A connection between 5-systolicity and CAT(−1) metrics on cubical complexes

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Abstract. We investigate the existence of a CAT(−1) metric on a piecewise hyperbolic cubical complex of edge lengths ε whose cells represent only finitely many isometry classes, and satisfying a combinatorial curvature condition called local 5-largeness. We will show that, for some ε, the star of any cell of a locally 5-large cubical complex is locally a CAT(−1) space, whereas any 5-systolic cubical complex is a CAT(−1) space. The key step of our proof is to show that the link of any vertex of an 5-large regular piecewise hyperbolic cubical complex is a flag piecewise spherical simplicial cell complex with no empty 4-circuits.

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1 Introduction

Considering tessellations of the hyperbolic 3-space by cubes, we note that the link of each vertex of such space is the boundary of an icosahedron. In this paper we show that the inverse of sorts of this statement holds as well, moreover, in any finite dimension. Namely, we prove that the piecewise hyperbolic metric for which the cells are regular hyperbolic with edge lengths $\varepsilon > 0$, on any finite dimensional 5-systolic cubical complex with finitely many shapes of cells is, for some ε, CAT(−1).

So in this paper we investigate connections between simplicial nonpositive curvature conditions and metric nonpositive curvature conditions on piecewise hyperbolic cubical complexes. The combinatorial curvature condition we consider is called local 5−largeness. It was introduced on simplicial complexes by T. Januszkiewicz and J. Swiatkowski in [11] and independently by F. Haglund in [10]. It is defined in terms of links in the complex by very simple combinatorial means. The metric curvature condition we have in mind is given by the so called CAT(k) inequality, $k \leq 0$ (see [1],

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[2], [4], [3]). The most important case is when $k = 0$; the case $k < 0$, however, is also interesting. We focus in this paper on this second case.

Links between the two curvature conditions were studied before (see [11], [12]). It turns out that in dimension two they are equivalent: a polyhedral space has curvature ≤ -1 (is nonpositively curved) if and only if every injective loop in the link of each of its vertices has length strictly greater than 2π (greater than 2π) (see [2], chapter II.5, page 216, page 207). Local 7-largeness (local 5-largeness, local 4-largeness, local 3-largeness) therefore coincides with the CAT(-1) property of piecewise hyperbolic metrics for which the cells are regular hyperbolic on simplicial (cubical, pentagonal, hexagonal) 2-complexes. Similarly, local 6-largeness (local 4-largeness, local 3-largeness) is equivalent to the CAT(0) property of standard piecewise Euclidean metrics on simplicial (cubical, hexagonal) 2-complexes. $4-$systolic cubical complexes are, according to M. Gromov's combinatorial description of nonpositively curved cubical complexes (see [8], Appendix I.6, page 516), CAT(0) spaces (see [13]). Moreover, under a technical condition, both CAT(0) and systolic simplicial complexes of dimension 3 or less endowed with the standard piecewise Euclidean metric, simplicially collapse to a point (see [6], [14]). Unfortunately, the equivalence no longer holds in higher dimensions (see [11], chapter 14, page 51).

Still, one implication remains true on simplicial complexes of dimensions greater than two. Namely, in [11] (chapter 14, page 52) it is shown that there exists a constant $k(n) \geq 6$ such that the piecewise hyperbolic (piecewise Euclidean) metric on a $k(n)$-systolic, n-dimensional simplicial complex whose simplices represent only finitely many isometry classes, is CAT(-1) (CAT(0)). It is also proven there that there exists a constant $k(n) \geq 6$ such that the piecewise spherical metric on a $k(n)$-large, n-dimensional simplicial complex with finitely many shapes of simplices, is CAT(1). M. Gromov's combinatorial characterization of nonpositively curved piecewise Euclidean cubical complexes given in [8] (Appendix I.6, page 516) guarantees that an implication similar to the one proven in the paper holds on $4-$systolic cubical complexes. Namely, the standard piecewise Euclidean metric on any 4-systolic cubical complex is CAT(0) (see [13]).

