol(M)$ and $\mathfrak{M}_\mu$ then using this decomposition they defined a Riemannian metric on $\mathfrak{M}$, and obtained the Levi-Civita connection and its geodesics [10]. There are enormous papers about different aspects of the space of Riemannian metrics, see [3, 4, 9, 10, 21, 12].

2.1 Met(M)

In this section we describe the decomposition of $\mathfrak{M}$ defined by Freed and Groisser[10], which we use in the next sections.

For any $g \in \mathfrak{M}$ an L^2 inner product on tensor fields is induced. For any $A, B \in \Gamma(T^*M \bigotimes T^*M)$, we set

$$\langle A, B\rangle_g = \int_M tr_g(AB^t)\mu(g),$$

where in local coordinates $\{x^i\}$, $A = A_{ij}dx^i \bigotimes dx^j$ (and similarly for B, g), $B^t = B_{ji}dx^i \bigotimes dx^j$ (the "transpose" of B), $\{g^{ij}\}$ is the inverse matrix of $\{g_{ij}\}$, $tr_g(AC) = A_{ij}g^{jk}C_{kl}g^{li}$, and $\mu(g)$ is the volume form $\sqrt{det(g_{ij})}dx^1 \wedge ... \wedge dx^n$. The restriction of this quadratic form to symmetric the symmetric tensor fields A and B is positive definite. Thus there is a metric on $\mathfrak{M}$.

Let $Vol(M) \subset \Omega^n(M)$ be the space of volume forms on M, consistent with the orientation. For $\alpha \in \Omega^n(M)$ and $\nu \in Vol(M)$, let (α/ν) be the function satisfying $\alpha = (\alpha/\nu)\nu$. Let $p : \mathfrak{M} \to Vol(M)$ be the projection carrying g to $\mu(g)$, and let $\mathfrak{M}_\nu = Met_\nu(M) = p^{-1}(\nu)$ for any $\nu \in Vol(M)$.

Each volume form μ determines a splitting

$$i_\mu : Vol(M) \times \mathfrak{M}_\mu \longrightarrow \mathfrak{M},$$

such that

$$(\nu, h) \mapsto (\nu/\mu(h))^{2/n}h. \tag{2.1}$$

Since $Vol(M)$ is an open subset of vector space $\Omega^n(M)$, the tangent bundle of $Vol(M)$ is canonically isomorphic to $Vol(M) \times \Omega^n(M)$. A vector field β over any subset $U \subset Vol(M)$ may therefore be naturally identified with a function $\beta : U \to \Omega^n(M)$, and we implicitly make this identification henceforth.

The tangent space to $\mathfrak{M}_\mu$ at h is the set of $h-$ traceless symmetric tensor fields; that is, $\{A \in \Gamma(S^2T^*M)| \quad h^{ij}A_{ij} \equiv 0\}$. Then for any $g = (\nu, h) \in \mathfrak{M} \cong Vol(M) \times \mathfrak{M}_\mu$, a tangent vector of $T_g\mathfrak{M}$ can be considered to be of the form $\alpha + A$ where $\alpha \in \Omega^n(M) \cong T_\nu Vol(M)$ and $A \in T_h\mathfrak{M}_\mu$.

The Koszul's formula of Levi-Civita connection, applied to constant vector fields B, C, E on $\mathfrak{M} \cong Vol(M) \times \mathfrak{M}_\mu$, quickly leads to

$$\nabla_B C|_g = -\frac{1}{2}(Bg^{-1}C + Cg^{-1}B) + \frac{1}{4}\{(tr_g(C))B + (tr_g(B))C - tr_g(BC)g\}.$$

3 $\mathfrak{M}$ as a projective limit manifold

In this section we give a brief definition of a certain type of infinite dimensional manifolds for which the space of all Riemannian metrics is an example.