In [11] (chapter 2, page 14) it is proven that the 1-skeleton of any 7-systolic simplicial complex with its standard hyperbolic metric is hyperbolic. More precisely, geodesic triangles in the 1-skeleton of such complex, with vertices at vertices of the complex, are $\frac{5}{2}$-slim. It is the paper's object to show that for some $\varepsilon > 0$, 5-systolicity on cubical complexes endowed with the piecewise hyperbolic metric for which the cells have edge lengths ε, also ensures hyperbolicity. Our proof relies on a combinatorial description of piecewise hyperbolic cubical complexes of curvature ≤ -1 given in [8] (see Appendix I.6, page 520). The description is an immediate consequence of L. Siebenmann's flag-no-square condition given in [8] (see Appendix I.6, page 518).

The paper is divided as follows. In section 3 we prove certain well known results in small cancelation theory on cubical complexes of arbitrary dimensions. Similar results were obtained by T. Januszkiewicz and J. Swiatkowski on simplicial complexes in [11]. In particular we show that any 5-systolic cubical complex is 5-large. This is a combinatorial analogue of the fact that simply connected spaces of curvature ≤ -1, are CAT(-1) spaces. In section 4 we show our main result.

2 Preliminaries

We present in this section the notions we shall work with and the results we shall refer to.

Let k be a real number. Let M_k^n denote a simply connected, complete, Riemannian $n-$manifold of constant curvature k. So X_0^2 is the Euclidean plane $\mathbb{R}^2$. If $k < 0$, M_k^2 is the hyperbolic plane. If $k > 0$, M_k^2 is the 2-sphere with its metric re-scaled so that its curvature is k (i.e., it is the sphere of radius $\frac{1}{\sqrt{k}}$).

Let (X, d) be a metric space. Given a path $\gamma : [a, b] \to X$ in X, its *length* is defined by $L(\gamma) = \sup\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i))\}$, where the supremum is taken over all possible subdivisions of $[a, b]$, $a = t_0 < t_1 < ... < t_n = b$.

We call (X, d) a *geodesic space* if given two points p, q in X, there is a path from p to q whose length equals $d(p, q)$. Such a distance minimizing path is called a *geodesic segment* and we denote it by $[p, q]$.

Let (X, d) be a geodesic space. A *geodesic triangle* in X consists of three points $p, q, r \in X$, called *vertices*, and a choice of three geodesic segments $[p, q], [q, r], [r, p]$ joining them, called *sides*. Such a geodesic triangle is denoted by $\triangle = \triangle(p, q, r)$. If a point $x \in X$ lies in the union of $[p, q], [q, r]$ and $[r, p]$, then we write $x \in \triangle$. A triangle $\overline{\triangle} = \triangle(\overline{p}, \overline{q}, \overline{r})$ in M_k^2 is called a *comparison triangle* for $\triangle = \triangle(p, q, r)$ if $d(p, q) = d_{M_k^2}(\overline{p}, \overline{q})$, $d(q, r) = d_{M_k^2}(\overline{q}, \overline{r})$ and $d(r, p) = d_{M_k^2}(\overline{r}, \overline{p})$. A point $\overline{x} \in [\overline{q}, \overline{r}]$ is called a *comparison point* for $x \in [q, r]$ if $d(q, x) = d_{M_k^2}(\overline{q}, \overline{x})$.

A metric space X is a CAT(k) space if it is a geodesic space all of whose geodesic triangles satisfy the so called *CAT(k)-inequality*. Namely, for any geodesic triangle $\triangle(p, q, r) \subset X$ (with $l(\triangle) < \frac{2\pi}{\sqrt{k}}$ if $k > 0$), and any two points $x, y \in \triangle$, we have: $d(x, y) \le d_{M_k^2}(\overline{x}, \overline{y})$, where $\overline{x}, \overline{y}$ are the corresponding points in the comparison triangle $\overline{\triangle}$.

A metric space X *has curvature* $\le k$ if the CAT(k)- inequality holds locally in X. If X has curvature ≤ 0, we say it is *nonpositively curved*.

Let $\delta > 0$. A geodesic triangle in a metric space is said to be $\delta-$*slim* if each of its sides is contained in the $\delta-$neighborhood of the union of the other two sides. A geodesic space is said to be $\delta-$*hyperbolic* if each of its geodesic triangles is $\delta-$slim. If $k < 0$ then every CAT(k) space is $\delta-$hyperbolic, where δ depends only on k (for the proof see [2], chapter III.H.1, page 399).

A *convex M_k^n polyhedral cell* P is the convex hull of a finite subset in M_k^n. The *support* of a point $x \in P$, denoted *supp*(x), is the unique face of P containing x in its interior. Suppose M_k^n has tangent space V. Let TP denote the tangent space of P, i.e., TP is the linear subspace of V consisting of all vectors of the form $t(x - y)$, where $x, y \in P$ and $t \in \mathbf{R}$. Given $x \in P$, define C_xP, the inward-pointing tangent cone at x, to be the set of all $v \in TP$ such that $x + tv \in P$ for all $t \in [0, \alpha)$ for some $\alpha > 0$. C_xP is a linear polyhedral cone in TP. Suppose F is a proper face of P. If x, y are both in $int(F)$, then C_xP and C_yP have the same image in TP/TF. This common image is an essential polyhedral cone, denoted by $Cone(F, P)$. The *link of F in P*, denoted $Lk(F, P)$, is the intersection of $Cone(F, P)$ with an affine hyperplane in TP/TF which intersects every nonzero face of $Cone(F, P)$. $Lk(F, P)$ is a convex polytope in this affine hyperplane of dimension $dimP - dimF - 1$.

The proof of the main result of the paper makes use of the following relation proven in [8], Appendix A.6, page 419.

Lemma 2.1. *Suppose v is a vertex of a cell F in some convex cell complex X. Then $Lk(\sigma_F, Lk(v, X)) = Lk(F, X)$, where σ_F denotes the cell of $Lk(v, X)$ corresponding to F.*

A *cubical complex* X is the quotient of a disjoint union of cubes $L = \bigcup_\Lambda J_\lambda$ by an equivalence relation $\sim$. The restrictions $p_\lambda : J_\lambda \to X$ of the natural projection $p : L \to X = L|_\sim$ are required to satisfy:

1. for every $\lambda \in \Lambda$, the map p_λ is injective;
2. if $p_\lambda(J_\lambda) \bigcap p_{\lambda'}(J_{\lambda'}) \neq \varnothing$, then there is an isometry $h_{\lambda,\lambda'}$ from a face $T_\lambda \subset J_\lambda$ onto a face $T_{\lambda'} \subset J_{\lambda'}$ such that $p_\lambda(x) = p_{\lambda'}(x')$ if and only if $x' = h_{\lambda,\lambda'}(x)$.

We note that the definition of a cubical complex mimicks the one of a simplicial complex. There are many interesting polyhedral complexes all of whose cells are simplices (cubes) but which do not satisfy all conditions from the definition of a simplicial (cubical) complex. We use the term simplicial cell complex (cubed complex) to describe this larger class of complexes and introduce it below.

Let $(P_\lambda : \lambda \in \Lambda)$ be a family of convex M^n_k-polyhedral cells and let $L = \cup_{\lambda \in \Lambda}(P_\lambda \times \{\lambda\})$ denote their disjoint union. Let $\sim$ be an equivalence relation on L and let $X = L|_\sim$. Let $p : L \to X$ be the natural projection and define $p_\lambda : P_\lambda \to X$ by $p_\lambda(x) := p(x, \lambda)$. X is called an n-dimensional *M^n_k-polyhedral complex* if:

1. for all $\lambda \in \Lambda$, the restriction of p_λ to the interior of each face of P_λ is injective;
2. for all $\lambda_1, \lambda_2 \in \Lambda$ and $x_1 \in P_{\lambda_1}$, $x_2 \in P_{\lambda_2}$, if $p_{\lambda_1}(x_1) = p_{\lambda_2}(x_2)$ then there is an isometry $h : supp(x_1) \to supp(x_2)$ such that $p_{\lambda_1}(y) = p_{\lambda_2}(h(y))$ for all $y \in supp(x_1)$.

$|X|$ denotes the underlying space of X, and $X^{(k)}$ denotes the k-skeleton of X.

A *simplicial cell complex* (*cubed complex*) is a M^n_k-polyhedral complex whose 2-cells have three (four) 1-dimensional faces. What distinguishes simplicial (cubical) complexes among simplicial cell complexes (cubed complexes) is that the intersection of any two simplices (cells) in such complex is either the empty set or a single common simplex (cell). In a simplicial cell complex (cubed complex) such intersection may be a union of faces.