It is well known (see for instance [11]) that every Fréchet space F can be identified with the limit of a projective system $\{E^i; \rho^{ji}\}_{i,j\in\mathbb{N}}$ of Banach spaces, $F \cong \varprojlim E^i$. This means that the mappings $\rho^{ji} : E^j \longrightarrow E^i$ $(j \geq i)$ are smooth and satisfy the following conditions for every (i, j, k) such that $j \geq i \geq k$:

$$\rho^{ik}\rho^{ji} = \rho^{jk}.$$

Definition 3.1. Let $\{M^i; \phi^{ji}\}_{i,j\in\mathbb{N}}$ be a projective system of smooth manifolds modeled on the Banach spaces $\{E^i\}_{i\in N}$ respectively. We assume that:

(1) The models $\{E^i\}_{i\in\mathbb{N}}$ form a projective system with connecting morphisms $\{\rho^{ji} : E^j \longrightarrow E^i; j \geq i\}$ and limit the Fréchet space $F = \varprojlim E^i$.

(2) For any element $x = (x^i)_{i\in\mathbb{N}} \in M = \varprojlim M^i$ there exists a family of charts $\{(U^i, \psi^i)\}_{i\in\mathbb{N}}$ of M^i-s, such that the limits $\varprojlim U^i, \varprojlim \psi^i$ can be defined and the sets $\varprojlim U^i, \varprojlim \psi^i(\varprojlim U^i)$ are open in M, F respectively.

Then the limit $M = \varprojlim M^i$ is called a PLB-manifold.

A PLB-manifold M is a Fréchet manifold modelled on F. The corresponding local structure is fully determined by the charts $(\varprojlim U^i, \varprojlim \psi^i)$. The differentiability of mappings involved can be either this of J. A. Leslie ([16]) or that of A.Kriegl-P.Michor ([15]). The tangent bundle TM of M has also a Fréchet manifold structure with model the Fréchet space $F \times F$ which is isomorphic to PLB-manifold $\varprojlim TM^i$.

Proposition 3.1. *[11] The tangent bundles$\{TM^i\}_{i\in N}$ form a projective system with limit set-theoretically isomorphic to $TM : TM \simeq \varprojlim TM^i$.*

Omori introduced inverse limit Hilbert manifolds and inverse limit Hilbert groups, which are a special type of projective limit Banach manifolds, those where connecting morphisms of factors are the natural embedding [17, 18].

The space $\mathfrak{M}$ is an inverse limit Hilbert manifold. In order to obtain some of results obtained by Palais in [19, 20] about space of sections of the vector bundles, we consider the bundles $T^p_q M$ of (p, q)-tensors as vector bundles E over M. Then the space of sections $\Gamma(E) = \Gamma(T^p_q)$ is the space of all smooth tensor fields of type (p, q) on M. The metric g on M determines an inner product on the bundles $T^p_q M$ in the usual way. Therefore, an inner product in the space $\Gamma(E)$ of tensor fields of type (p, q) is defined. If T and U are tensor fields of type (p, q), then

$$\langle T, U\rangle_g = \int_M g(x)(T(x), U(x))\mu(g)(x), \tag{3.1}$$

where

$$g(x)(T(x), U(x)) = g^{i_1k_1}...g^{i_qk_q}g_{j_1l_1}...g_{j_pl_p}T^{j_1...j_p}_{i_1...i_q}U^{l_1...l_p}_{k_1...k_q}.$$

The C^k-norm is defined in the space $\Gamma(E)$ as follows: for a nonnegative integer k and a tensor field T of type (p, q), set

$$|T|_k = \sum_{i=0}^{k} sup_{x\in M}\|\nabla^{(i)}T(x)\|, \tag{3.2}$$

where $\nabla^{(i)} = \nabla \circ ... \circ \nabla$ is the ith power of the covariant derivative and $\|\nabla^{(i)}T(x)\| = \langle\nabla^{(i)}T, \nabla^{(i)}T\rangle_g^{1/2}$ is the tensor norm at a point $x \in M$. We denote by $C^k(E)$ the completion of the space $\Gamma(E)$ with respect to the topology defined by the norm $|T|_k$. The Banach space $C^k(E)$ consists of tensor fields of class C^k. However, in many problems, Hilbert spaces are more convenient, then we define on the space $\Gamma(E)$ inner products stronger than (3.1). Let s be a nonnegative integer and T and U be tensor fields of type (p, q). We set

$$\langle T, U\rangle_{g,s} = \sum_{i=0}^{s}\langle\nabla^{(i)}T, \nabla^{(i)}U\rangle_g = \sum_{i=0}^{s}\int_M g(\nabla^{(i)}T, \nabla^{(i)}U)\mu(g), \tag{3.3}$$

where $\langle\nabla^{(i)}T, \nabla^{(i)}U\rangle_g$ is the inner product (3.1). We denote by $H^s(E)$ the completion of the space $\Gamma(E)$ with respect to the topology defined by inner product(3.3). The space $H^s(E)$ is called the space of Sobolev smoothness class H^s; it is a Hilbert space.