Suppose X is a M^n_k-polyhedral complex. A path $\gamma : [a, b] \to X$ is *piecewise geodesic* if there is a subdivision $a = t_0 < t_1 < ... < t_n = b$ so that for $1 \leq i \leq n$, $\gamma([t_{i-1}, t_i])$ is contained in a single (closed) cell of X and so that the restriction of γ to $[t_{i-1}, t_i]$ is a geodesic segment in that cell. The length of the piecewise geodesic γ is defined by $L(\gamma) = \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i))$. X has a natural length metric $d(x, y) := \inf\{L(\gamma)|\gamma$ is a piecewise geodesic from x to $y\}$. The length space X is called a *piecewise constant curvature polyhedron.* As k equals $+1, 0$ or 1, we say that it is, respectively, *piecewise spherical, piecewise Euclidean*, or *piecewise hyperbolic.*

A piecewise spherical simplex is *all right* if each of its edges has length $\frac{\pi}{2}$. It has size $< \frac{\pi}{2}$ if each of its edges has length $< \frac{\pi}{2}$. A piecewise spherical simplicial cell complex is *all right* (has simplices of size $< \frac{\pi}{2}$) if the corresponding property holds for each of its simplices.

The link of any cell in a piecewise hyperbolic cubical complex is a piecewise spherical simplicial cell complex with simplices of size $< \frac{\pi}{2}$.

Suppose X is a simplicial cell complex (cubed complex). A set V of vertices in X *spans a complete graph* if any two distinct elements of V span an edge in X. We call X a *flag complex* if any set of vertices which spans a complete graph actually spans a simplex (cell).

Let σ be a cell of X. The (closed) *star* of σ in X, denoted $St(\sigma, X)$, is the union of all cells of X that contain σ. A *cycle* in X is a subcomplex γ of X isomorphic to a triangulation of $S^{(1)}$. We denote by $|\gamma|$ the number of 1-cells contained in γ and we call $|\gamma|$ the *length* of γ. A subcomplex L in X is called *full* (in X) if any cell of X spanned by a set of vertices in L, is a cell of L. A *full cycle* in X is a cycle that is full as subcomplex of X. We define the *systole* of X by

$$sys(X) = \min\{|\gamma| : \gamma \text{ is a full cycle in } X\}.$$

Let $k \geq 4$ ($k \geq 3$) be a natural number. We call X *k-large* if $sys(X) \geq k$ ($sys(X) \geq 2k$) and $sys(Lk(\sigma, X)) \geq k$ ($sys(Lk(\sigma, X)) \geq 2k$) for each simplex (cell) σ of X. We call X *locally k-large* if the star of every simplex (cell) of X is k-large. We call X *k-systolic* if it is connected, simply connected and locally k-large. We abbreviate 6-systolic to systolic.

Note that a (locally) k-large complex is (locally) m-large for $k \geq m$. Note further that a simplicial cell complex is flag if and only if it is 4-large whereas a cubed complex is flag if and only if it is 3-large.

Remark 2.1. Every cycle of length less than k in a k-large simplicial cell complex X has some two consecutive edges contained in a common 2-simplex of X (see [11], chapter 1, page 10).

An empty $4-$circuit in a simplicial cell complex X is a circuit of 4 edges such that neither pair of opposite vertices is connected by an edge. In other words, the 4-circuit is a full subgraph of $X^{(1)}$. If X is a flag complex, a 4-circuit is empty if and only if it is not the boundary of two adjacent 2-simplices. We say X *satisfies the no-square condition* if it has no empty 4-circuits.

The main result of the paper uses the following result proven in [8], chapter I.6, page $518 - 520$.

Theorem 2.2. *Suppose X is a locally finite cubical complex endowed with the hyperbolic metric obtained by declaring each cube to be isometric to a regular cube of edge length $\varepsilon > 0$ in the hyperbolic space. Further suppose that there are only finitely many isomorphism types of links of vertices in X. Then X has curvature ≤ 1 for some ε if and only if the link of each of its vertices is a flag complex satisfying the no-square condition.*

We shall study a cubed complex X by associating to each cycle γ in X a diagram in the Euclidean plane, called a van Kampen diagram, which contains all the essential information about γ (see [15]).

A *combinatorial map* $f : X_1 \to X_2$ between two cubed complexes X_1 and X_2 is a homeomorphism which maps each open cell of X_1 onto an open cell of X_2. We call a combinatorial map *nondegenerate* if it is injective on each cell of the cellulation. A *combinatorial 2-complex* is a 2-dimensional cell complex whose 2-cells are attached along continuous maps from $S^{(1)}$ to the 1-skeleton of the complex. A *combinatorial disk* is a combinatorial 2-complex homeomorphic to a disk.