We denote by $\|.\|_s$ the norm of this space. In particular, for $s = 0$, the space $H^0(E)$ is the completion of the space $\Gamma(T^p_q M)$ with respect to inner space (3.1).(For detail, see ([19], Chap. IX).) For $l \geq s$, we have $H^l(E) \subset H^s(E)$; this embedding is continuous. Obviously, $C^k(E) \subset H^k(E)$. The inverse embedding is stated by the following Sobolev embedding theorem (for the proof, see ([19], Chap. X)).

Theorem 3.2. *[19] If $s \geq n/2 + 1 + k$, then $H^s(E) \subset C^k(E)$ and the embedding mapping $H^s(E) \to C^k(E)$ is completely continuous.*

Therefore, for $s \geq n/2+1+k$, we can assume that a tensor field T of Sobolev class H^s is differentiable of class C^k. Further restrictions on s are related with smoothness conditions for tensor fields of the space $H^s(E)$.

Theorem 3.3. *[20] Let E and F be vector bundles over M and $f : E \to F$ be a C^∞-mapping preserving fibers. If $s \geq n/2+1$, then the mapping $\phi : H^s(E) \to H^s(F)$ defined by the formula $\phi(\alpha) = f \circ \alpha$ is a mapping of class C^∞.*

Now we see that the space $\mathfrak{M}$ is a projective (inverse) limit Hilbert manifold. Assume $S^s_2 = H^s(S_2M)$ is the Hilbert space of symmetric 2-forms of class H^s, $s > n/2$. Let $C^0\mathfrak{M}$ be the space of continuous Riemannian metrics. For $s > n/2$, let $\mathfrak{M}^s = H^s(S^2M) \bigcap C^0\mathfrak{M}$. Since $H^s \subset C^0(S_2(M))$ and the embedding is continuous, we have that $\mathfrak{M}^s$ is open in $S^s_2 = H^s(S_2M)$ and is an open, convex, positive cone. In particular, the space $\mathfrak{M}^s$ is a smooth Hilbert manifold. The system $\{\mathfrak{M}, \mathfrak{M}^s\}$ forms a strong ILH-manifold (inverse limit Hilbert manifold)[21].

3.1 Integral curves

In an infinite dimensional manifold a given vector filed need not have integral curves locally, and if there exist they need not be unique for a given initial value. This is due to the fact that inverse function theorem, the implicit function theorem don't hold for all of the infinite dimensional manifolds.

We have the following theorem about existence and uniqueness of integral curves for those Fréchet manifolds which can be obtained as a projective limits of Banach manifolds.[1]

Theorem 3.4. *[1] Let ξ be a vector field on the Fréchet manifold $M = \underleftarrow{lim} M^i$ which can be considered as the projective limit of vector fields $\{\xi^i\}_{i\in\mathbb{N}}$ of $\{M^i\}_{i\in\mathbb{N}}$ respectively. Then ξ admits locally a unique integral curve θ, satisfying an initial condition of the form $\theta(0) = x$ for $x \in M$.*

4 Ricci flow and space of Riemannian metrics

Hamilton introduced in [14] the Ricci flow as follows:

Definition 4.1. Let M be a manifold with an initial metric g_0, a Ricci flow is a family of Riemannian metrics $\{g(t)\}$ on M satisfying the PDE:

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2Rc(g(t)) \\ g(0) = g_0, \end{cases}$$

where $Ric(g(t))$ denotes the Ricci curvature of the metric $g(t)$.

The traceless part of the Ricci tensor on a Riemannian n-manifold (M, g) is the tensor $E_{ij} := R_{ij} - \frac{1}{n}Rg_{ij}$, where R_{ij} is coordinate expression of Ricci curvature and R is the scalar curvature defined as the trace of the Ricci tensor, i.e. $R := g^{ij}R_{ij}$. A metric g is called Einstein if the traceless part of the Ricci tensor is identically 0.
The various geometric quantities evolve when the metric evolves [2]. Suppose that $g_{ij}(t)$ is a time-dependent Riemannian metric, and $\frac{\partial}{\partial t}g_{ij}(t) = h_{ij}(t)$. Then the volume element as a geometric quantities evolves according to the following equation:

$$\frac{\partial}{\partial t}\mu(g(t)) = \frac{1}{2}tr_g h\mu(g(t)).$$

4.1 Short-Time existence

An important foundational step in the study of any system of evolutionary partial differential equations is to show short-time existence and uniqueness. As the PDE of Ricci flow is not strictly parabolic, we can't deduce directly the existence and uniqueness of short time solution for Ricci flow. Hamilton using Nash-Moser implicit function theorem proved short-time existence theorem for Ricci flow[14]. DeTurck later gave a more direct proof by modifying the flow by a time-dependent change of variables to make it parabolic [6].
Since $\mathfrak{M}$ is a projective limit Banach manifold, we can find another approach for the proof of the short time existence theorem of Ricci flow equation.

Theorem 4.1. *If (M, g_0) is a compact Riemannian manifold, there exists a unique solution $g(t)$, defined for time $t \in [0, \varepsilon)$, to the Ricci flow such that $g(0) = g_0$ for some $\varepsilon > 0$.*

Proof. Let X be the vector filed on $\mathfrak{M}$ such that $g \longrightarrow -2Rc(g)$. Since $\mathfrak{M}$ is a type of projective limit Banach manifold, the vector filed V can be considered as projective limit. Using theorem (3.4) the Ricci flow curve can be considered as the integral curve of this vector filed and its existence can be proved as follows.
By proposition (3.1) the vector field $X \in T\mathfrak{M}$ can be considered as $X = \underleftarrow{lim} X^i$. We show that X admit locally a unique integral curve α with initial condition as $\alpha(0) = g_0$ for $g_0 \in \mathfrak{M}$. Since for $i \in \mathbb{N}$, X^i is a vector field on Hilbert manifold $\mathfrak{M}^i$, there exist a unique integral curve α^i such that

$$(\psi_k^i \circ \alpha^i)' = \Psi_k^{2,i}(X^i(\alpha^i(t))); \qquad k \in I$$

and $\alpha^i(0) = g_0^i = \varphi^i(g_0)$ where $\varphi^i : \mathfrak{M} \to \mathfrak{M}^i$ is the canonical projection and $\{(\pi^{-1}(U_k^i), \Psi_k)\}_{k\in I}$ is the corresponding trivialization for $T\mathfrak{M}^i$, $\Psi_k^{2,i}$ is the projection of Ψ_k^i onto its second factor. Now $\alpha = \underleftarrow{lim}\alpha^i$ is integral curve for X. We must show $\phi^{ji} \circ \alpha^j = \alpha^i$ for $j \geq i$. It is sufficient to show $\phi^{ji} \circ \alpha^j$ is also an integral curve of X^i. Then we have

$$\begin{aligned}
(\psi_k^i \circ (\phi^{ji} \circ \alpha^j)' &= \rho^{ji} \circ (\psi_k^j \circ \alpha^j))'(t) = \rho^{ji}(\psi_k^j \circ \alpha^j)'(t) \\
&= \rho^{ji}(\Psi_k^{2,j}(X^j(\alpha^j(t)))) = \Psi_k^{2,j}(T\phi^{ji}(X^j(\alpha^j(t))) \\
&= \Psi_k^{2,j}(X^j(\phi^{ji} \circ \alpha^j(t)))
\end{aligned}$$

and also $\phi^{ji} \circ \alpha^j(g_0^j) = g_0^i$. Therefore by uniqueness of integral curve $\phi^{ji} \circ \alpha^j = \alpha^i$. This shows $\{\alpha^i\}_{i\in\mathbb{N}}$ is a projective system of curves and $\alpha = \underleftarrow{lim}\alpha^i$ exists. α satisfies the conditions for theorem because

$$(\psi\circ\alpha)'(t)=((\psi_k^i\circ\alpha^i)'(t))_{i\in\mathbb{N}}=(\Psi_k^{2,i}(X^i(\alpha^i(t))))_{i\in\mathbb{N}}=\Psi_k^2(X(\alpha(t))).$$