Let $\gamma = e_0 e_1 ... e_n$ be a cycle in X. A *van Kampen diagram* for γ is a pair (D, ϕ). D is a finite, simply connected combinatorial disk embedded in the Euclidean plane, bounded by the cycle $\beta = f_0 f_1 ... f_n$. $\phi : D \to K$ is a combinatorial map assigning to each edge f_i of β in D an edge $\phi(f_i) = e_i$ of γ in X such that $\phi(f_i^{-1}) = \phi(f_i)^{-1}$ for all $0 \leq i \leq n$. A *region* is a 2-cell of D. The *area* of the diagram is given by the number of regions of D.

Let v be a vertex of X. The *degree* of v, denoted by $\deg v$, is the number of edges having v as initial vertex.

An *almost cubed 2-complex* is a cubed complex whose cells are glued to lower dimensional skeleta through nondegenerate maps, i.e. such that multiple edges and loops are allowed in the 1-skeleton of the complex, and the interior of each boundary edge of a 2-cell is glued through homeomorphisms to the 1-skeleton on the interior of some 1-cell. A cell map from an almost cubed 2-complex to a cubed complex is determined by its values at the vertices as an ordinary cell map, i.e. a loop is mapped to a vertex.

3 The geometry of 5−systolic cubical complexes

In this section we proof certain well known results in small cancelation theory (see [15], chapter V, page $237 - 242$) on 5-systolic cubical complexes of higher dimensions. The main purpose of the section is to show that any 5-systolic cubical complex is 5-large. Similar results were obtained by T. Januszkiewicz and J. Swiatkowski in [11] (see chapter 1, page $10 - 14$) on systolic simplicial complexes. Our approach is based on their considerations.

We start by proving a combinatorial Gauss-Bonnet theorem on cubical disks.

Lemma 3.1. *Let D be a cubical disk. Then:*

$$4 = \sum_{v \in int(D)} (4 - \ deg\, v) + \sum_{v \in \partial D} (3 - \ deg\, v),$$

where we take the sums over the interior vertices of D and over the boundary vertices of D, respectively.

Proof. We denote the set of interior vertices of D by $int(D)$. As well, V, V_{int}, V_{ext}, E, E_{ext} and F will accordingly represent the number of vertices, interior vertices, exterior vertices, edges, exterior edges and 2-cells of D. It is known that the following relations hold true, in any cubical disk:

$$1 = V - E + F, \;\; 2E - E_{ext} = 4F, \;\; V_{ext} = E_{ext}, \;\; \sum_{v} \deg v = 2E.$$

Using these relations, we obtain:

$$\begin{aligned} 6 &= 6(V - E + F) = 6V - \tfrac{3}{2} \cdot 2E - \tfrac{3}{2} V_{ext} \\ &= 6V - \tfrac{3}{2}\Big(\sum_{v \in int(D)} \deg v + \sum_{v \in \partial D} \deg v\Big) - \tfrac{3}{2} V_{ext} \\ &= \tfrac{3}{2}\Big(4V_{int} - \sum_{v \in int(D)} \deg v\Big) + \tfrac{3}{2}\Big(3V_{ext} - \sum_{v \in \partial D} \deg v\Big). \end{aligned}$$

Hence we infer $4 = \sum_{v\in int(D)} (4 - \deg v) + \sum_{v\in\partial D} (3 - \deg v)$. □

The purpose of the following lemma is analogous to reducing van Kampen diagrams in small cancelation theory.

Lemma 3.2. *Given $k \geq 3$ natural, let X be a k-large cubed complex. Let $S^{(1)}_{2m}$ denote the triangulation of $S^{(1)}$ with $2m$ 1-cells. If $m < k$ then any combinatorial map $f : S^{(1)}_{2m} \to X$ extends to a combinatorial map from the disk $D^{(2)}$, triangulated so that triangulation on the boundary is $S^{(1)}_{2m}$ and so that there are no interior vertices in $D^{(2)}$.*