Uniqueness of α follows from uniqueness of α^i. □

4.2 Geodesics on $\mathfrak{M}$

Geodesics on $\mathfrak{M}$ with initial conditions have been calculated in [10].Freed and Groisser fixed $\mu\in Vol(M)$ and implicity identified $\mathfrak{M}$ with $Vol(M)\times\mathfrak{M}_\mu$ as in (2.1) and found the geodesics. In this section we explain geodesic equation on $\mathfrak{M}$.

Proposition 4.2. *[10] The geodesic in $Vol(M)$ with initial position μ and initial velocity $\alpha\in T_\mu(Vol(M)=\Omega^n(M)$ is*

$$\mu(t)=\mu_t=(1+\frac{1}{2}(\frac{\alpha}{\mu})t)\mu.$$

Proposition 4.3. *[10] The geodesic in $\mathfrak{M}_\mu$ with initial position g and initial velocity $A\in T_g(\mathfrak{M}_\mu)$ is*

$$g(t)=g_t=ge^{t(g^{-1}A)}.$$

Theorem 4.4. *[10] the geodesic in $\mathfrak{M}$ with initial position (μ,g) and initial velocity $(\alpha,A)\in\Omega^n(M)\times sym_0(M,g)$ is*

$$i_\mu(g_t)=(q(t)^2+r^2t^2)^{\frac{2}{n}}g\exp(\frac{\tan^{-1}(rt/q)}{r}g^{-1}A),$$

where $q(t)=1+\frac{1}{2}(\alpha/\mu)t$, $r=\frac{1}{4}(ntr((g^{-1}A)^2))^{\frac{1}{2}}$ and i_μ is defined as in previous section. If $r=0$ after replacing the exponential term by 1, then the change in volume form of $g(t)$ is given by the formula

$$\mu(g(t))=(q(t)^2+r^2t^2)\mu.$$

Sketch of proof. Let $\{g(t)\}$ be a geodesic and $B=g'=\frac{dg}{dt}$ then the above formula of Levi-Civita connection has been applied to the geodesic equation $\nabla_{g'}g'=0$ to obtain (μ_t,h_t). Finally using splitting map i_μ for $\mathfrak{M}\cong Vol(M)\times\mathfrak{M}_\mu$, geodesics are obtained, you can see [10] for more details. □

Theorem 4.5. *The velocity vector of geodesics on $\mathfrak{M}$ is as follows:*

$$\alpha'(t)=(\alpha(1+\frac{1}{2}\frac{\alpha}{\mu}t)+2r^2t\mu,\frac{A}{q^2+r^2t^2}\exp(\frac{\tan^{-1}(\frac{rt}{q})}{r}g^{-1}A))$$

also for $r=0$

$$\alpha'(t)=(\alpha(1+\frac{1}{2}\frac{\alpha}{\mu}t),0).$$

Proof. According to $\mathfrak{M}\cong Vol(M)\times\mathfrak{M}_\mu$, a geodesic is of the form $\alpha(t)=(\mu_t,h_t)$ where

$$\mu_t=\mu(g(t))=(q(t)^2+r^2t^2)\mu$$

and

$$h_t = g\exp\left(\frac{\tan^{-1}(\frac{rt}{q})}{r}g^{-1}A\right).$$

Therefore

$$\frac{d}{dt}\mu_t = (2q(t)\acute{q}(t)+2r^2t)\mu = \left(2\left(1+\frac{1}{2}\frac{\alpha}{\mu}t\right)\frac{1}{2}\frac{\alpha}{\mu}+2r^2t\right)\mu = \left(1+\frac{1}{2}\frac{\alpha}{\mu}t\right)\alpha + 2r^2t\mu.$$

As well, we get

$$\begin{aligned}
\frac{d}{dt}h_t &= g\exp\left(\frac{\tan^{-1}(\frac{rt}{q}}{r}g^{-1}A\right)g^{-1}A\times\frac{1}{r}\frac{\frac{d}{dt}(\frac{rt}{q})}{1+(\frac{rt}{q})^2}\\
&= g\exp\left(\frac{\tan^{-1}(\frac{rt}{q})}{r}g^{-1}A\right)g^{-1}A\times\frac{1}{r}\times\frac{q^2}{q^2+r^2t^2}\times\frac{rq-\frac{1}{2}\frac{\alpha}{\mu}tr}{q^2}\\
&= \exp\left(\frac{\tan^{-1}(\frac{rt}{q})}{r}g^{-1}A\right)A\times\frac{q-\frac{1}{2}\frac{\alpha}{\mu}t}{q^2+r^2t^2}\\
&= \frac{A}{q^2+r^2t^2}\exp\left(\frac{\tan^{-1}(\frac{rt}{q})}{r}g^{-1}A\right)
\end{aligned}$$

thus for $r \neq 0$ we have

$$\alpha'(t) = \left(\alpha\left(1+\frac{1}{2}\frac{\alpha}{\mu}t\right)+2r^2t\mu, \frac{A}{q^2+r^2t^2}\exp\left(\frac{\tan^{-1}(\frac{rt}{q})}{r}g^{-1}A\right)\right),$$

and also for $r = 0$ according to $\alpha(t) = (q(t)^2\mu, g)$ the velocity vector of geodesic is as follows:

$$\alpha'(t) = (\alpha(1+\frac{1}{2}\frac{\alpha}{\mu}t), 0),$$

which concludes the proof. □

Remark 4.2. By Definition 4.1, a Ricci flow equation can be considered as a curve on $\mathfrak{M}$. We call these curves as *Ricci flow curve.*

We will study the behavior of them on the manifold $\mathfrak{M}$. In special cases we obtain the following result:

Theorem 4.6. *A Ricci flow curve starting from an Einstein metric g_0 is not a geodesic.*

Proof. Let g_0 be an Einstein metric: $Ric(g_0) = \lambda g_0$, where λ is a constant. Then $g(t) = (1-2\lambda t)g_0$.Then by Theorem 4.4 this curve is not the geodesic. □

Moreover using Theorem 4.5, we see that velocity vector of geodesic on $\mathfrak{M}$ is different from velocity vector of Ricci flow curve

Proposition 4.7. *If M is a compact manifold with strictly positive scalar curvature R_0 then the space of Riemannian metrics on the compact manifold M is foliated by the Ricci flow.*

Proof. Since M is a compact manifold with strictly positive scalar curvature R_0, Ricci flow equation is a non-vanishing vector field on $\mathfrak{M}$ also solution of Ricci flow are integral curves on the $\mathfrak{M}$. It is obvious that this integral curve foliate the $\mathfrak{M}$. □

4.3 Ricci solitons

Ricci solitons are special solution of Ricci flow such that every $g(t)$ is the form $g(t) = \sigma(t)\varphi_t^*(g_0)$ where $\sigma(t)$ and $\varphi(t)$ are respectively scalar and diffeomorphisms of M and g_0 is initial metric and $\varphi_0 = Id, \quad \sigma(0) = 1$[5]. We call these curve, the Ricci soliton curve.

Theorem 4.8. *Ricci soliton curve results in an equation which consists of an initial metric and a vector field and a scalar as follows:*

$$-2Ricg_0 = 2\lambda g_0 + \pounds_V g_0.$$

Proof. For arbitrary curve $\alpha(t) = \sigma(t)\varphi_t^*(g_0)$ according to decomposition $\mathfrak{M} \cong Vol(M) \times \mathfrak{M}_\mu$ we have:

$$\alpha(t) = \left(\mu_{\sigma(t)\varphi_t^*(g_0)}, \left(\frac{\mu}{\mu_{\sigma(t)\varphi_t^*(g_0)}}\right)^{\frac{2}{n}} \sigma(t)\varphi_t^*(g_0)\right).$$