Proof. Note first that, because $k \geq 3$, X is flag.
The proof is by induction on m. For $m = 2$, because X is flag, the result follows. For $m \geq 3$, take some non-consecutive vertices u, v of $S^{(1)}_{2m}$ such that the vertices $f(u)$ and $f(v)$ are joined by none or at least one edge of X. The case when $f(u)$ and $f(v)$ are joined by at least two edges, is clear due to flagness of X. Consider further the other two cases. Dividing $S^{(1)}_{2m}$ into two edge-paths, say A and B, with endpoints u and v and adding the edge $[u, v]$ to A and B, respectively, we get new triangulations of $S^{(1)}_{2m}$ denoted by $S^{(1)}_A$ and $S^{(1)}_B$. By the choice of u and v, restrictions of the map f to A and B extend uniquely to the combinatorial maps $f_A : S^{(1)}_A \to X$ and $f_B : S^{(1)}_B \to X$, respectively. The choice of u and v further implies that neither $S^{(1)}_A$ nor $S^{(1)}_B$ consists of more than $2m$ edges. Therefore, by the inductive assumption, there are two 2-disks $D^{(2)}_A$ and $D^{(2)}_B$ with no interior vertices, and triangulated so that triangulation on the boundary is $S^{(1)}_A$ and $S^{(1)}_B$, respectively, and there are combinatorial maps $F_A : D^{(2)}_A \to X$ and $F_B : D^{(2)}_B \to X$ extending the maps f_A and f_B, respectively. We glue $D^{(2)}_A$ to $D^{(2)}_B$ along the edge $[u, v]$ and we take as F the union of the maps F_A and F_B. Because u and v are joined by a unique edge, and $D^{(2)}_A$ and $D^{(2)}_B$ have no interior vertices, so does their union. So the map F is the extension of f from the disk $D^{(2)}$ to X as required. □

The following theorems represent higher dimensional versions of certain well known results in small cancellation theory. They will be useful when showing the main result of the section.

Theorem 3.3. *Let X be a simply connected cubical complex and let γ be a cycle in X. Then there exists a nondegenerate van Kampen diagram (D, ϕ) for γ (i.e. ϕ is a nondegenerate combinatorial map) such that ϕ is an isomorphism from the boundary of D to γ.*

Proof. Let (D_0, ϕ_0) be a van Kampen diagram for γ. We will modify this diagram such that it becomes nondegenerate. We do this by constructing another van Kampen diagram (D_0', ϕ_0') for γ such that D_0' is an almost cubical disk. Let e be an edge in D_0 which is mapped by ϕ_0 to a vertex. Then there are two 2-cells in D_0 adjacent to e. We delete the interior of the union of these two 2-cells from D_0 such that the two distinct vertices of e are identified. We obtain an almost cubical 2-complex D' and a combinatorial map $\phi' : D' \to X$ induced from ϕ_0. We repeat the modification procedure with the new cellulation. Now, due to the fact that the cellulation is almost cubical, we must consider two more cases.

The first case is that e is a loop. It then bounds a sub-disk $\triangle$ of D'. There exists a 2-cell C outside $\triangle$ which is adjacent to e. If all edges of C are loops, then we have a nested family of disks bounded by them; take the outermost loop e^* and repeat the argument with e^* in place of e. We eventually arrive at the situation where the three remaining edges of C are embedded. We delete further in D' the interior of the union of some other 2-cells with at least one 1-dimensional face mapped by ϕ' to a vertex and get a new almost cubical disk D'' with the combinatorial map $\phi'' : D'' \to X$ induced from ϕ'.

The second case is that e is adjacent on both sides to the same 2-cell C of D'. Then e is not a loop, and plays the role of two out of the four boundary edges of C. The remaining two edges of C are necessarily loops; thus we are in the situation as in the previous case, and we perform, for each one of the two loops, the modifications as above.

Since a modification reduces the number of 2-cells in D', we eventually obtain a van Kampen diagram (D'_0, ϕ'_0) for γ such that D'_0 is almost cubical, and ϕ'_0 is a nondegenerate map on the 1-skeleton of D'_0, and therefore nondegenerate.

The next step is to further modify the van Kampen diagram so that it remains nondegenerate but it is no longer almost cubical, i.e. such that the 1-skeleton of D'_0 does not contain loops or multiple edges (edges sharing both endpoints). Because ϕ'_0 is nondegenerate, D'_0 does not have loop edges. So it is sufficient to eliminate multiple edges, while keeping induced maps to X nondegenerate.