The velocity vector $\alpha'(t)$ is as follows:

$$\begin{aligned}
\alpha'(t) &= \Big(\tfrac{1}{2}tr_{\sigma(t)\varphi_t^*(g_0)}(\tfrac{\partial}{\partial t}\sigma(t)\varphi_t^*(g_0))\mu_{\sigma(t)\varphi_t^*(g_0)},\\
&\quad (\tfrac{\mu}{\mu_{\sigma(t)\varphi_t^*(g_0)}})^{\frac{2}{n}}(\tfrac{\partial}{\partial t}\sigma(t)\varphi_t^*(g_0) - \tfrac{1}{n}tr_{\sigma(t)\varphi_t^*(g_0)}(\tfrac{\partial}{\partial t}\sigma(t)\varphi_t^*(g_0))\sigma(t)\varphi_t^*(g_0)\Big)\\
&= \Big(\tfrac{1}{2}\tfrac{1}{\sigma(t)}(\varphi^*)_t^{-1}(g_0)(\tfrac{d\sigma(t)}{dt}\varphi_t^*(g_0) + \sigma(t)\tfrac{\partial}{\partial}\varphi_t^*(g_0))\mu_{\sigma(t)\varphi_t^*(g_0)},\\
&\quad (\tfrac{\mu}{\mu_{\sigma(t)\varphi_t^*(g_0)}})^{\frac{2}{n}}(\tfrac{d\sigma(t)}{dt}\varphi_t^*(g_0) + \sigma(t)\tfrac{\partial}{\partial}\varphi_t^*(g_0) - \tfrac{1}{n}\tfrac{1}{\sigma(t)}(\varphi^*)_t^{-1}(g_0)(\tfrac{d\sigma(t)}{dt}\varphi_t^*(g_0)\\
&\quad +\sigma(t)\tfrac{\partial}{\partial}\varphi_t^*(g_0))\sigma(t)\varphi_t^*(g_0)\Big)
\end{aligned}$$

and at $t = 0$ we have

(4.1)
$$\alpha'(0) = (\frac{1}{2}g_0^{-1}(\sigma'(0)g_0 + \pounds_V g_0)\mu_{g_0}, (\frac{\mu}{\mu_{g_0}})^{\frac{2}{n}}(\sigma'(0)g_0 + \pounds_V g_0 - \frac{1}{n}(g_0^{-1}(\sigma'(0)g_0 + \pounds_V g_0)g_0)),$$

where $\pounds$ denotes the Lie derivative, V is the time-dependent vector field such that $V(\varphi_t(p)) = \frac{d}{dt}\varphi_t(p))$ for any $p \in M$. On the other hand for $\alpha(t)$ as a Ricci flow curve according to (3.4), the velocity vector $\alpha'(t)$ is as follows:

$$\begin{aligned}
\alpha'(t) &= (-R\mu_{\sigma(t)\varphi_t^*(g_0)}, -2(\tfrac{\mu}{\mu_{\sigma(t)\varphi_t^*(g_0)}})^{\frac{2}{n}}[Ric(\sigma(t)\varphi_t^*(g_0)) - \tfrac{R}{n}\sigma(t)\varphi_t^*(g_0)]\\
&= (-\tfrac{1}{\sigma(t)}\varphi_t^*(R_0)\sigma(t)^{\frac{n}{2}}\mu_{\varphi_t^*(g_0)}, -2\tfrac{1}{\sigma(t)}(\tfrac{\mu}{\mu_{\sigma(t)\varphi_t^*(g_0)}})^{\frac{2}{n}}\cdot\\
&\quad \cdot[\varphi_t^*(Ricg_0) - \tfrac{1}{n}\tfrac{1}{\sigma(t)}\varphi_t^*(R_0)\sigma(t)\varphi_t^*(g_0)])\\
&= (-\varphi_t^*(R_0)\sigma(t)^{\frac{n}{2}-1}\mu_{\varphi_t^*(g_0)}, \tfrac{-2}{\sigma(t)}(\tfrac{\mu}{\mu_{\sigma(t)\varphi_t^*(g_0)}})^{\frac{2}{n}}[\varphi_t^*(Ricg_0) - \tfrac{\varphi_t^*(R_0)}{n}\varphi_t^*(g_0)]);
\end{aligned}$$

at $t = 0$ we have

$$\alpha'(0) = \left(-R_0\mu_{g_0}, -2(\frac{\mu}{\mu_{g_0}})^{\frac{2}{n}}[Ricg_0 - \frac{R_0}{n}g_0]\right). \tag{4.2}$$

By comparing (4.1) and (4.2) have:

$$\begin{cases} -R_0 = \frac{1}{2}g_0^{-1}(\sigma'(0)g_0 + \pounds_V g_0), \\ -2(Ricg_0 - \frac{R_0}{n}g_0) = \sigma'(0)g_0 + \pounds_V g_0 - \frac{1}{n}g_0^{-1}(\sigma'(0)g_0 + \pounds_V g_0)g_0, \end{cases}$$

Therefore $-2Ricg_0 = \sigma'(0)g_0 + \pounds_V g_0$,
Let $\sigma'(0) = 2\lambda$; then $-2Ricg_0 = 2\lambda g_0 + \pounds_V g_0$. □

Remark 4.3. The above theorem coincides with the well-known result on Ricci soliton - see for example [5].

The Ricci flow curve has certain properties which will be included in another forthcoming paper.

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References

[1] M. Aghasi, A. Suri, *Ordinary differential equations on infinite dimensional manifolds*, Balkan J. Geom. Appl. 12, 1 (2007), 1-8.

[2] B. Andrews, C. Hopper, *The Ricci Flow in Riemannian Geometry: A Complete Proof of the Differentiable 1/4-Pinching Sphere Theorem*, Springer-Verlag, 2011.

[3] D. E. Blair, *Spaces of metrics and curvature functionals*, in Handbook of Differential Geometry, I, North Holland, Amsterdam 2000, 153-185.

[4] D. E. Blair, *On the space of Riemannian metrics on surfaces and contact manifolds*, Lect. Notes. Math. 792 (1980), 203-212.

[5] B. Chow, D. Knopf, *The Ricci Flow: An Introduction*, AMS, Providence, Surveys and Monographs, United States of America. 2004.

[6] D. M. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, J. Differ. Geom. 18, 1 (1983), 157-162.

[7] C. T. J. Dodsona, G. N. Galanis, E. A. Vassiliouc, *A generalized second-order frame bundle for Fréchet manifolds*, J. Geom. Phys. 55, 3 (2005), 291-305.

[8] J. Eells, J. H. Sampson *Harmonic mappings of Riemannian manifolds*, AM. J. MATH. 86, 1 (1964), 109-160.

[9] D. Ebin, The manifold of Riemannian metrics, Proc. Symp. Pure Math. 15 (1970), 11-40.

[10] D. S. Freed, D. Groisser, *The basic geometry of the manifold of Riemannian metrics and of its quotient by the diffeomorphism group*, Mich. Math. J.36, 3 (1989), 323-344.

[11] G. N. Galanis, *Differential and geometric structure for the tangent bundle of a projective limit manifold*, Rend. Sem. Mat. Univ. Padova. 112 (2004), 103115.

[12] O. Gil-Medrano, P. Michor, *The Riemannian manifold of all Riemannian metrics*, Quart. J. Math. Oxford. 42, 2 (1991), 183-202.

[13] R. Hamilton, *The inverse function theorem of Nash and Moser*, B. Am. Math. Soc 7 (1982), 65-222.

[14] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differ. Geom. 17, 2 (1982), 255-306.

[15] A. Kriegl, P. Michor, *The convenient setting of global analysis*, Mathematical Surveys and Monographs 53, American Mathematical Society 1997.

[16] J. A. Leslie, *On a differential structure for the group of diffeomorphisms*, Topology. 46, (1967), 263-271.

[17] H. Omori. *On the group of diffeomorphisms on a compact manifold*, Proc. Symp. Pure Appl. Math. XV, (1970), 167-183.

[18] H. Omori, *Infinite-Dimensional Lie Groups*, Translations of Mathematical Monographs 158, American Mathematical Society, Berlin, 1997.

[19] R. S. Palais, *Seminar on the Atiyah-Singer Index Theorem*, Princeton Univ. Press, Princeton, New Jersey 1965.

[20] R. S. Palais, *Foundations of Global Nonlinear Analysis*, Benjamin, New York 1968.

[21] N. K. Smolentsev, *Space of Riemannian metrics*, J. Math. Sci. (N. Y.) 142, 5 (2007), 2436-2519.

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