Let e_1, e_2 be edges of D'_0 with common endpoints. Their union bounds a subdisk $\triangle$ of D'_0. Remove the interior of $\triangle$ from D'_0 by gluing the resulting two free edges with each other, getting a new disk D_1 with a new nondegenerate combinatorial map ϕ_1 to X induced from the previous one. The procedure terminates because the number of 2-cells in D'_0 decreases. The applied algorithm does not change the map ϕ_1 on the boundary. (D_1, ϕ_1) is therefore a nondegenerate van Kampen diagram for γ. □

Theorem 3.4. *Given $k \geq 3$, let X be a k-systolic cubical complex and let γ be a cycle in X. Let (D, ϕ) be a nondegenerate van Kampen diagram for γ. If D has minimal area, then D is k-systolic. If moreover γ is full, then D has at least one interior vertex and any second boundary vertex of D is contained in at least two 2-cells of D.*

Proof. Let (D_1, ϕ_1) be a nondegenerate van Kampen diagram for γ. Let v be an interior vertex of D_1 with degree less than k. We construct another van Kampen diagram (D'_1, ϕ'_1) for γ such that D'_1 has one interior vertex less than D_1. Namely, we delete the interior of the subdisk $\mathrm{St(v, D_1)}$, replace it by some cellulation given by Lemma 3.2, and define ϕ'_1 so that it coincides with ϕ_1 on $D_1 \setminus [\mathrm{St(v, D_1)}]$. The resulting van Kampen diagram is not necessarily nondegenerate and it has fewer cells. We apply to it procedure used in the proof of the previous theorem and we produce a nondegenerate van Kampen diagram with fewer cells. Because D'_1 is finite, the procedure terminates after finitely many steps yielding a cubical disk D_2 of minimal area and a combinatorial map $\phi_2 : D_2 \to X$ which coincides with ϕ'_1 on the boundary of D_2. The pair (D_2, ϕ_2) is therefore another van Kampen diagram for γ which is nondegenerate, of minimal area, and with no interior vertices of degree less than k.

Suppose that every exterior vertex of D_2 has degree exactly 2. Every exterior vertex of D_2 is then contained in a single 2-cell of D_2. Suppose further that D_2 has no interior vertices. In both cases the boundary ∂D_2 is not full in D_2 implying that γ is not a full cycle in X which is a contradiction. So every second exterior vertex of D has degree at least 3, and D has at least one interior vertex. Note that every exterior vertex of D lying between two exterior vertices with degree at least 3, has degree exactly 2. □

We give further the main result of the section.

Theorem 3.5. *For $k \geq 4$ natural, any k-systolic cubical complex X is k-large.*

Proof. We need to show that $sys(X) \geq 2k$. Let γ be a full cycle in X. Because X is simply connected, there exists a van Kampen diagram (D, ϕ) for γ. Using relative Simplicial Approximation Theorem for cell complexes (see [5], chapter II.4, page 146), we can arrange that D is a cubical disk and ϕ is a combinatorial map (which is a cell homeomorphism on the boundary). We choose the disk D to be of minimal area. Theorem 3.4 guarantees that the degree of every interior vertex of D is at least k, and the degree of every second exterior vertex of D is at least 3. Note that there exists, between any two exterior vertices of D with degree is at least 3, an exterior vertex with degree exactly 2. D being cubical, the number of exterior vertices of D with degree exactly 2 equals the number of exterior vertices of D with degree at least 3 and it is equal to $\frac{\sharp(\partial D)}{2}$. Lemma 3.1 implies that

$$4 = \sum_{v \in int(D)} (4 - \deg v) + \sum_{v \in \partial D} (3 - \deg v).$$

The first sum of the above equation is at most $4 - k$, while the second sum is at most $\frac{\sharp(\partial D)}{2}$. So $\frac{\sharp(\partial D)}{2} \geq k$ and therefore $|\gamma| = \sharp(\partial D) \geq 2k$. Thus $sys(X) \geq 2k$. □

4 The main result

In this section we show that the piecewise hyperbolic structure on a 5-systolic cubical complex in which each cube is a regular hyperbolic cube of edge length $\varepsilon > 0$, and with only finitely many shapes of cells is, for some ε, CAT(-1).

We start by giving a combinatorial characterization of the link of a vertex in a 5-large cubical complex.

Lemma 4.1. *Let X be a $p-$dimensional 5-large cubical complex. Then the link of any vertex of X is a 5-large simplicial cell complex.*

Proof. Let v be a vertex of X. We note that $\mathrm{Lk(v,X)}$ is a piecewise spherical simplicial cell complex of size $< \frac{\pi}{2}$ and of dimension $p-1$. Because X is 5-large, we have

$$sys(\mathrm{Lk(v,X)}) \geq 5. \tag{4.1}$$

We note further that for any cell $\alpha^{(j+1)} \in X$ such that v is one of its vertices, there exists a simplex $\sigma_\alpha^{(j)} \in \mathrm{Lk(v,X)}$, $j \in \{0, ..., p-1\}$. Lemma 2.1 guarantees that

$$\mathrm{Lk}(\sigma_\alpha, \mathrm{Lk(v,X)}) = \mathrm{Lk}(\alpha, \mathrm{X}).$$

X being 5−large, we have $sys(\mathrm{Lk}(\alpha, \mathrm{X})) \geq 5$. Thus it follows that

$$sys(\mathrm{Lk}(\sigma_\alpha, \mathrm{Lk}(\mathrm{v}, \mathrm{X}))) \geq 5, \tag{4.2}$$

for any simplex $\sigma_\alpha^{(j)} \in \mathrm{Lk}(\mathrm{v}, \mathrm{X})$, $j \in \{0, ..., p-1\}$. In conclusion, the relations 4.1 and 4.2 ensure that $\mathrm{Lk}(\mathrm{v}, \mathrm{X})$ is 5-large. □

The link L of a vertex of a cubical complex of edge lengths $\varepsilon > 0$ in hyperbolic space is a regular piecewise spherical simplex cell complex whose edges have length $< \frac{\pi}{2}$. As ε increases from 0, each simplex in L is a deformation of the all right regular piecewise spherical simplex. So, the piecewise spherical structure on L deforms from its usual all right structure. For small ε, this deformation remains CAT(1) if and only if in the all right structure on L, for each simplex $\sigma \in L$, the infimum of the lengths of all closed geodesics in $Lk(\sigma, L)$ is strictly greater than 2π, condition called extra largeness. Any all right, piecewise spherical flag simplicial complex is, according to L. Siebenmann, extra large if and only if it satisfies the no-square condition (see [8], Appendix I.6, page 518). So extra largeness holds for L if and only if it satisfies the flag-no-square condition. A polyhedral complex with finitely many shapes of cells, has curvature ≤ -1 if and only if the link of each of its vertices is a CAT(1) space (see [8], Appendix I.3, page 509). Thus, the piecewise hyperbolic cubical structure on a complex of edge lengths ε has curvature ≤ -1, for some ε, if and only if the link of each of its vertices satisfies the flag-no-square condition.

Theorem 4.2. *Let X be a finite dimensional locally 5-large cubical complex endowed with the piecewise hyperbolic metric obtained by declaring each cube to be isometric to a regular hyperbolic cube of edge length $\varepsilon > 0$, and whose simplices represent only finitely many isometry classes. Then the star of any cell of X has, for some ε, curvature ≤ -1.*

Proof. Let τ be any cell of X. We denote the star of τ in X, $\mathrm{St}(\tau, \mathrm{X})$, by S. We note that S is a 5-large cubical complex. Let v be any vertex in S and let L denote the link of v in S. According to Lemma 4.1, L is a 5-large piecewise spherical simplicial cell complex of size $< \frac{\pi}{2}$. So L is 4-large and therefore flag. L being 5-large, any cycle in L of length less than 5 has, according to Remark 2.1, some two consecutive edges contained in a common 2-simplex of L. Because in a flag complex such cycle is not an empty 4-circuit, and because $sysL \geq 5$, L contains no empty 4-circuits. So the link of any vertex of S is a flag simplicial cell complex with no empty 4-circuits. According to Theorem 2.2, this guarantees that, for some $\varepsilon > 0$, S has curvature ≤ -1. □

Because an 5-systolic cubical complex is, according to Theorem 3.5, 5-large, the above theorem implies that such complex has curvature ≤ -1. Because a simply connected, locally CAT(-1) space, is a CAT(-1) space, the main result of the paper follows.

Corollary 4.3. *The piecewise hyperbolic metric obtained by declaring each cube to be isometric to a regular hyperbolic cube of edge length $\varepsilon > 0$ on any finite dimensional 5-systolic cubical complex whose simplices represent only finitely many isometry classes, is, for some ε, a CAT(-1) metric.*

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